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Qualitative stability analysis of multicriteria combinatorial minimin problems*

by

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Abstract: A multicriteria combinatorial problem with minimin partial criteria is considered. Necessary and sufficient conditions for the five known stability types of the problem are obtained. These stability types describe in different ways the behavior of the Pareto and lexicographic sets of the problem under initial data perturbations of the vector criteria.

Keywords: combinatorial optimization, Pareto set, lexicographic set, stability analysis, stability types.

1. Introduction

As a rule, when solving many applied problems, the information for mathematical model construction is given inaccurately. This inaccuracy is caused by various factors of uncertainty and randomness such as inadequacy of mathematical models to real processes, rounding off, calculation errors etc. In these cases a mathematical problem cannot be well posed and solved without using the results of stability theory at least implicitly. Investigation of stability of multicriteria discrete optimization problems is usually connected with discrete analogues of the Hausdorff continuity (semi-continuity) of set-valued mappings which put into correspondence each initial data set of a problem with the set of optimal solutions (see, e.g., Tanino, 1988; Miettinen, 1999).

Despite numerous approaches to stability analysis of discrete optimization problems (a comprehensive survey of numerous results is given in annotated bibliographies, Greenberg, 1998; Ehrgott and Gandibleux, 2000) two major directions of investigation can be singled out: quantitative and qualitative.

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The quantitative direction aims to derive bounds for feasible initial data changes preserving some preassigned properties of optimal solutions and create algorithms for the calculation of bounds (see, e.g., Sotskov et al., 1995, 2006, 2010; Chakravarti and Wagelmans, 1998; Libura et al., 1998, 2004, 2007; van Hoesel and Wagelmans, 1999; Emelichev et al., 2002, 2004, 2005, 2010b; Kozeratska et al., 2004; Lai et al., 2004).

The qualitative direction aims to obtain conditions under which the set of optimal solutions of the problem possesses a certain preassigned property of invariance to external influence on initial data of the problem. A number of results in this direction is connected with deriving necessary and sufficient conditions for various stability types of multicriteria integer linear and quadratic programming problems, which consist in finding of Pareto optimal, Slater optimal and Smale optimal solutions (see, e.g., Sergienko and Shilo, 2003; Lebedeva et al., 2005, 2008), as well as boolean and integer problems of sequential minimization of linear function modules (see, e.g., Emelichev et al. 2007, 2010a), multicriteria combinatorial bottleneck problems (see, Emelichev and Kuzmin, 2008) and those with other nonlinear criteria (see Emelichev et al., 2009, 2011).

In this work, we address investigation of qualitative characteristics of stability of discrete multicriteria optimization problems. Analysis of five stability types has been carried out for two multicriteria minimin combinatorial problems: with Pareto and lexicographic principles of optimality. As a result, necessary and at the same time sufficient conditions for each stability type are obtained, and interrelations between these types are revealed.

2. Basic definitions and notations

Let A_i be the i-th row of matrix $A = [a_{ij}] \in \mathbf{R}^{n\times m}$, $n \geq 1$, $m \geq 2$, T be a family of nonempty subsets of $N_m = \{1, 2, \ldots, m\}$ (called trajectories), i.e. $T \subseteq 2^{N_m} \setminus \{\emptyset\}$, $|T| \geq 2$. Let $f : T \times \mathbf{R}^{n\times m} \to \mathbf{R}^n$ be a vector-valued function, where, $f(t, A) = (f_1(t, A_1), f_2(t, A_2), \ldots, f_n(t, A_n))$ for any $t \in T,\ A \in \mathbf{R}^{n\times m}$ and $f_i(t, A_i) = \min\limits_{j\in t} a_{ij}$ for $i \in N_n$. The MINMIN problem can be written down as follows:

$$\min_{t\in T} f(t, A).$$

On the set of trajectories T we define two binary relations of domination

$$t \underset{P,A}{\succ} t' \ \Leftrightarrow\ f(t, A) \geq f(t', A) \ \wedge\ f(t, A) \neq f(t', A),$$

here and henceforth

$$f(t, A) \geq f(t', A) \ \Leftrightarrow\ f_i(t, A_i) \geq f_i(t', A_i) \ \ \forall i \in N_n,$$

$$t \underset{L,A}{\succ} t' \ \Leftrightarrow\ \exists k \in N_n$$
$$\big(f_k(t, A_k) > f_k(t', A_k) \ \wedge\ k = \min\{i \in N_n : f_i(t, A_i) \neq f_i(t', A_i)\}\big).$$

Using these relations we specify the Pareto set (the set of efficient trajectories)

$$P^n(A) = \big\{t \in T : \ \forall t' \in T \ \ (t \underset{P,A}{\overline{\succ}} t')\big\}$$

and the lexicographic set (the set of lexicographically optimal trajectories)

$$L^n(A) = \big\{t \in T : \ \forall t' \in T \ \ (t \underset{L,A}{\overline{\succ}} t')\big\}.$$

Here and further on the line over a binary relation means the negation of the relation, i.e.

$$t \underset{P,A}{\overline{\succ}} t' \ \Leftrightarrow \ f(t,A) = f(t',A) \ \vee \ \exists k \in N_n \ \ (f_k(t,A_k) < f_k(t',A_k)),$$

$$t \underset{L,A}{\overline{\succ}} t' \ \Leftrightarrow \ f(t,A) = f(t',A) \vee \ \exists k \in N_n \ \ \big(f_k(t,A_k) < f_k(t',A_k)$$

$$\wedge \ k = \min\{i \in N_n : \ f_i(t,A_i) \neq f_i(t',A_i)\}\big).$$

Thus, two n-criteria combinatorial problems with minimin criteria arise: the problem $Z_P^n(A)$ of finding the Pareto set $P^n(A)$ and the problem $Z_L^n(A)$ of finding the lexicographic set $L^n(A)$.

Since $2 \leq |T| < \infty$ then $\emptyset \neq L^n(A) \subseteq P^n(A)$ for any matrix $A \in \mathbf{R}^{n\times m}$. Moreover, it is evident that in scalar case ($n = 1$) the Pareto set coincides with the lexicographic set, i.e. $P^1(A) = L^1(A)$ is the set of optimal trajectories, $A \in \mathbf{R}^m$.

It is easy to see that many extreme problems on graphs such as the traveling salesman problem, the assignment problem, the shortest path problem etc. are included in the similar scheme of scalar combinatorial problems (with linear, minimax and other criteria types). Therefore, the elements of T are usually called trajectories.

It is known (see, e.g., Ehrgott, 2005) that the set of lexicographically optimal trajectories $L^n(A)$ can be specified as a result of solving the sequence of n scalar problems

$$L_i^n(A) = \operatorname{Arg\,min}\{f_i(t,A_i) : \ t \in L_{i-1}^n(A)\}, \quad i \in N_n,$$

where $L_0^n(A) = T$, $\operatorname{Arg\,min}\{\cdot\}$ is the set of all optimal trajectories of a corresponding minimization problem. Hence the following inclusions

$$T \supseteq L_1^n(A) \supseteq L_2^n(A) \supseteq \ldots \supseteq L_n^n(A) = L^n(A) \tag{1}$$

are valid.

Let us put into consideration the Smale set (Smale, 1974) and the Slater set (Slater, 1950) correspondingly:

$$Sm^n(A) = \big\{t \in T : \ \forall t' \in T \setminus \{t\} \ \ (t \underset{Sm,A}{\overline{\succ}} t')\big\},$$

$$Sl^n(A) = \big\{t \in T : \ \forall t' \in T \setminus \{t\} \ \ (t \underset{Sl,A}{\overline{\succ}} t')\big\},$$

where

$$t \underset{Sm,A}{\succ} t' \;\Leftrightarrow\; f(t,A) \geq f(t',A),$$

$$t \underset{Sl,A}{\succ} t' \;\Leftrightarrow\; \forall i \in N_n \;\; (f_i(t,A_i) > f_i(t',A_i)).$$

It is evident that $Sm^n(A) \subseteq P^n(A) \subseteq Sl^n(A)$ for any $A \in \mathbf{R}^{n\times m}$ and $Sm^n(A)$ can be empty.

Note that the elements of the Smale and Slater sets are strictly efficient trajectories and weakly efficient trajectories, respectively.

Let us denote for brevity any of the sets $P^n(A)$ or $L^n(A)$ by $M^n(A)$ and the multicriteria problem of finding $M^n(A)$ by $Z^n_M(A)$.

We will investigate five stability types (see, e.g., Emelichev et al., 2002, 2010a) of the multicriteria problem $Z^n_M(A)$. The problem $Z^n_M(A)$ is called S_1-stable if

$$\exists \varepsilon > 0 \;\; \forall A' \in \Omega(\varepsilon) \quad (M^n(A + A') \subseteq M^n(A)),$$

– S_2-stable if

$$\exists \varepsilon > 0 \;\; \forall A' \in \Omega(\varepsilon) \quad (M^n(A) \cap M^n(A + A') \neq \emptyset),$$

– S_3-stable if

$$\exists \varepsilon > 0 \;\; \forall A' \in \Omega(\varepsilon) \quad (M^n(A) \subseteq M^n(A + A')),$$

– S_4-stable if

$$\exists \varepsilon > 0 \;\; \forall A' \in \Omega(\varepsilon) \quad (M^n(A) = M^n(A + A')),$$

and S_5-stable if

$$\exists \varepsilon > 0 \;\; \exists t^0 \in M^n(A) \;\; \forall A' \in \Omega(\varepsilon) \quad (t^0 \in M^n(A + A')).$$

Here

$$\Omega(\varepsilon) = \{A' \in \mathbf{R}^{n\times m} : \; \|A'\| < \varepsilon\}$$

is the set of perturbing matrices $A' = [a'_{ij}]$ with rows A'_i, $i \in N_n$,

$$\|A'\| = \max\{\|A'_i\| : \; i \in N_n\} = \max\{|a'_{ij}| : \; (i,j) \in N_n \times N_m\}.$$

It is easy to understand that S_1- and S_3-stability can be interpreted as discrete analogue of the Hausdorff upper and lower semi-continuity, correspondingly, of the set-valued mapping at a point A, which defines a corresponding principle of optimality. By analogy, S_4-stability can be interpreted as a property of continuity by Hausdorff.

Remark 1 *Directly from the given definitions it follows that:*
1) if the problem $Z_M^n(A)$ is S_1-stable, then it is S_2-stable,
2) if the problem $Z_M^n(A)$ is S_3-stable, then it is S_5-stable,
3) the problem $Z_M^n(A)$ is S_4-stable if and only if it is S_1- and S_3-stable simultaneously,
4) if the problem $Z_M^n(A)$ is S_5-stable, then it is S_2-stable.

We define a positive value

$$\Delta(A) = \min\big\{|a_{ij} - a_{i'j'}| > 0 :\ (i,j) \in N_n \times N_m,\ (i',j') \in N_n \times N_m\big\}$$

for a matrix $A \in \mathbf{R}^{n\times m}$, which contains at least one pair of different elements. The set of such matrices will be denoted by Ξ.

Put

$$N_i(t, A_i) = \operatorname{Arg}\min\{a_{ij} : j \in t\}, \qquad i \in N_n.$$

Let us introduce a notation for Cartesian product of sets $N_i(t, A_i)$, $i \in I \subseteq N_n$:

$$\overline{N}(t, A, I) = N_{i_1}(t, A_{i_1}) \times N_{i_2}(t, A_{i_2}) \times \ldots \times N_{i_k}(t, A_{i_k}),$$

where $I = \{i_1, i_2, \ldots, i_k\} \subseteq N_n$, $i_1 < i_2 < \ldots < i_k$.

For the notations given above the following properties hold.

Property 1 *Let $A \in \Xi$ and $A' \in \Omega(\Delta(A)/2)$. Then for any $i \in N_n$, $t \in T$ the following relations hold*
(a) $N_i(t, A_i + A'_i) \subseteq N_i(t, A_i)$,
(b) $f_i(t, A_i + A'_i) = f_i(t, A_i) + \min\{a'_{ij} :\ j \in N_i(t, A_i)\}$.

Proof. Since, taking into account $A' \in \Omega(\Delta(A)/2)$, $j \notin N_i(t, A_i + A'_i)$ for $j \notin N_i(t, A_i)$, then inclusion (a) is evident. Using this inclusion and $A' \in \Omega(\Delta(A)/2)$, we deduce

$$\begin{aligned}
&f_i(t, A_i + A'_i) = \min\{a_{ij} + a'_{ij} :\ j \in N_i(t, A_i + A'_i)\} = \\
&= \min\{a_{ij} :\ j \in N_i(t, A_i + A'_i)\} + \min\{a'_{ij} :\ j \in N_i(t, A_i + A'_i)\} = \\
&= \min\{a_{ij} :\ j \in N_i(t, A_i)\} + \min\{a'_{ij} :\ j \in N_i(t, A_i)\} = \\
&f_i(t, A_i) + \min\{a'_{ij} :\ j \in N_i(t, A_i)\}.
\end{aligned}$$

The next statement is valid by virtue of continuity of the functions $f_i(t, A_i)$, $i \in N_n$, at A_i.

Property 2 *Let $k \in N_n$. If the inequality $f_k(t, A_k) < f_k(t', A_k)$ is valid, then*

$$\exists \varepsilon > 0 \ \ \forall A' \in \Omega(\varepsilon) \quad (t \underset{P, A+A'}{\succ} t').$$

PROPERTY 3 *If the inclusion* $N_i(t, A_i) \supseteq N_i(t', A_i)$ *is valid for any* $i \in N_n$, *then*

$$\exists \varepsilon > 0 \ \ \forall A' \in \Omega(\varepsilon) \quad (t \underset{M,A+A'}{\succeq} t').$$

Proof. Indeed, if $N_i(t, A_i) \supseteq N_i(t', A_i)$, then there exists $\varepsilon > 0$ such that for any perturbing matrix $A' \in \Omega(\varepsilon)$ we have $N_i(t, A_i + A'_i) \supseteq N_i(t', A_i + A'_i)$. Therefore

$$f_i(t, A_i + A'_i) = f_i(t', A_i + A'_i).$$

Since this equality is valid for any $i \in N_n$, then $t \underset{M,A+A'}{\succeq} t'$. ∎

3. The problem with the Pareto principle of optimality

For vector $v = (v_1, v_2, \dots, v_n) \in \mathbf{R}^n$ and set $I = \{i_1, i_2, \dots, i_k\} \subseteq N_n$, $i_1 < i_2 < \dots < i_k$, we introduce notation

$$v_I = (v_{i_1}, v_{i_2}, \dots, v_{i_k}).$$

We put

$$P^n(t, A) = \{t' \in P^n(A) : \ f(t, A) \geq f(t', A)\},$$
$$I(t, t') = \{i \in N_n : \ f_i(t, A_i) = f_i(t', A_i)\}.$$

It is evident that $I(t, t') \neq \emptyset$ for any trajectories $t \in Sl^n(A)$ and $t' \in P^n(t, A)$. Since the set T is finite, then $P^n(t, A) \neq \emptyset$ for any trajectory $t \in T$. In addition, $t \in P^n(t, A)$ if and only if $t \in P^n(A)$.

THEOREM 1 *The problem* $Z^n_P(A)$, $n \geq 1$, *is* S_1*-stable if and only if for any trajectory* $t \in Sl^n(A)$ *the formula*

$$\forall v \in \overline{N}(t, A, N_n) \ \ \exists t^* \in P^n(t, A) \quad \big(v_{I(t,t^*)} \in \overline{N}(t^*, A, I(t, t^*))\big) \tag{2}$$

is valid.

Formula (2) indicates that for any weakly efficient trajectory t there exists trajectory $t^* \in P^n(t, A)$ which is invariant to small perturbations of problem parameters.

Proof. Necessity. Let the problem be S_1-stable. Let us show that for any $t \in Sl^n(A)$ formula (2) holds.

If $t \in P^n(A)$, then by virtue of $t \in P^n(t, A)$ and $I(t, t) = N_n$, inclusion $v_{I(t,t)} \in \overline{N}(t, A, I(t, t))$ is valid for any vector $v \in \overline{N}(t, A, N_n)$. Thus, formula (2) is valid for $t \in P^n(A)$.

We prove that (2) holds for any trajectory $t \in Sl^n(A) \setminus P^n(A)$ by contradiction. Let there exist $t^0 \in Sl^n(A) \setminus P^n(A)$ and $v^0 = (v_1^0, v_2^0, \ldots, v_n^0) \in \overline{N}(t^0, A, N_n)$ such that

$$\forall t \in P^n(t^0, A) \quad \left(v^0_{I(t^0,t)} \notin \overline{N}(t, A, I(t^0, t))\right),$$

i.e. for any trajectory $t \in P^n(t^0, A)$ different from t^0, there exists $q \in I(t^0, t)$ such that $v_q^0 \notin N_q(t, A_q)$. Then

$$f_q(t^0, A_q) = f_q(t, A_q), v_q^0 \in N_q(t^0, A_q) \setminus N_q(t, A_q).$$

Therefore, defining the elements of perturbing matrix $A^0 = [a_{ij}^0] \in \Omega(\varepsilon)$ according to the rule

$$a_{ij}^0 = \begin{cases} -\alpha, & \text{if } i \in N_n,\ j = v_i^0, \\ 0 & \text{otherwise,} \end{cases}$$

where $0 < \alpha < \varepsilon$, we find

$$f_q(t^0, A_q + A_q^0) = f_q(t^0, A_q) - \alpha < f_q(t, A_q) = f_q(t, A_q + A_q^0).$$

Thus, for any trajectory $t \in P^n(t^0, A)$ we have

$$t^0 \underset{P, A+A^0}{\succeq} t. \tag{3}$$

Now, let $t \in T \setminus P^n(t^0, A)$. Then there exists an index $k \in N_n$ such that $f_k(t^0, A_k) < f_k(t, A_k)$. From this inequality, taking into account the construction of matrix A^0, we have

$$f_k(t^0, A_k + A_k^0) = f_k(t^0, A_k) - \alpha < f_k(t, A_k) - \alpha \leq f_k(t, A_k + A_k^0),$$

which reduces to (3).

Summarizing the above, we conclude that $t^0 \in P^n(A + A^0)$. Therefore, taking into account $t^0 \notin P^n(A)$, we deduce that

$$\forall \varepsilon > 0 \;\; \exists A^0 \in \Omega(\varepsilon) \quad (P^n(A + A^0) \not\subseteq P^n(A)).$$

Thus, problem $Z_P^n(A)$ is not S_1-stable.

Sufficiency. If $P^n(A) = T$, then it is evident that problem $Z_P^n(A)$ is S_1-stable. Therefore, further we assume $P^n(A) \neq T$. Then $A \in \Xi$, i.e. $\Delta(A) > 0$. Further, supposing $t \notin P^n(A)$ and $A' \in \Omega(\Delta(A)/2)$ we consider two possible cases.

Case 1: $t \in T \setminus Sl^n(A)$. Then, according to the definition of the Slater set, there exists a trajectory $t^* \in T$ such that for any index $i \in N_n$ the inequality

$$f_i(t, A_i) > f_i(t^*, A_i) \tag{4}$$

is true and therefore the inequality $f_i(t, A_i) - f_i(t^*, A_i) \geq \Delta(A)$ is also true. From here, taking into account property 1(b), for any index $i \in N_n$ we deduce

$$f_i(t, A_i + A'_i) - f_i(t^*, A_i + A'_i) = f_i(t, A_i) + \min\{a'_{ij} : \ j \in N_i(t, A_i)\} -$$
$$-f_i(t^*, A_i) - \min\{a'_{ij} : \ j \in N_i(t^*, A_i)\} \geq \Delta(A) - 2\|A'_i\| > 0.$$

From here it follows that $t \underset{P,A+A'}{\succ} t^*$.

Case 2: $t \in Sl^n(A)$. Let $v = (v_1, v_2, \ldots, v_n) \in \overline{N}(t, A + A', N_n)$. Then, according to property 1(a), inclusion $v \in \overline{N}(t, A, N_n)$ holds. Therefore, by virtue of (2) there exists trajectory $t^* \in P^n(t, A)$ with condition $v_{I(t,t^*)} \in \overline{N}(t^*, A, I(t, t^*))$, where $I(t, t^*) \neq \emptyset$, i.e. $v_i \in N_i(t, A_i) \cap N_i(t^*, A_i)$ for any $i \in I(t, t^*)$. Wherefrom, taking into account property 1(b), we deduce

$$f_i(t, A_i + A'_i) = a_{iv_i} + a'_{iv_i} = f_i(t, A_i) + a'_{iv_i} = f_i(t^*, A_i) + a'_{iv_i} \geq$$
$$\geq f_i(t^*, A_i) + \min\{a'_{ij} : \ j \in N_i(t^*, A_i)\} = f_i(t^*, A_i + A'_i), \quad i \in I(t, t^*). \quad (5)$$

Taking into account $t^* \in P^n(t, A)$, it is easy to see that set $N_n \setminus I(t, t^*)$ is non empty and for any index $i \in N_n \setminus I(t, t^*)$ inequality (4) is valid. Further, using the same reasons as in the first case we conclude that

$$f_i(t, A_i + A'_i) > f_i(t^*, A_i + A'_i), \quad i \in N_n \setminus I(t, t^*).$$

Hence, by virtue of relations (5), obtained above, we again arrive at $t \underset{P,A+A'}{\succ} t^*$.

Thus,

$$\exists \varepsilon = \Delta(A)/2 > 0 \ \ \forall A' \in \Omega(\varepsilon) \ \ \forall t \notin P^n(A) \quad (t \notin P^n(A + A')).$$

Hence the problem $Z^n_P(A)$ is S_1-stable. ■

THEOREM 2 *The problem $Z^n_P(A)$, $n \geq 1$, is S_2-stable for any matrix $A \in \mathbf{R}^{n\times m}$.*

Proof. The statement of Theorem 2 is evident for $P^n(A) = T$.

Let $P^n(A) \neq T$, $t \in P^n(A)$. Then, for any trajectory $t' \notin P^n(A)$ there exists an index $k \in N_n$, such that $f_k(t, A_k) < f_k(t', A_k)$. Therefore, by virtue of Property 2 there exists a number $\varepsilon(t') > 0$ such that for any perturbing matrix $A' \in \Omega(\varepsilon(t'))$ the relation $t \underset{P,A+A'}{\overline{\succ}} t'$ is true. From here, assuming $\varepsilon = \min\{\varepsilon(t') : \ t' \notin P^n(A)\}$, we obtain

$$\exists \varepsilon > 0 \ \ \forall A' \in \Omega(\varepsilon) \ \ \forall t' \notin P^n(A) \quad (t \underset{P,A+A'}{\overline{\succ}} t'). \quad (6)$$

Further two cases are possible.

Case 1: $t \notin P^n(A + A')$. Then, by virtue of external stability of the Pareto set $P^n(A + A')$ (see, e.g., Ehrgott, 2005) for any trajectory $t' \in T$ there exists a trajectory $t^0 \in P^n(A + A')$ such that $t' \underset{P,A+A'}{\succ} t^0$. Therefore, the formula

$$\exists t^0 \in P^n(A + A') \quad (t \underset{P,A+A'}{\succ} t^0)$$

is valid. From here and (6) we obtain $t^0 \in P^n(A)$. Thus, $t^0 \in P^n(A) \cap P^n(A+A') \neq \emptyset$.

Case 2: $t \in P^n(A+A')$. Then, it is evident that $t \in P^n(A) \cap P^n(A+A') \neq \emptyset$.

Summarizing the above, we conclude that

$$\exists \varepsilon > 0 \ \ \forall A' \in \Omega(\varepsilon) \quad (P^n(A) \cap P^n(A+A') \neq \emptyset),$$

and therefore the problem $Z^n_P(A)$ is S_2-stable. This proves Theorem 2. ∎

For trajectory $t \in P^n(A)$ we introduce now a set of equivalent trajectories

$$Q(t,A) = \{t' \in T : \ f(t,A) = f(t',A)\}.$$

THEOREM 3 *The problem $Z^n_P(A)$, $n \geq 1$, is S_3-stable if and only if*

$$\forall t \in P^n(A) \ \ \forall t' \in Q(t,A) \ \ \forall i \in N_n \quad \big(N_i(t,A_i) \supseteq N_i(t',A_i)\big). \tag{7}$$

Condition (7) indicates that for any two equivalent trajectories t and t' the equality $\overline{N}(t,A,N_n) = \overline{N}(t',A,N_n)$ must hold.

Proof. NecessityAssume the converseLet, contrary to (7), trajectories $t^* \in P^n(A)$, $t^0 \in Q(t^*,A)$ and index $k \in N_n$ exist such that $p \in N_k(t^0,A_k) \setminus N_k(t^*,A_k)$. The existence of this index allows for constructing the perturbing matrix $A^0 = [a^0_{ij}] \in \Omega(\varepsilon)$ with elements

$$a^0_{ij} = \begin{cases} -\alpha, & \text{if } i = k, \ j = p, \\ 0 & \text{otherwise,} \end{cases}$$

where $0 < \alpha < \varepsilon$. Then, taking into account $t^0 \in Q(t^*,A)$, we obtain

$$\begin{aligned} & f_k(t^0, A_k + A^0_k) = a_{kp} - \alpha = f_k(t^0,A_k) - \alpha < f_k(t^0,A_k) \\ & = f_k(t^*,A_k) = f_k(t^*, A_k + A^0_k), \\ & f_i(t^0, A_i + A^0_i) = f_i(t^0,A_i) = f_i(t^*,A_i) = f_i(t^*, A_i + A^0_i) \quad \text{for} \ \ i \neq k. \end{aligned}$$

Therefore, we have

$$t^* \underset{P,A+A^0}{\succ} t^0,$$

i.e. $t^* \notin P^n(A+A^0)$. Hence, in view of $t^* \in P^n(A)$, we deduce

$$\forall \varepsilon > 0 \ \ \exists A^0 \in \Omega(\varepsilon) \quad (P^n(A) \not\subseteq P^n(A+A^0)),$$

which contradicts S_3-stability of the problem $Z^n_P(A)$. ∎

Sufficiency. Let $t \in P^n(A)$ and $t' \in T$. We consider two possible cases.

Case 1: $f(t,A) = f(t',A)$. Then, according to (7) (in view of $t' \in Q(t,A)$), the inclusion $N_i(t,A_i) \supseteq N_i(t',A_i)$ is valid for any index $i \in N_n$. Therefore, by virtue of Property 3 we have

$$\exists \varepsilon(t') > 0 \;\; \forall A' \in \Omega(\varepsilon(t')) \quad (t \underset{P,A+A'}{\not\succ} t'). \tag{8}$$

Case 2: $f(t,A) \neq f(t',A)$. Then there exists an index $k \in N_n$ such that $f_k(t,A_k) < f_k(t',A_k)$. From this inequality, by virtue of Property 2, formula (8) is valid. Wherefrom it follows that each trajectory $t \in P^n(A)$ remains efficient for the problem $Z^n_P(A+A')$ for any perturbing matrix $A' \in \Omega(\varepsilon)$ if $\varepsilon = \min\{\varepsilon(t') : \; t' \in T\}$. Hence, the problem $Z^n_P(A)$ is S_3-stable.

The next result follows from Theorems 1 and 3 by virtue of Remark 1.

Theorem 4 *The problem $Z^n_P(A)$, $n \geq 1$, is S_4-stable if and only if both statements hold:*
(i) $\forall t \in Sl^n(A) \;\; \forall v \in \overline{N}(t,A,N_n) \;\; \exists t^* \in P^n(t,A) \quad \big(v_{I(t,t^*)} \in \overline{N}(t^*,A,I(t,t^*))\big)$,
(ii) $\forall t \in P^n(A) \;\; \forall t' \in Q(t,A) \;\; \forall i \in N_n \quad \big(N_i(t,A_i) \supseteq N_i(t',A_i)\big)$.

Theorem 5 *The problem $Z^n_P(A)$, $n \geq 1$, is S_5-stable if and only if*

$$\exists t^0 \in P^n(A) \;\; \forall t \in Q(t^0,A) \;\; \forall i \in N_n \quad \big(N_i(t^0,A_i) \supseteq N_i(t,A_i)\big). \tag{9}$$

Condition (9) indicates the existence of an efficient trajectory t^0 such that for all trajectories t equivalent to it the inclusion $\overline{N}(t^0,A,N_n) \supseteq \overline{N}(t,A,N_n)$ holds.

Proof. Necessity. Let the problem $Z^n_P(A)$ be S_5-stable. Assume that, contrary to formula (9), for any trajectory $t \in P^n(A)$ there exist $t^* \in Q(t,A)$, $k \in N_n$ and $p \in N_m$ such that $p \in N_k(t^*,A_k) \setminus N_k(t,A_k)$. Then, using the same arguments as in the proof of necessity in Theorem 3, we conclude that

$$\forall \varepsilon > 0 \;\; \exists A^0 \in \Omega(\varepsilon) \quad (t \notin P^n(A+A^0)).$$

Wherefrom we obtain contradiction to S_5-stability of the problem $Z^n_P(A)$.

Sufficiency. Let formula (9) hold. By reasoning analogously as in the proof of sufficiency in Theorem 3 it is easy to show that

$$\exists \varepsilon > 0 \;\; \forall A' \in \Omega(\varepsilon) \;\; \forall t \in T \quad (t^0 \underset{P,A+A'}{\not\succ} t).$$

This formula proves S_5-stability of the problem $Z^n_P(A)$. ■

Summarizing the results obtained in Theorems 1 – 5 and taking into account Remark 1, we conclude that relations between different stability types of the problem $Z^n_P(A)$ are described by the following scheme:

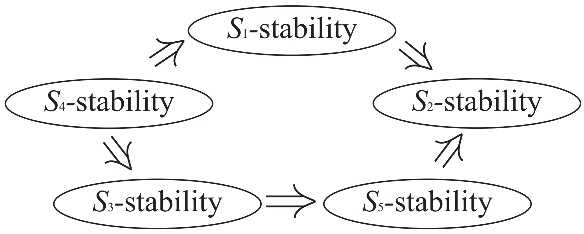

Figure 1.

4. The problem with lexicographic principle of optimality

Let us introduce a set of indexes

$$M(t) = \{i \in N_n : \ t \in L_i^n(A_i)\}.$$

It is easy to see that for $t \in L_1^n(A)$

$$\emptyset \neq M(t) = N_q \subseteq N_n,$$

where $q = \max\{i \in N_n : t \in L_i^n(A)\} = |M(t)|$. Moreover, it is clear that $M(t) = N_n$ if $t \in L^n(A)$.

Theorem 6 *For the problem $Z_L^n(A)$, $n \geq 1$, the following statements are equivalent:*
(i) *the problem $Z_L^n(A)$ is S_1-stable,*
(ii) *the problem $Z_L^n(A)$ is S_2-stable,*
(iii) *for any trajectory $t \in L_1^n(A)$*

$$\forall v \in \overline{N}(t, A, M(t)) \ \ \exists t^* \in L^n(A) \quad \big(v \in \overline{N}(t^*, A, M(t))\big). \tag{10}$$

Statement (iii) indicates that for any non lexicographic trajectory $t \in L_1^n(A)$ there exists trajectory $t^* \in L^n(A)$ that will not allow trajectory t to become lexicographically optimal under small perturbations.

Proof. (i)⇒(ii). This implication is evident (see Remark 1).

(ii)⇒(iii). Let the problem $Z_L^n(A)$ be S_2-stable. We will show that formula (10) is valid for any trajectory $t \in L_1^n(A)$. If $t \in L^n(A)$, then, assuming $t^* = t$, we conclude that formula (10) is true.

We will prove that formula (10) is valid for any trajectory in $L_1^n(A) \setminus L^n(A)$ by contradiction. Let there exist $t^0 \in L_1^n(A) \setminus L^n(A)$ and $v^0 = (v_1^0, v_2^0, \dots, v_q^0) \in \overline{N}(t^0, A, M(t^0))$ such that

$$\forall t \in L^n(A) \quad (v^0 \notin \overline{N}(t, A, M(t^0))). \tag{11}$$

Here, $q = |M(t^0)| < n$, i.e. $t^0 \in L_i^n(A)$ for $i \in N_q$. Therefore, in view of $t \in L^n(A)$ we have

$$f_i(t^0, A_i) = f_i(t, A_i), \quad i \in M(t^0). \tag{12}$$

Since $v_i^0 \in N_i(t^0, A_i)$ for $i \in M(t^0)$, then, according to the definition of $N_i(t, A_i)$, we have

$$f_i(t^0, A_i) = a_{iv_i^0}, \quad i \in M(t^0). \tag{13}$$

From (11) it follows that

$$\forall t \in L^n(A) \ \ \exists k = k(t) \in M(t^0) \ \ (v_k^0 \in N_k(t^0, A_k) \setminus N_k(t, A_k)). \tag{14}$$

Supposing $\varepsilon > 0$ we define the elements of the perturbing matrix $A^0 = [a_{ij}^0] \in \Omega(\varepsilon)$ by the rule

$$a_{ij}^0 = \begin{cases} -\alpha, & \text{if } i \in M(t^0), \ j = v_i^0, \\ 0 & \text{otherwise,} \end{cases}$$

where $0 < \alpha < \varepsilon$. Then, in view of (13), it is evident that

$$f_i(t, A_i + A_i^0) \geq f_i(t, A_i) - \alpha, \quad i \in M(t^0), \tag{15}$$

$$f_i(t^0, A_i + A_i^0) = f_i(t^0, A_i) - \alpha = a_{iv_i^0} - \alpha, \quad i \in M(t^0). \tag{16}$$

Further, we prove that

$$t \underset{L, A+A^0}{\succ} t^0. \tag{17}$$

From (12), (15) and (16), taking into account (14), we obtain

$$f_k(t, A_k + A_k^0) > f_k(t^0, A_k + A_k^0). \tag{18}$$

And if $k > 1$, then, by consequently applying (15), (12), (13), (16) and $N_{k-1} \subset M(t^0)$, we find

$$f_i(t, A_i + A_i^0) \geq f_i(t, A_i) - \alpha = f_i(t^0, A_i) - \alpha = a_{iv_i^0} - \alpha$$
$$= f_i(t^0, A_i + A_i^0), i \in N_{k-1}.$$

From here and (18) we obtain (17).

Thus we have

$$\forall \varepsilon > 0 \ \ \exists A^0 \in \Omega(\varepsilon) \ \ \forall t \in L^n(A) \quad (t \notin L^n(A + A^0)).$$

Hence, the problem $Z_L^n(A)$ is not S_2-stable. This contradiction proves the implication (ii)$\Rightarrow$(iii).

(iii)$\Rightarrow$(i). Statement (i) is evident if $L^n(A) = T$. Therefore, further we assume that $L^n(A) \neq T$. Then $A \in \Xi$, i.e. $\Delta(A) > 0$. Hence, by virtue of property 1(a), for any index $p \in N_i(t, A_i + A'_i)$ the inclusion $p \in N_i(t, A_i)$ is valid. Wherefrom, according to the definition of $N_i(t, A_i)$, we obtain

$$f_i(t, A_i + A'_i) = a_{ip} + a'_{ip} = f_i(t, A_i) + a'_{ip}. \tag{19}$$

It means that

$$f_i(t, A_i) - \|A'_i\| \leq f_i(t, A_i + A'_i) \leq f_i(t, A_i) + \|A'_i\|, \quad i \in N_n, \quad t \in T. \tag{20}$$

Let us show that if $t \notin L^n(A)$ then $t \notin L^n(A + A')$ for any matrix $A' \in \Omega(\Delta(A)/2)$.

Assuming $t \notin L^n(A)$ we consider two possible cases.

Case 1: $t \in T \setminus L^n_1(A)$. Then, there exists t^* such that $f_1(t, A_1) > f_1(t^*, A_1)$. Therefore, the equality $f_1(t, A_1) - f_1(t^*, A_1) \geq \Delta(A)$ is valid. Hence, applying (20) yields the inequalities

$$f_1(t, A_1 + A'_1) - f_1(t^*, A_1 + A'_1) \geq \Delta(A) - 2\|A'_1\| > 0,$$

which imply

$$t \underset{L, A+A'}{\succ} t^*. \tag{21}$$

Case 2: $t \in L^n_1(A)$. Let $v = (v_1, v_2, \ldots, v_q) \in \overline{N}(t, A + A', M(t))$, where $q = |M(t)|$. Then, according to Property 1(a), the inclusion $v \in \overline{N}(t, A, M(t))$ holds. Therefore, in view of (10) there exists $t^* \in L^n(A)$ such that $v \in \overline{N}(t^*, A, M(t))$, i.e. $v_i \in N_i(t, A_i) \cap N_i(t, A_i + A'_i) \cap N_i(t^*, A_i)$, $i \in M(t)$. Hence, taking into account Property 1(b) and (19), we deduce

$$\begin{aligned} &f_i(t, A_i + A'_i) = f_i(t, A_i) + a'_{iv_i} = f_i(t^*, A_i) + a'_{iv_i} \geq \\ &\geq f_i(t^*, A_i) + \min\{a'_{ij} : j \in N_i(t^*, A_i)\} = f_i(t^*, A_i + A'_i), \\ &i \in M(t) = N_q. \end{aligned} \tag{22}$$

Since $t \notin L^n(A)$, then $q = |M(t)| < n$, i.e. $t \notin L^n_{q+1}(A)$, and therefore, taking into account $t^* \in L^n_{q+1}(A)$ (in view of (1)), we conclude that

$$f_{q+1}(t, A_{q+1}) > f_{q+1}(t^*, A_{q+1}).$$

By carrying out the reasoning analogously to the first case further on, we obtain

$$f_{q+1}(t, A_{q+1} + A'_{q+1}) > f_{q+1}(t^*, A_{q+1} + A'_{q+1}),$$

which, together with (7), gives (21).

Thus, it is proved that if $t \notin L^n(A)$, then $t \notin L^n(A + A')$ for any matrix $A' \in \Omega(\Delta(A)/2)$. Hence

$$\exists \varepsilon = \Delta(A)/2 \ \ \forall A' \in \Omega(\varepsilon) \quad (L^n(A + A') \subseteq L^n(A)).$$

Therefore, the problem $Z^n_L(A)$ is S_1-stable. $\blacksquare$

THEOREM 7 *For the problem $Z^n_L(A)$, $n \geq 1$, the following statements are equivalent:*
(i) *the problem $Z^n_L(A)$ is S_3-stable,*
(ii) *the problem $Z^n_L(A)$ is S_4-stable,*
(iii)

$$\forall t \in L^n(A) \;\; \forall i \in N_n \;\; \forall t' \in L^n_i(A) \quad \big(N_i(t, A_i) \supseteq N_i(t', A_i)\big). \tag{23}$$

Formula (23) indicates that any trajectory $t \in L^n(A)$ must not be dominated by trajectories $L^n_i(A)$, $i \in N_n$, under small perturbations of problem parameters.

Proof. (i)⇒(ii). Let the problem $Z^n_L(A)$ be S_3-stable. Then it is S_2-stable and therefore by virtue of Theorem 6 it is S_1-stable. Hence the problem $Z^n_L(A)$ is S_4-stable.

(ii)⇒(iii). Assume the contrary, i.e. the problem $Z^n_L(A)$ is S_4-stable but formula (23) does not hold. Then there exist $t^* \in L^n(A)$, $k \in N_n$ and $t^0 \in L^n_k(A)$ such that $N_k(t^*, A_k) \not\supseteq N_k(t^0, A_k)$ and $f_i(t^0, A_i) = f_i(t^*, A_i)$, $i \in N_k$. Let us fix the index $p \in N_k(t^0, A_k) \setminus N_k(t^*, A_k)$. Then, $a_{kp} = f_k(t^0, A_k) = f_k(t^*, A_k)$. Therefore, assuming $\varepsilon > 0$ and building the elements of the perturbing matrix $A^0 = [a^0_{ij}] \in \mathbf{R}^{n \times m}$ by the rule

$$a^0_{ij} = \begin{cases} -\alpha, & \text{if } i = k, \; j = p, \\ 0 & \text{otherwise,} \end{cases}$$

where $0 < \alpha < \varepsilon$, we conclude that

$$\begin{aligned} & f_k(t^0, A_k + A^0_k) = \min\{a_{kj} + a^0_{kj} : j \in t^0\} = a_{kp} - \alpha < a_{kp} \\ & = f_k(t^*, A_k) = f_k(t^*, A_k + A^0_k), \\ & f_i(t^0, A_i + A^0_i) = f_i(t^0, A_i) = f_i(t^*, A_i) = f_i(t^*, A_i + A^0_i), \quad i \in N_{k-1}. \end{aligned}$$

From here it follows that

$$t^* \underset{L, A + A^0}{\succ} t^0,$$

i.e. $t^* \notin L^n(A + A^0)$. Therefore, in view of $t^* \in L^n(A)$ we deduce that

$$\forall \varepsilon > 0 \;\; \exists A^0 \in \Omega(\varepsilon) \quad (L^n(A) \not\subseteq L^n(A + A^0)).$$

This contradicts S_4-stability of the problem $Z^n_L(A)$.

(iii)⇒(i). Let formula (23) be valid. We will show that for any $t \in L^n(A)$ the formula

$$\exists \varepsilon = \varepsilon(t) > 0 \;\; \forall A' \in \Omega(\varepsilon) \quad (t \in L^n(A + A')) \tag{24}$$

holds.

For $t' \in T$ we consider the following three cases under the condition that $t \in L^n(A)$.

Case 1: $t' \in L^n(A)$. Then, according to (23), for any index $i \in N_n$ the equality $N_i(t, A_i) = N_i(t', A_i)$ is valid. Therefore, according to Property 3 the formula

$$\exists \varepsilon = \varepsilon(t') > 0 \ \ \forall A' \in \Omega(\varepsilon) \quad (t \underset{L, A+A'}{\not\succ} t') \tag{25}$$

holds.

Case 2: $t' \in L_1^n(A) \setminus L^n(A)$. Then, there exists an index $k = k(t') \in N_n \setminus \{1\}$ such that $t' \notin L_k^n(A)$, $t' \in L_i^n(A)$ for $i \in N_{k-1}$. Thus, $f_k(t', A_k) > f_k(t, A_k)$ and, in view of (23) $N_i(t, A_i) \supseteq N_i(t', A_i)$, $i \in N_{k-1}$. Now it is easy to see that, formula (25) holds.

Case 3: $t' \in T \setminus L_1^n(A)$. Then $f_1(t', A_1) > f_1(t, A_1)$. Consequently by virtue of continuity (at A_1) of the function $f_1(t, A_1)$, formula (25) holds.

Thus, formula (25) is valid for any $t' \in T$. Therefore, assuming $\varepsilon(t) = \min\{\varepsilon(t') : t' \in T\}$ it is easy to see that formula (24) is valid. Hence, by choosing $\varepsilon^* = \min\{\varepsilon(t) : \ t \in L^n(A)\}$ we have

$$\exists \varepsilon^* > 0 \ \ \forall A' \in \Omega(\varepsilon^*) \quad (L^n(A) \subseteq L^n(A + A')).$$

Thus, the problem $Z_L^n(A)$ is S_3-stable. ∎

THEOREM 8 *The problem $Z_L^n(A)$, $n \geq 1$, is S_5-stable if and only if*

$$\exists t^0 \in L^n(A) \ \ \forall i \in N_n \ \ \forall t \in L_i^n(A) \quad \big(N_i(t^0, A_i) \supseteq N_i(t, A_i)\big). \tag{26}$$

Formula (26) indicates the existence of the lexicographically optimal trajectory t^0 which must not be dominated by trajectories $L_i^n(A)$, $i \in N_n$, under small perturbations of problem parameters.

Proof. Necessity. Let formula (26) not hold. Then, for any $t \in L^n(A)$ there exist $k \in N_n$ and $t^0 \in L_k^n(A)$ such that $N_k(t, A) \not\supseteq N_k(t^0, A)$. Therefore, carrying out the reasoning analogously as in the proof of implication (ii)⇒(iii) in Theorem 7 we conclude that

$$\forall \varepsilon > 0 \ \ \forall t \in L^n(A) \ \ \exists A^0 \in \Omega(\varepsilon) \quad (t \notin L^n(A + A^0)).$$

Hence, the problem is not S_5-stable.

Sufficiency. Let formula (26) hold. Then as shown in the proof of implication (iii)⇒(i) in Theorem 7 for $t^0 \in L^n(A)$ the formula

$$\exists \varepsilon > 0 \ \ \forall A' \in \Omega(\varepsilon) \quad (t^0 \in L^n(A + A'))$$

is valid (see (24)). Thus the problem $Z_L^n(A)$ is S_5-stable. ∎

Summarizing the results obtained in Theorems 6, 7 and 8, taking into account Remark 1, we conclude that the relations between different stability types of the problem $Z_L^n(A)$ are described by the following scheme:

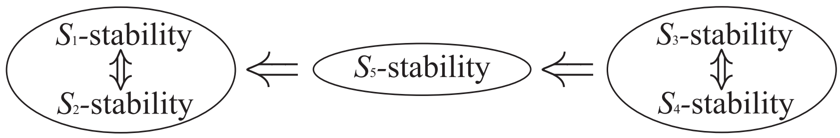

Figure 2.

5. Corollaries

The next four sufficient conditions for different stability types of the problem $Z^n_P(A)$ obviously follow from Theorems 1, 3–5.

COLLORARY 1 *If $P^n(A) = Sl^n(A)$, then the problem $Z^n_P(A)$ is S_1-stable.*

COLLORARY 2 *If $Sm^n(A) = P^n(A)$, then the problem $Z^n_P(A)$ is S_3-stable.*

COLLORARY 3 *If $Sm^n(A) = Sl^n(A)$, then the problem $Z^n_P(A)$ is S_4-stable.*

COLLORARY 4 *If $Sm^n(A) \neq \emptyset$, then the problem $Z^n_P(A)$ is S_5-stable.*

COLLORARY 5 *If the problem $Z^n_L(A)$ is S_1-stable or S_2-stable, then*

$$\forall t \in L^n_1(A) \;\; \exists t^* \in L^n(A) \quad (t \cap t^* \neq \emptyset). \tag{27}$$

Proof. Let formula (27) not hold, i.e. there exists trajectory $t^0 \in L^n_1(A)$ such that $t \cap t^0 = \emptyset$ for any $t \in L^n(A)$. Then $N_i(t, A_i) \cap N_i(t^0, A_i) = \emptyset$ for any $i \in N_n$. Therefore $\overline{N}(t^0, A, M(t^0)) \cap \overline{N}(t, A, M(t^0)) = \emptyset$. Thus

$$\forall t \in L^n_1(A) \;\; \forall v \in \overline{N}(t^0, A, M(t^0)) \;\; (v \notin \overline{N}(t, A, M(t^0))).$$

This formula, by virtue of Theorem 6, proves that the problem $Z^n_L(A)$ is not S_1- and not S_2-stable. ∎

COLLORARY 6 *If*

$$\forall t \in L^n_1(A) \;\; \exists t^* \in L^n(A) \;\; \forall i \in M(t) \;\; (N_i(t, A_i) \subseteq N_i(t^*, A_i)), \tag{28}$$

then the problem $Z^n_L(A)$ is S_1- and S_2-stable simultaneously.

Indeed, if for any index $i \in M(t)$ the inclusion $N_i(t, A_i) \subseteq N_i(t^*, A_i)$ holds, then $\overline{N}(t, A, M(t)) \subseteq \overline{N}(t^*, A, M(t))$. Therefore, formula (10) follows from (28). Hence by virtue of Theorem 6 the problem $Z^n_L(A)$ is S_1- and S_2-stable.

By Corollary 6 it is clear that the following statement is valid.

COLLORARY 7 *If $L^n(A) = L^n_1(A)$, then the problem $Z^n_L(A)$ is S_1- and S_2-stable.*

By virtue of Theorem 7 we have

Collorary 8 *If* $|L_1^n(A)| = 1$*, then the problem* $Z_L^n(A)$ *is* S_3*-,* S_4*- and* S_5*-stable.*

As it was mentioned above, the problems $Z_P^1(A)$ and $Z_L^1(A)$ are the same for the scalar case. This problem consists in finding the set of optimal trajectories $M^1(A) = P^1(A) = L^1(A)$. We denote the problem by $Z^1(A)$.

The following statement obviously follows from Corollary 7.

Collorary 9 *The problem* $Z^1(A)$ *is* S_1*- and* S_2*-stable for any* $A \in \mathbf{R}^m$*.*

It follows from Theorem 7 that

Collorary 10 *The following statements are equivalent for the problem* $Z^1(A)$*:*
(i) *the problem* $Z^1(A)$ *is* S_3*-stable,*
(ii) *the problem* $Z^1(A)$ *is* S_4*-stable,*
(iii)

$$\forall t, t' \in M^1(A) \quad (N_1(t, A_1) = N_1(t', A_1)).$$

If each row of matrix A consists of pairwise different elements then for any two trajectories $t, t' \in T$ and any $i \in N_n$ the equality $f_i(t, A_i) = f_i(t', A_i)$ is equivalent to $N_i(t, A_i) = N_i(t', A_i)$. Therefore, by virtue of Theorem 4 the problem $Z_P^n(A)$ is S_4-stable, wherefrom it follows (see Scheme 1) that the problem is characterised by four other stability types. On the other hand, by virtue of Theorem 7, the problem $Z_L^n(A)$ is S_3- and S_4-stable. Hence (see Scheme 2) the problem is S_1-, S_2- and S_5-stable. Thus it is true that

Collorary 11 *If any row of matrix A consists of pairwise different elements, then the problems* $Z_L^n(A)$ *and* $Z_P^n(A)$ *are* S_k*-stable for any* $k \in N_5$*.*

Remark 1 *All the results in the current paper are valid for any norm in matrix space* $\mathbf{R}^{m\times n}$ *by virtue of the equivalence of any two norms in a finite-dimensional linear space (see, e.g., Suhubi, 2003).*

6. Examples

Let us present a number of examples illustrating the results stated above.

At first we consider examples of problems with Pareto principle of optimality $Z_P^3(A)$.

Example 1 *Problem* $Z_P^3(A)$*. Let* $n = 3$*,* $m = 3$*,* $T = \{t^1, t^2, t^3\}$*,* $t^1 = \{1, 3\}$*,* $t^2 = \{2, 3\}$*,* $t^3 = \{1, 2\}$ *and*

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 3 & 3 \\ 1 & 4 & 3 \end{pmatrix}.$$

Then $f(t^1, A) = (1, 2, 1)$, $f(t^2, A) = (1, 3, 3)$, $f(t^3, A) = (1, 2, 1)$. *Therefore,* $P^3(A) = \{t^1, t^3\}$ *and* $Sl^3(A) = T$. *Observe that*

$$N_1(t^1, A_1) = \{1, 3\},\quad N_1(t^2, A_1) = \{3\},\quad N_1(t^3, A_1) = \{1\},$$
$$N_2(t^1, A_2) = \{1\},\quad N_2(t^2, A_2) = \{2, 3\},\quad N_2(t^3, A_2) = \{1\},$$
$$N_3(t^1, A_3) = \{1\},\quad N_3(t^2, A_3) = \{3\},\quad N_3(t^3, A_3) = \{1\}.$$

Hence

$$\overline{N}(t^1, A, N_3) = \{1, 3\} \times \{1\} \times \{1\},$$
$$\overline{N}(t^2, A, N_3) = \{3\} \times \{2, 3\} \times \{3\},$$
$$\overline{N}(t^3, A, N_3) = \{1\} \times \{1\} \times \{1\}.$$

In addition $I(t^1, t^2) = \{1\}$ *and* $I(t^1, t^3) = \{1, 2, 3\}$. *Therefore, for any trajectory* $t \in Sl^3(A)$ *and any vector* $v \in \overline{N}(t, A, N_3)$ *there exists trajectory* $t^1 \in P^3(t, A)$ *such that* $v_{I(t,t^1)} \in \overline{N}(t^1, A, I(t, t^1))$. *Thus, in view of Theorem 1, problem* $Z_P^3(A)$ *is* S_1*-stable.*

By virtue of Theorem 2 problem $Z_P^3(A)$ *is* S_2*-stable.*

Since $Q(t^1, A) = Q(t^3, A) = \{t^1, t^3\}$ *and* $N_1(t^3, A_1) \supset N_1(t^1, A_1)$, *the condition in Theorem 3 does not hold and therefore problem* $Z_P^3(A)$ *is not* S_3*-stable. Hence, taking into account Scheme 1 the problem is not* S_4*-stable. However, there exists trajectory* $t^1 \in P^3(A)$ *such that for any trajectory* $t \in Q(t^1, A)$ *and any index* $i \in N_3$ *inclusion* $N_i(t^1, A_i) \supseteq N_i(t, A_i)$ *holds. Thus, by virtue of Theorem 5 problem* $Z_P^3(A)$ *is* S_5*-stable.*

EXAMPLE 2 *Problem* $Z_P^3(A)$. *Let* $n = 3$, $m = 3$, $T = \{t^1, t^2, t^3\}$, $t^1 = \{1, 3\}$, $t^2 = \{2, 3\}$, $t^3 = \{1\}$ *and*

$$A = \begin{pmatrix} 1 & 3 & 3 \\ 1 & 1 & 4 \\ 2 & 2 & 4 \end{pmatrix}.$$

Then $f(t^1, A) = (1, 1, 2)$, $f(t^2, A) = (3, 1, 2)$, $f(t^3, A) = (1, 1, 2)$. *Therefore,* $P^3(A) = \{t^1, t^3\}$ *and* $Sl^3(A) = T$. *Observe that*

$$N_1(t^1, A_1) = \{1\},\quad N_1(t^2, A_1) = \{2, 3\},\quad N_1(t^3, A_1) = \{1\},$$
$$N_2(t^1, A_2) = \{1\},\quad N_2(t^2, A_2) = \{2\},\quad N_2(t^3, A_2) = \{1\},$$
$$N_3(t^1, A_3) = \{1\},\quad N_3(t^2, A_3) = \{2\},\quad N_3(t^3, A_3) = \{1\}.$$

Hence

$$\overline{N}(t^1, A, N_3) = \{1\} \times \{1\} \times \{1\},$$
$$\overline{N}(t^2, A, N_3) = \{2, 3\} \times \{2\} \times \{2\},$$
$$\overline{N}(t^3, A, N_3) = \{1\} \times \{1\} \times \{1\}.$$

Moreover, $I(t^1,t^2)=\{2,3\}$ *and* $P^3(t^2,A)=\{t^1,t^3\}$. *Then we have trajectory* $t^2\in Sl^3(A)$ *and vector* $v^0=(2,2,2)\in\overline{N}(t^2,A,N_3)$ *such that for any trajectory* $t\in P^3(t^2,A)$ *relations* $v_{I(t,t^2)}=(2,2)\notin\overline{N}(t,A,I(t,t^2))$ *are valid. Thus, the condition in Theorem 1 does not hold. Consequently the considered problem is not* S_1*-stable.*

By virtue of Theorem 2 problem $Z_P^3(A)$ *is* S_2*-stable.*

Since $Q(t^1,A)=Q(t^3,A)=\{t^1,t^3\}$, *then for any trajectories* $t\in P^3(A)$, $t'\in Q(t,A)$ *and any index* $i\in N_3$ *inclusion* $N_i(t,A_i)\supseteq N_i(t',A_i)$ *is valid. Therefore, according to Theorem 3 problem* $Z_P^3(A)$ *is* S_3*-stable.*

Summarizing the above, in view of Scheme 1 we conclude that problem $Z_P^3(A)$ *is* S_5*-stable and is not* S_4*-stable.*

Further, we consider two examples of problems with lexicographic principle of optimality $Z_L^3(A)$.

Example 3 *Problem* $Z_L^3(A)$. *Let* $n=3$, $m=4$, $T=\{t^1,t^2,t^3\}$, $t^1=\{1,2,3\}$, $t^2=\{1,2,4\}$, $t^3=\{1,4\}$ *and*

$$A=\begin{pmatrix}1&2&1&3\\2&2&3&1\\2&3&3&2\end{pmatrix}.$$

Then $f(t^1,A)=(1,2,2)$, $f(t^2,A)=(1,1,2)$, $f(t^3,A)=(1,1,2)$. *Therefore,* $L^3(A)=\{t^2,t^3\}$ *and* $L_1^3(A)=T$. *Observe that*

$$\overline{N}(t^1,A,N_3)=\{1,3\}\times\{1,2\}\times\{1\},$$
$$\overline{N}(t^2,A,N_3)=\{1\}\times\{4\}\times\{1,4\},$$
$$\overline{N}(t^3,A,N_3)=\{1\}\times\{4\}\times\{1,4\}.$$

Moreover, $M(t^1)=\{1\}$. *Hence we deduce that*

$$\exists t^1\in L_1^3(A)\ \ \exists v^0=3\in\overline{N}(t^1,A,M(t^1))\ \ \forall t\in L^3(A)\ \ \big(v^0\notin\overline{N}(t,A,M(t^1))\big).$$

Thus, according to Theorem 6 problem $Z_L^3(A)$ *neither* S_1*-, nor* S_2*-stable. Consequently by virtue of Scheme 2 problem is not* S_3*-,* S_4*- or* S_5*-stable.*

Example 4 *Problem* $Z_L^3(A)$. *Let* $n=3$, $m=4$, $T=\{t^1,t^2,t^3\}$, $t^1=\{1,2,3\}$, $t^2=\{1,2,4\}$, $t^3=\{2,3,4\}$ *and*

$$A=\begin{pmatrix}1&2&1&3\\2&2&3&1\\2&3&3&2\end{pmatrix}.$$

Then $f(t^1,A)=(1,2,2)$, $f(t^2,A)=(1,1,2)$, $f(t^3,A)=(1,1,2)$. *Therefore,* $L^3(A)=\{t^2,t^3\}$ *and* $L_1^3(A)=T$. *Observe*

$$N_1(t^1,A_1)=\{1,3\},\quad N_1(t^2,A_1)=\{1\},\quad N_1(t^3,A_1)=\{3\},$$
$$N_2(t^1,A_2)=\{1,2\},\quad N_2(t^2,A_2)=\{4\},\quad N_2(t^3,A_2)=\{4\},$$
$$N_3(t^1,A_3)=\{1\},\quad N_3(t^2,A_3)=\{1,4\},\quad N_3(t^3,A_3)=\{4\}.$$

Hence

$$\overline{N}(t^1, A, N_3) = \{1, 3\} \times \{1, 2\} \times \{1\},$$
$$\overline{N}(t^2, A, N_3) = \{1\} \times \{4\} \times \{1, 4\},$$
$$\overline{N}(t^3, A, N_3) = \{3\} \times \{4\} \times \{4\}.$$

In addition, $M(t^1) = \{1\}$ *and* $M(t^2) = M(t^3) = \{1, 2, 3\}$. *Therefore for any trajectory* $t \in L_1^3(A)$ *and any vector* $v \in \overline{N}(t, A, M(t))$ *there exists a lexicographically optimal trajectory* t^* *such that inclusion* $v \in \overline{N}(t^*, A, M(t))$ *is valid. Thus, according to Theorem 6 the considered problem is both* S_1*- and* S_2*-stable.*

However, there exist trajectories $t^2 \in L^3(A)$, $t^3 \in L_1^3(A)$ *and index* $k = 1$ *such that* $N_1(t^2, A_1) \not\supseteq N_1(t^3, A_1)$. *Therefore, by virtue of Theorem 7 problem* $Z_L^3(A)$ *is not* S_3*-stable or* S_4*-stable.*

Finally, since there exists $t^1 \in L_1^n(A)$ *such that for any* $t \in L^n(A)$ *it is true that* $N_1(t, A_1) \not\supseteq N_1(t^1, A_1)$, *then by virtue of Theorem 8 the problem is not* S_5*-stable.*

Now we provide an example for special case of the multicriteria combinatorial center and median location problems (Christofides, 1975; Daskin, 1995), which is included in the scheme of multicriteria combinatorial minimin problem.

Example 5 *The problem* $Z_P^2(A)$ *of consumer assignment to the facilities. Given a graph which represents a road network with its vertices representing communities, one may have the problem of locating optimally a hospital, police station, fire station, or any other "emergency" service facility. Suppose optimal locations of emergency centers for the given communities have already been found. We want to add one new community to the network. The problem consists in finding among the existing facilities one that optimally serves the new community (consumer).*

Consider the following numerical example of problem $Z_P^2(A)$. *Let* $N_4 = \{1, 2, 3, 4\}$ *be facility centers and*

$$A = \begin{pmatrix} 1 & 1 & 2 & 2 \\ 2 & 3 & 3 & 1 \end{pmatrix}$$

be a matrix where numbers in the first row represent the distance from facility centers to the consumer and the second row represents the service cost. Further, let

$$t^1 = \{1, 2\}, \quad t^2 = \{1, 3\}, \quad t^3 = \{2, 3\}, \quad t^4 = \{3, 4\}$$

be possible sets of centers which can serve the consumer. Then

$$f(t^1, A) = (1, 2), \quad f(t^2, A) = (1, 2), \quad f(t^3, A) = (1, 3), \quad f(t^4, A) = (2, 1)$$

and therefore

$$P^2(A) = \{t^1, t^2, t^4\} \quad \text{and} \quad Sl^2(A) = \{t^1, t^2, t^3, t^4\}.$$

Since $I(t^3, t^1) = I(t^3, t^2) = \{1\}$, $P^2(t^1, A) = P^2(t^2, A) = P^2(t^3, A) = \{t^1, t^2\}$ *and* $P^2(t^4, A) = \{t^4\}$, *then taking into account the sets found above it is easy to see that the condition of Theorem 1 holds. Consequently, the problem is* S_1*-stable and therefore (see Remark 1) it is* S_2*-stable.*

Further, we deduce

$$N_1(t^1, A_1) = \{1, 2\}, \quad N_1(t^2, A_1) = \{1\}, \quad N_1(t^3, A_1) = \{2\}, \quad N_1(t^4, A_1) = \{3, 4\},$$
$$N_2(t^1, A_2) = \{1\}, \quad N_2(t^2, A_2) = \{1\}, \quad N_2(t^3, A_2) = \{2, 3\}, \quad N_2(t^4, A_2) = \{4\}.$$

Hence we have

$$N_1(t^1, A_1) \not\subseteq N_1(t^2, A_1).$$

Therefore, taking into account equivalence of trajectories t^1 *and* t^2*, by virtue of Theorem 3 we conclude that the problem* $Z_P^2(A)$ *is not* S_3*-stable and consequently it is not* S_4*-stable (see Scheme 1).*

Finally, since formula

$$\exists t^1 \in P^2(A) \quad \forall t \in Q(t^1, A) = \{t^1, t^2\} \quad \forall i \in N_2 \quad \big(N_i(t^1, A_i) \supseteq N_i(t, A_i)\big)$$

is valid, then by virtue of Theorem 5 we conclude that problem $Z_P^2(A)$ *is* S_5*-stable.*

7. Conclusions

In the present work we proposed a general theoretical approach to qualitative analysis of the multicriteria combinatorial minimin problems with Pareto and lexicographic principles of optimality. Necessary and sufficient conditions of five stability types of the problems are obtained and interrelations between stability types are revealed (Schemes 1 and 2). The proved theorems allow for analyzing and predicting the behavior of the Pareto and lexicographic sets under different types of uncertainty without solving the perturbed variant of the considered problem.

Many extreme problems on graphs such as the traveling salesman problem, the assignment problem, the shortest path problem etc. are included in the similar scheme of scalar combinatorial problems. In addition, the multicriteria minimin problem is a special case of multicriteria variant of the well known median and center location problems.

One more issue which has to be emphasized is that practical verification of conditions of Theorems 1 – 8 and their straightforward application for general case can be as hard as to solve the problem itself. Nevertheless we believe that more methodological results might be developed and implemented for special cases of the multicriteria combinatorial minimin problem with restrictions of some factors, such as structure of initial data, perturbations of particular problem parameters etc. As possible continuation of the research within this topic, it would be interesting to explore these classes of problems.

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