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Probabilistic solution to a two-dimensional LQG homing problem*

by

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Abstract: Let x_t be an arbitrary one-dimensional diffusion process and y_t be a one-dimensional controlled diffusion process starting from $y_0 = y \in (a, b)$. The process is controlled until y_t crosses either $y = a$ or $y = b$ for the first time. Our aim is to find the control u_t^* that minimizes an expected cost functional with both quadratic control and boundary crossing costs. An explicit form for the optimal control is obtained under certain conditions.

Keywords: optimal control, dynamic programming, Whittle's theorem, diffusion process, hitting time.

1. Introduction

We consider a two-dimensional controlled process (x_t, y_t) defined by the stochastic differential equations

$$\begin{aligned} dx_t &= \mu(x_t)dt + \sigma(x_t)dB_t^1, \\ dy_t &= m(y_t)dt + \beta(x_t, y_t)u_t dt + \{v(y_t)\}^{1/2} dB_t^2, \end{aligned} \tag{1}$$

where $\mu(\cdot)$, $\sigma(\cdot) > 0$, $m(\cdot)$, $\beta(\cdot,\cdot)$ and $v(\cdot) > 0$ are Borel measurable functions, B_t^1 and B_t^2 are independent Brownian motions defined on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$, and $u_t = u(x_t, y_t)$ for all t is a measurable $u : \mathbf{R}^2 \to \mathbf{R}$ control variable.

Assume that the process (x_t, y_t) starts from (x, y) in the domain D defined by

$$D = \{(x, y) \in \mathbf{R}^2 : a < y < b\}, \tag{2}$$

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where a, b are some fixed constants, and define

$$\tau^u = \inf\{t > 0 : y_t = a \text{ or } y_t = b \mid (x, y) \in D\}. \tag{3}$$

That is, τ^u denotes the first time the controlled process y_t crosses the barrier a or b.

Our aim is to find the control u_t^* that minimizes the expected cost functional

$$J(x, y) := \mathbf{E}^{x,y}\left[\int_0^{\tau^u}\left(\frac{1}{2}q(x_t, y_t)u_t^2 + \lambda\right)dt + H(x_{\tau^u}, y_{\tau^u})\right] \tag{4}$$

where $q(\cdot, \cdot)$ is a positive function, $\lambda > 0$ is a positive constant, and $H(\cdot, \cdot)$ is a general terminal cost function.

The problem treated in this note is a particular LQG homing problem, as defined by Whittle (1982) and considered, in particular, by Lefebvre (1987, 1991, 1994), and by Makasu (2009).

An LQG homing problem is a problem in which the optimizer tries to bring a controlled diffusion process into a given stopping set as quickly as possible, while taking the control costs into account. However, if the parameter λ in (4) is negative, the aim is actually to maximize the survival time in the continuation region. Moreover, even if Whittle used the term "LQG homing", the problems considered need not be linear (respectively, quadratic) in u_t in the plant equation (1) (respectively, the cost functional (4)).

An application of such problems is the following: imagine that the controlled process describes the path of an aircraft and that the objective is to land this aircraft in minimum time. In the case of the model considered here, $y = a$ could be the height at which a military aircraft is likely to be detected by a radar. Therefore, the controller wants to increase the aircraft's height to $y = b$ rapidly, and without ever hitting $y = a$.

In the next section, we will obtain the solution to our problem, although, as will be seen, we cannot appeal to the theorem proved in Whittle (1982). Important particular cases will be solved explicitly in Section 3.

2. Optimal solution for the LQG homing problem

Let

$$\psi(x, y) := \inf_{u_t, 0 \le t \le \tau^u} J(x, y).$$

Assuming that the function $\psi(x, y)$ exists and is twice differentiable, it turns out that it satisfies the Hamilton-Jacobi-Bellman equation (see Fleming and Rishel, 1975, and Whittle, 1982)

$$\inf_u \left\{\frac{1}{2}\sigma^2(x)\psi_{xx}(x, y) + \frac{1}{2}v(y)\psi_{yy}(x, y) + \mu(x)\psi_x(x, y) + m(y)\psi_y(x, y) \right.$$
$$\left. + \beta(x, y)u\psi_y(x, y) + \frac{1}{2}q(x, y)u^2 + \lambda\right\} = 0 \tag{5}$$

for $(x,y) \in D$, where $u := u_0$. Furthermore, we assume that the boundary conditions are

$$\psi(x,y) = \begin{cases} h(x) \text{ if } y = b, \\ \\ +\infty \text{ if } y = a, \end{cases} \tag{6}$$

where $h(\cdot)$ is a twice continuously differentiable function on $(0,\infty)$. Hence, we force the process y_t to exit (a,b) through the right end of the interval.

The control u^* that minimizes the expression in (5) is

$$u^* = -\beta(x,y)\frac{\psi_y(x,y)}{q(x,y)}.$$

Substituting u^* into (5), it follows that

$$\frac{1}{2}\sigma^2(x)\psi_{xx}(x,y) + \frac{1}{2}v(y)\psi_{yy}(x,y) + \mu(x)\psi_x(x,y) + m(y)\psi_y(x,y) \\ -\frac{\beta^2(x,y)}{2q(x,y)}\psi_y^2(x,y) + \lambda = 0. \tag{7}$$

Let y_t be the process given in (1), and let

$$\tau^0 := \inf\{t > 0 : y_t^0 = b \text{ and } y_s^0 > a \text{ for } s \in [0,t]\}, \tag{8}$$

where y_t^0 is the uncontrolled process that corresponds to y_t, starting from $y \in (a,b)$.

We shall prove the following proposition.

PROPOSITION 2.1 *Assume that*

A_0: *there exists a positive constant α such that*

$$\alpha = \frac{v(y)q(x,y)}{\beta^2(x,y)} \quad \forall\,(x,y) \in D; \tag{9}$$

A_1: *the function $h(x)$ satisfies the second order ordinary differential equation*

$$\frac{1}{2}\sigma^2(x)h''(x) + \mu(x)h'(x) = k,$$

where k is a constant and

A_2: $\mathbf{P}[\tau^0 < \infty] = 1$ *a.s., where the first passage time τ^0 is given by (8).*

Then the optimal control u^* that minimizes the expected cost functional (4) is given by

$$u^*(x,y) = \frac{v(y)}{\beta(x,y)}\frac{\phi'(y)}{\phi(y)} \quad \forall\,(x,y) \in D, \tag{10}$$

where

$$\phi(y) := \mathbf{E}^y\left[\exp\left\{-\left(\frac{k+\lambda}{\alpha}\right)\tau^0\right\}\right], \tag{11}$$

in which the expectation is taken over the first passage time τ^0.

Proof. Whittle (1982) has shown that if the relation (in our case)

$$\begin{pmatrix} \sigma^2(x) & 0 \\ 0 & v(y) \end{pmatrix} = \alpha \begin{pmatrix} 0 & 0 \\ 0 & \beta(x,y) \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1/q(x,y) \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & \beta(x,y) \end{pmatrix}$$

holds for a positive constant α, then the transformation

$$\Phi(x,y) := e^{-\psi(x,y)/\alpha}$$

linearizes the partial differential equation (7). Notice, however, that here the relation in question is *not* satisfied. Nevertheless, using this transformation and Assumption A_0, we obtain the new non-linear partial differential equation

$$\frac{1}{2}\sigma^2(x)\Phi_{xx}(x,y) + \frac{1}{2}v(y)\Phi_{yy}(x,y) + \mu(x)\Phi_x(x,y) + m(y)\Phi_y(x,y) \\ -\frac{1}{2}\sigma^2(x)\frac{\Phi_x^2(x,y)}{\Phi(x,y)} - \frac{\lambda}{\alpha}\Phi(x,y) = 0 \qquad (12)$$

for all $(x,y) \in D$. The boundary conditions are now of the form

$$\Phi(x,y) = \begin{cases} e^{-h(x)/\alpha} \text{ if } y = b, \\ 0 \text{ if } y = a. \end{cases} \qquad (13)$$

Next, assume that the solution $\Phi(x,y)$ is of the form

$$\Phi(x,y) = \eta(x)\phi(y), \qquad (14)$$

where $\eta(x) := e^{-h(x)/\alpha}$ and $\phi(y) : \mathbf{R} \to (0,1)$ is a twice continuously differentiable function.

Then, using the transformation (14), Eq. (12) reduces to

$$\frac{1}{2}v(y)\phi''(y) + m(y)\phi'(y) - \frac{1}{\alpha}\left(\frac{1}{2}\sigma^2(x)h''(x) + \mu(x)h'(x) + \lambda\right)\phi(y) = 0. \qquad (15)$$

Finally, with the help of Assumption A_1, it turns out that $\phi(y)$ is a solution to

$$\frac{1}{2}v(y)\phi''(y) + m(y)\phi'(y) = \gamma\phi(y), \qquad (16)$$

where $\gamma := (k+\lambda)/\alpha$, subject to $\phi(a) = 0$ and $\phi(b) = 1$, from which we deduce the probabilistic representation (11). ■

Remarks.

1. The constant γ should be such that the mathematical expectation in (11) exists.

2. Notice that Assumption A_0 is always satisfied when the functions v, q and β are all constants.
3. The random variable τ^0 denotes the time taken by the uncontrolled process y_t^0 to hit $y = b$, without ever hitting the barrier at a.
4. The condition $\mathbf{P}[\tau^0 < \infty] = 1$, which is needed to ensure uniqueness of the solution to our problem, is not restrictive when the interval (a, b) is finite.

COROLLARY 2.1 *Let*

$$\phi(y) = c_1 p_1(y) + c_2 p_2(y)$$

be the general solution of (16), where p_1 and p_2 are the fundamental solutions and c_1 and c_2 are arbitrary constants. Then, the optimal control u^* is given explicitly by

$$u^*(x, y) = \frac{v(y)}{\beta(x, y)} \left\{ \frac{p_1(a)p_2'(y) - p_1'(y)p_2(a)}{p_1(a)p_2(y) - p_1(y)p_2(a)} \right\} \quad \forall\, (x, y) \in D. \tag{17}$$

In the next section, some important particular cases will be solved explicitly.

3. Particular cases

(a) In the first particular case, we consider the process (x_t, y_t) in (1) with $\mu(x_t) \equiv 0$, $\sigma^2(x_t) \equiv 1$, $m(y_t) \equiv 0$, $\beta(x_t, y_t) \equiv 1$ and $v(y_t) \equiv 1$. Then, x_t is a standard Brownian motion, and y_t is a controlled standard Brownian motion.
Moreover, we take $q(x_t, y_t) \equiv q_0 > 0$. It follows that the constant α in (9) is given by

$$\alpha \equiv q_0.$$

Next, the function $h(x)$ must satisfy the second order ordinary differential equation

$$\frac{1}{2} h''(x) = k.$$

We find at once that $h(x)$ must be of the form

$$h(x) = kx^2 + k_1 x + k_0,$$

where k_1 and k_0 are arbitrary constants. In particular, we can take $h(x) = kx^2$.
Finally, the function $\phi(y)$ is such that

$$\frac{1}{2} \phi''(y) = \gamma \phi(y).$$

That is, if γ is positive, we have that

$$\phi(y) = c_1 e^{\sqrt{2\gamma} y} + c_2 e^{-\sqrt{2\gamma} y}.$$

Making use of the boundary conditions $\phi(a) = 0$ and $\phi(b) = 1$, we find that

$$\phi(y) = \frac{e^{\sqrt{2\gamma}(y-a)} - e^{-\sqrt{2\gamma}(y-a)}}{e^{\sqrt{2\gamma}(b-a)} - e^{-\sqrt{2\gamma}(b-a)}} = \frac{\sinh[\sqrt{2\gamma}(y-a)]}{\sinh[\sqrt{2\gamma}(b-a)]}.$$

We then deduce from (10) that the optimal control is

$$u^* = \sqrt{2\gamma} \coth[\sqrt{2\gamma}(y-a)] \quad \text{for } y \in (a, b).$$

Notice that u^* tends to ∞ as y decreases to a, which is logical, because the optimizer wants to avoid receiving the infinite penalty incurred when $y_{\tau^u} = a$.

(b) Let $\mu(x_t) = x_t$, $\sigma^2(x_t) = x_t^2$, $m(y_t) = y_t$, $\beta(x_t, y_t) = y_t$ and $v(y_t) = y_t^2$, so that x_t is a geometric Brownian motion, and y_t is a controlled geometric Brownian motion.
As above, we take $q(x_t, y_t) \equiv q_0 > 0$. Then, the constant α in (9) is again equal to q_0.
This time, the function $h(x)$ must satisfy

$$\frac{1}{2} x^2 h''(x) + x h'(x) = k. \tag{18}$$

It is a simple matter to show that

$$h(x) = 2k \ln(x) + k_1 \frac{1}{x} + k_0 \tag{19}$$

does satisfy (18).
Thus, setting for instance $k_1 = k_0 = 0$, it follows that $h(x) = 2k \ln(x)$. Notice that here x is always positive, because a geometric Brownian motion has a natural boundary at the origin.
Next, the function $\phi(y)$ is such that

$$\frac{1}{2} y^2 \phi''(y) + y \phi'(y) = \gamma \phi(y). \tag{20}$$

If $\gamma > -1/8$, then the solution that satisfies $\phi(a) = 0$ and $\phi(b) = 1$ (with $a > 0$) is

$$\phi(y) = \frac{\sqrt{b}}{\sqrt{y}} \frac{a^{-\delta} y^{\delta} - a^{\delta} y^{-\delta}}{a^{-\delta} b^{\delta} - a^{\delta} b^{-\delta}}, \tag{21}$$

where

$$\delta := \frac{1}{2} \sqrt{1 + 8\gamma}.$$

Finally, (10) implies that the optimal control is given by

$$u^* = -\frac{1}{2} + \delta \frac{y^{2\delta} + a^{2\delta}}{y^{2\delta} - a^{2\delta}} \quad \text{for } y \in (a, b).$$

As in the previous case, u^* tends to ∞ when y decreases to a, as it should. Furthermore, if $\gamma = 0$, we have

$$u^* = \frac{a}{y-a} \quad \text{for } y \in (a,b).$$

Notice that it is not essential that the processes x_t and y_t^0, as in the above cases, should have the same infinitesimal parameters. We shall illustrate this point in the two examples that follow.

(c) In the third particular case that we treat, we take $\mu(x_t) = x_t$ and $\sigma^2(x_t) = x_t^2$, as in the previous case, so that x_t is a geometric Brownian motion, which is always positive. Moreover, we let $m(y_t) \equiv 0$, $\beta(x_t, y_t) = (x_t y_t)^{1/2}$ and $v(y_t) = y_t$, and we assume that the interval (a,b) is $(0,1)$. Finally, we set $q(x_t, y_t) = x_t > 0$. In this particular case, the constant α in (9) is equal to 1.
As in (b), we find that we can also take $h(x) = 2k\ln(x)$. However, the function $\phi(y)$ is now a solution of the second order linear differential equation

$$\frac{1}{2} y\phi''(y) = \gamma\phi(y).$$

We assume that the constant γ is strictly positive. Then, we find that the unique solution satisfying the conditions $\phi(0) = 0$ and $\phi(1) = 1$ is

$$\phi(y) = \sqrt{y}\,\frac{I_1(2\sqrt{2\gamma y})}{I_1(2\sqrt{2\gamma})},$$

where $I_1(\cdot)$ is a modified Bessel function of the first kind. Hence, from (10), we deduce that the optimal control is given by

$$u^* = \frac{\sqrt{2\gamma}}{\sqrt{x}}\,\frac{I_0(2\sqrt{2\gamma y})}{I_1(2\sqrt{2\gamma y})} \quad \text{for } y \in (0,1).$$

We know that we should have $\lim_{y\downarrow 0} u^* = \infty$. Because $I_0(0) = 1$ and $I_1(0) = 0$, the optimal control does indeed tend to ∞ as y decreases to 0.

(d) Consider, finally, the two-dimensional controlled process (x_t, y_t) in (1), where $\mu(x_t) = 1$, $\sigma^2(x_t) = 1$, $m(y_t) = y_t$, $v(y_t) = y_t^2$ and $\beta(x_t, y_t) = y_t\sqrt{x_t^2 + y_t^2}$. Assume that $q(x_t, y_t) = x_t^2 + y_t^2$, then $\alpha = 1$ in this case.
Let $h(x) = e^{-2x}$ such that $k \equiv 0$. Now observe that, in the present case, $\phi(y)$ is still a solution of Eq. (20). Using Proposition 2.1, it follows that the optimal control u^* is of the form

$$u^* = \frac{y}{\sqrt{x^2+y^2}}\,\frac{\phi'(y)}{\phi(y)} \text{ for } y \in (a,b),$$

where $\phi(y)$ is given explicitly by (21) and $a > 0$.

4. Conclusion

We have solved explicitly a two-dimensional LQG homing problem under certain conditions. It is shown that the optimal control can be expressed as a mathematical expectation for the corresponding uncontrolled process, even if the relation that would enable us to appeal to a theorem proved by Whittle (1982) is not satisfied.

The present work can be generalized in various ways. We could consider the same type of problem, but in more than two dimensions. We could also assume that the first passage time τ^u is replaced by

$$\tau^{u,t_0} = \min\{\tau^u, t_0\},$$

where $t_0 > 0$ is a fixed constant, so that we would control the process at most until time t_0. The case with a risk-sensitive cost criterion is yet another possible generalization.

Finally, we could, of course, solve other important special cases. For instance, the case when x_t and y_t^0 are Ornstein-Uhlenbeck processes, or Bessel processes.

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