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An approach to multiple attribute decision making with combined weight information in interval-valued intuitionistic fuzzy environment*

by

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Abstract: With respect to multiple attribute decision making problems with interval-valued intuitionistic fuzzy information, some operational laws of interval-valued intuitionistic fuzzy numbers, score function and accuracy function of interval-valued intuitionistic fuzzy numbers are introduced. A combined optimization model based on the deviation method, by which the attribute weights can be determined, is established. For special situations, in which information about attribute weights is completely unknown, we establish another combined optimization model. By solving this model, we get a simple and exact formula, which can be used to determine the attribute weights. We utilize the interval-valued intuitionistic fuzzy weighted averaging (IIFWA) operator to aggregate the intuitionistic fuzzy information corresponding to each alternative, and then rank the alternatives and select the most desirable one(s) according to the score and accuracy functions. Finally, an illustrative example is given to verify the developed approach and to demonstrate its practicality and effectiveness.

Keywords: multiple attribute decision making; interval-valued intuitionistic fuzzy number; interval-valued intuitionistic fuzzy weighted averaging (IIFWA); weight information.

1. Introduction

Multiple attribute decision making (MADM) problems are very frequent in real life decision situations (Cholewa, 1985; Herrera, et al. 1997, 1998; Dubois and Prade, 1985, 1986; Wei, 2009a,b, 2010c, 2011a,b; Wei, et al., 2011a-e). A MADM problem is to find a most desirable alternative from all feasible alternatives assessed on multiple attributes, both quantitative and qualitative. In

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order to deal with the qualitative attribute effectively, Atanassov (1986, 1989) introduced the concept of intuitionistic fuzzy set (IFS), which is a generalization of the concept of fuzzy set (Zadeh, 1965). The intuitionistic fuzzy set has been given increasing attention since its appearance (Lin, 2007; Liu, 2007; Ye, 2009a, b; Li, 2008, 2009, 2010; Liu, 2009; Zhang and Liu, 2010). Xu and Yager (2006) developed geometric aggregation operators with intuitionistic fuzzy information. Xu (2007a) further developed some arithmetic aggregation operators with intuitionistic fuzzy information. Wei (2008a) utilized the maximizing deviation method for intuitionistic fuzzy multiple attribute decision making with incomplete weight information. Wei (2010b) developed the GRA method for intuitionistic fuzzy multiple attribute decision making with incomplete weight information. Later, Atanassov and Gargov (1989) introduced the interval-valued intuitionistic fuzzy set (IVIFS), a generalization of the IFS. The fundamental characteristic of the IVIFS is that the values of membership and non-membership functions are intervals rather than exact numbers. Xu (2007b) and Xu and Chen (2007) developed some aggregation operators with interval-valued intuitionistic fuzzy information. Xu (2008) and Wei (2009a) proposed some aggregation functions for dynamic multiple attribute decision making in intuitionistic fuzzy setting or interval-valued intuitionistic fuzzy setting. Wei (2010a) developed some induced geometric aggregation operators with intuitionistic fuzzy information or interval-valued intuitionistic fuzzy information. Li (2010) proposed linear programming method for MADM with interval-valued intuitionistic fuzzy sets. Wei, Wang & Lin (2011) developed correlation coefficient for interval-valued intuitionistic fuzzy multiple attribute decision making with incomplete weight information.

In the process of MADM with interval-valued intuitionistic fuzzy information, sometimes, the attribute values take the form of interval-valued intuitionistic fuzzy numbers, and the information about attribute weights is incompletely known or completely unknown because of time pressure, lack of knowledge or data, and the expert's limited knowledge of the problem domain. All of the above methods, however, will be unsuitable for dealing with such situations. Therefore, it is necessary to pay attention to this issue. Xu (2007c) investigated the interval-valued intuitionistic fuzzy MADM with attribute weights incompletely known or completely unknown, and a method based on the ideal solution was proposed. The aim of this paper is to develop another combined method based on the deviation method, to overcome this limitation.

The remainder of this paper is set out as follows. In the next section, we introduce some basic concepts related to interval-valued intuitionistic fuzzy sets. In Section 3 we introduce the MADM problem with interval-valued intuitionistic fuzzy information, in which the information about attribute weights is incompletely known, and the attribute values take the form of interval-valued intuitionistic fuzzy numbers. To determine the attribute weights, a combined optimization model based on the deviation method, by which the attribute weights can be determined, is established. For situations, in which information about attribute weights is completely unknown, we establish another combined optimization model. By solving this model, we get a simple and exact formula, which can be used to determine the attribute weights. We utilize the interval-valued intuitionistic fuzzy weighted averaging (IIFWA) operator to aggregate the interval-valued intuitionistic fuzzy information corresponding to each alternative, and then rank the alternatives and select the most desirable one(s) according to the score function and accuracy function. In Section 4, an illustrative example is shown. In Section 5 we conclude the paper and give some remarks.

2. Preliminaries

In the following, we introduce some basic concepts related to intuitionistic fuzzy sets and interval-valued intuitionistic fuzzy sets.

Definition 1 *(Zadeh, 1965). Let X be a universe of discourse, a fuzzy set is defined as:*

$$A = \{\langle x, \mu_A(x)\rangle \,|x \in X\} \tag{1}$$

and is characterized by a membership function $\mu_A : X \to [0,1]$, where $\mu_A(x)$ denotes the degree of membership of the element xto the set A.

Atanassov (1986) extended the fuzzy set to the IFS, as follows:

Definition 2 *(Atanassov, 1986). An IFS A in X is given by*

$$A = \{\langle x, \mu_A(x), \nu_A(x)\rangle \,|x \in X\} \tag{2}$$

where $\mu_A : X \to [0,1]$ and $\nu_A : X \to [0,1]$, with the condition

$$0 \leqslant \mu_A(x) + \nu_A(x) \leqslant 1, \quad \forall\, x \in X.$$

The numbers $\mu_A(x)$ and $\nu_A(x)$ represent, respectively, the membership and non-membership degrees of the elementxto the set A.

Definition 3 *(Atanassov, 1986). For each IFS A in X, if*

$$\pi_A(x) = 1 - \mu_A(x) - \nu_A(x), \quad \forall\, x \in X, \tag{3}$$

then $\pi_A(x)$ is called the degree of indeterminacy of x to A.

Definition 4 *(Atanassov and Gargov, 1989)**.** *Let X be a universe of discourse, an IVIFS $\tilde{A}$ over X is an object having the form:*

$$\tilde{A} = \{\langle x, \tilde{\mu}_A(x), \tilde{\nu}_A(x)\rangle \,|x \in X\} \tag{4}$$

where $\tilde{\mu}_A(x) \subset [0,1]$ and $\tilde{\nu}_A(x) \subset [0,1]$ are interval numbers, and

$$0 \leqslant \sup(\tilde{\mu}_A(x)) + \sup(\tilde{\nu}_A(x)) \leqslant 1, \forall\, x \in X$$

For convenience, let $\tilde{\mu}_A(x) = [a,b]$,$\tilde{\nu}_A(x) = [c,d]$, so $\tilde{A} = ([a,b],[c,d])$.

DEFINITION 5 *(Xu, 2007b). Let $\tilde{a} = ([a,b],[c,d])$ be an interval-valued intuitionistic fuzzy number, a score function S of an interval-valued intuitionistic fuzzy value can be represented as follows:*

$$S(\tilde{a}) = \frac{a-c+b-d}{2}, \quad S(\tilde{a}) \in [-1,1]. \tag{5}$$

DEFINITION 6 *(Xu, 2007ba). Let $\tilde{a} = ([a,b],[c,d])$ be an interval-valued intuitionistic fuzzy number, an accuracy function H of an interval-valued intuitionistic fuzzy value can be represented as follows:*

$$H(\tilde{a}) \in [0,1], \quad H(\tilde{a}) = \frac{a+b+c+d}{2} \tag{6}$$

to evaluate the degree of accuracy of the interval-valued intuitionistic fuzzy value $\tilde{a} = ([a,b],[c,d])$. The larger the value of $H(\tilde{a})$, the higher the degree of accuracy of the interval-valued intuitionistic fuzzy value $\tilde{a}$.

As presented above, the score function $S$and the accuracy function $H$are, respectively, defined as the difference and the sum of the functions of membership $\tilde{\mu}_A(x)$and non-membership $\tilde{\nu}_A(x)$. Xu showed that relation between the score function $S$and the accuracy function H is similar to the relation between mean and variance in statistics. Based on the score and the accuracy functions, Xu (2007b) gave an order relation between two interval-valued intuitionistic fuzzy values, defined as follows:

DEFINITION 7 *(Xu, 2007b). Let $\tilde{a}_1 = ([a_1,b_1],[c_1,d_1])$ and $\tilde{a}_2 = ([a_2,b_2],[c_2,d_2])$ be two interval-valued intuitionistic fuzzy values, $S(\tilde{a}_1)$ and $S(\tilde{a}_2)$ be the scores of $\tilde{a}_1$ and $\tilde{a}_2$, respectively, and let $H(\tilde{a}_1)$ and $H(\tilde{a}_2)$ be the accuracy degrees of $\tilde{a}_1$ and $\tilde{a}_2$, respectively, if $S(\tilde{a}_1) < S(\tilde{a}_2)$, then $\tilde{a}_1$ is smaller than $\tilde{a}_2$, denoted by $\tilde{a}_1 < \tilde{a}_2$; if $S(\tilde{a}_1) = S(\tilde{a}_2)$, then*

(1) if $H(\tilde{a}_1) = H(\tilde{a}_2)$, then $\tilde{a}_1$ and $\tilde{a}_2$ represent the same information, denoted by $\tilde{a}_1 = \tilde{a}_2$;

(2) if $H(\tilde{a}_1) < H(\tilde{a}_2)$, then $\tilde{a}_1$ is smaller than $\tilde{a}_2$, denoted by $\tilde{a}_1 < \tilde{a}_2$.

DEFINITION 8 *(Xu, 2007b). Let $\tilde{a}_j = ([a_j,b_j],[c_j,d_j])$ $(j = 1,2,\cdots,n)$ be a collection of interval-valued intuitionistic fuzzy values, and let IIFWA*

$$\begin{aligned} &IIFWA_\omega(\tilde{a}_1,\tilde{a}_2,\cdots,\tilde{a}_n) = \sum_{j=1}^{n} \omega_j \tilde{a}_j \\ &= \left(\left[1 - \prod_{j=1}^{n} (1-a_j)^{\omega_j}, 1 - \prod_{j=1}^{n} (1-b_j)^{\omega_j} \right], \left[\prod_{j=1}^{n} c_j^{\omega_j}, \prod_{j=1}^{n} d_j^{\omega_j} \right] \right) \end{aligned} \tag{7}$$

where $w = (w_1, w_2, \cdots, w_n)^T$ be the weight vector of $\tilde{a}_j$ $(j = 1,2,\cdots,n)$, and $w_j \geqslant 0$, $\sum_{j=1}^{n} w_j = 1$, the IIFWAA is called the interval-valued intuitionistic fuzzy weighted arithmetic aggregation (IIFWA) operator.

Definition 9 *(Xu, 2007c). Let* $\tilde{a}_1 = ([a_1, b_1], [c_1, d_1])$ *and* $\tilde{a}_2 = ([a_2, b_2], [c_2, d_2])$ *be two interval-valued intuitionistic fuzzy values, the normalized Hamming distance* $d(\tilde{a}_1, \tilde{a}_2)$ *between* $\tilde{a}_1 = ([a_1, b_1], [c_1, d_1])$ *and* $\tilde{a}_2 = ([a_2, b_2], [c_2, d_2])$ *is defined as follows:*

$$\begin{array}{l} d(\tilde{a}_1, \tilde{a}_2) = \frac{1}{4}(|a_1 - a_2| + |b_1 - b_2| + |c_1 - c_2| + |d_1 - d_2|), \\ d(\tilde{a}_1, \tilde{a}_2) \in [0, 1] \end{array} \tag{8}$$

3. A method for interval-valued intuitionistic fuzzy multiple attribute decision making with combined weight information

The following assumptions or notations are used to represent the interval-valued intuitionistic fuzzy MADM problems with incomplete weight information:

(1) The alternatives are known. Let $A = \{A_1, A_2, \cdots, A_m\}$ be a discrete set of alternatives;

(2) The attributes are known. Let $G = \{G_1, G_2, \cdots, G_n\}$ be a set of attributes;

(3) The information about attribute weights is incompletely known. Let $w = (w_1, w_2, \cdots, w_n) \in H$ be the weight vector of attributes, where $w_j \geqslant 0, j = 1, 2, \cdots, n$, $\sum_{j=1}^{n} w_j = 1$, H is a set of the known weight information, which can be constructed by the following forms (Alonso et al., 2009, 2010; Cabrerizo et al., 2010; Herrera-Viedma et al., 2007; Porcel and Herrera-Viedma, 2010; Park and Kim, 1997; Kim and Choi, 1999; Kim and Ahn, 1999), for $i \neq j$: **Form 1.** A weak ranking: $w_i \geqslant w_j$; **Form 2.** A strict ranking: $w_i - w_j \geqslant \alpha_i$, $\alpha_i > 0$; **Form 3.** A ranking of differences: $w_i - w_j \geqslant w_k - w_l$, for $j \neq k \neq l$; **Form 4.** A ranking with multiples: $w_i \geqslant \beta_i w_j$, $0 \leqslant \beta_i \leqslant 1$; **Form 5.** An interval form: $\alpha_i \leqslant w_i \leqslant \alpha_i + \varepsilon_i$, $0 \leqslant \alpha_i < \alpha_i + \varepsilon_i \leqslant 1$.

Suppose that $\tilde{R} = (\tilde{r}_{ij})_{m \times n} = ([a_{ij}, b_{ij}], [c_{ij}, d_{ij}])_{m \times n}$ is the interval-valued intuitionistic fuzzy decision matrix, where $[a_{ij}, b_{ij}]$ indicates the degree to which the alternative A_i satisfies the attribute G_j given by the decision maker, $[c_{ij}, d_{ij}]$ indicates the degree to which the alternative A_i does not satisfy the attribute G_j given by the decision maker, $[a_{ij}, b_{ij}] \subset [0, 1]$, $[c_{ij}, d_{ij}] \subset [0, 1]$, $b_{ij} + d_{ij} \leqslant 1$, $i = 1, 2, \cdots, m$, $j = 1, 2, \cdots, n$.

Definition 10 *Let* $\tilde{R} = (\tilde{r}_{ij})_{m \times n} = ([a_{ij}, b_{ij}], [c_{ij}, d_{ij}])_{m \times n}$ *be an interval-valued intuitionistic fuzzy decision matrix,* $(\tilde{r}_{i1}, \tilde{r}_{i2}, \cdots, \tilde{r}_{in})$ *be the vector of attribute values corresponding to the alternative* A_i $(i = 1, 2, \cdots, m)$, *we call*

$$\begin{array}{l} \tilde{r}_i = ([a_i, b_i], [c_i, d_i]) = IIFWA_w(\tilde{r}_{i1}, \tilde{r}_{i2}, \cdots, \tilde{r}_{in}) \\ = \left(\left[1 - \prod_{j=1}^{n} (1 - a_{ij})^{w_j}, 1 - \prod_{j=1}^{n} (1 - b_{ij})^{w_j} \right], \left[\prod_{j=1}^{n} c_{ij}^{w_j}, \prod_{j=1}^{n} d_{ij}^{w_j} \right] \right) \\ i = 1, 2, \cdots, m, \end{array} \tag{9}$$

the overall value of the alternative A_i, where $w = (w_1, w_2, \cdots, w_n)^T$ is the weight vector of attributes.

In the situation when the information about attribute weights is completely known, i.e., each attribute weight can be provided by the expert with crisp numerical value, we can weight each attribute value and aggregate all the weighted attribute values corresponding to each alternative into an overall one by using Eq. (6). Based on the overall attribute values $\tilde{r}_i$ of the alternatives $A_i\,(i = 1, 2, \cdots, m)$, we can rank all these alternatives and then select the most desirable one(s). The greater$\tilde{r}_i$, the better the alternative A_i will be.

The maximizing deviation method was proposed by Wang (1998) and Wei (2008a) to deal with MADM problems with numerical information and intuitionistic fuzzy information. For a MADM problem, we need to compare the collective preference values to rank the alternatives, the larger the ranking value$\tilde{r}_i$, the better the corresponding alternative A_i. If the performance values of each alternative have little differences under an attribute, then such an attribute plays a small role in the priority procedure. If, however, some attribute implies obvious differences in the performance values among the alternatives, this attribute plays an important role in choosing the best alternative. So, if one attribute has similar values across alternatives, it should be assigned a small weight, while the attribute, entailing larger deviations should be assigned a bigger weight, in spite of the degree of its own importance. Especially, if all available alternatives score about equally with respect to a given attribute, then such an attribute will be judged unimportant by most decision makers, and it should be assigned a very small weight. Wang (1998) and Wei (2008a) suggest that zero be assigned to the attribute of this kind.

The deviation method is selected here to compute the differences of the performance values of each alternative. For the attribute $G_j \,.\!\in G$, the deviation of alternative A_i with respect to all the other alternatives can be defined as follows:

$$D_{ij}(w) = \sum_{k=1}^{m} d\left(\tilde{r}_{ij}, \tilde{r}_{kj}\right) w_j, i = 1, 2, \cdots, m, j = 1, 2, \cdots, n.$$

Let $D_j(w) = \sum\limits_{i=1}^{m} D_{ij}(w) = \sum\limits_{i=1}^{m} \sum\limits_{k=1}^{m} d\left(\tilde{r}_{ij}, \tilde{r}_{kj}\right) w_j, j = 1, 2, \cdots, n.$

Then, $D_j(w)$ represent the deviation values of all alternatives relative to other alternatives for the attribute $G_j \in G$, $d\left(\tilde{r}_{ij}, \tilde{r}_{kj}\right) = \frac{1}{4}\left(|a_{ij} - a_{kj}| + |b_{ij} - b_{kj}| + |c_{ij} - c_{kj}| + |d_{ij} - d_{kj}|\right)$.

Similarly, $S_j(w)$, $V_j(w)$ represent the standard deviation value and average deviation value of all alternatives to other alternatives for the attribute $G_j \in G$, respectively.

$$S_j(w) = \sqrt{\frac{1}{m}\sum_{i=1}^{m} d^2\left(\tilde{r}_{ij}w_j, \frac{1}{m}\sum_{k=1}^{m}\tilde{r}_{kj}w_j\right)} = w_j\sqrt{\frac{1}{m}\sum_{i=1}^{m} d^2\left(\tilde{r}_{ij}, \tilde{r}_j^*\right)},\ j = 1, 2, \cdots, n, \tag{10}$$

$$V_j(w) = \frac{1}{m}\sum_{i=1}^{m} d\left(\tilde{r}_{ij}w_j, \frac{1}{m}\sum_{k=1}^{m}\tilde{r}_{kj}w_j\right) = \frac{w_j}{m}\sum_{i=1}^{m} d\left(\tilde{r}_{ij}, \tilde{r}_j^*\right),\ j = 1, 2, \cdots, n, \tag{11}$$

where $\tilde{r}_j^*$ denotes the average value of $\tilde{r}_{ij}$ $(i = 1, 2, \cdots, m)$, and $\tilde{r}_j^*$

$$= \left(\left[\sum_{i=1}^{m} a_{ij} \Big/ m, \sum_{i=1}^{m} b_{ij} \Big/ m\right], \left[\sum_{i=1}^{m} c_{ij} \Big/ m, \sum_{i=1}^{m} d_{ij} \Big/ m\right]\right)$$

.

Based on the above analysis, we have to choose the weight vector w to maximize all combined deviation values for all the attributes. To do so, a combined optimization model (M-1) is established as follows:

$$\begin{cases} \max\ C(w) = \sum_{j=1}^{n} \left(D_j(w) + S_j(w) + V_j(w)\right) \\ = \sum_{j=1}^{n} w_j \left[\sum_{i=1}^{m}\sum_{k=1}^{m} d\left(\tilde{r}_{ij}, \tilde{r}_{kj}\right) + \sqrt{\frac{1}{m}\sum_{i=1}^{m} d^2\left(\tilde{r}_{ij}, \tilde{r}_j^*\right)} + \frac{1}{m}\sum_{i=1}^{m} d\left(\tilde{r}_{ij}, \tilde{r}_j^*\right)\right] \\ \text{Subject to} \quad w \in H, \sum_{j=1}^{n} w_j = 1, w_j \geqslant 0, j = 1, 2, \cdots, n \end{cases}$$

By solving the model (M-1), we get the optimal solution $w = (w_1, w_2, \cdots, w_n)$, which can be used as the weight vector of attributes.

If the information about attribute weights is completely unknown, we can establish another combined programming model (M-2):

$$\begin{cases} \max\ C(w) = \\ \sum_{j=1}^{n} w_j \left[\sum_{i=1}^{m}\sum_{k=1}^{m} d\left(\tilde{r}_{ij}, \tilde{r}_{kj}\right) + \sqrt{\frac{1}{m}\sum_{i=1}^{m} d^2\left(\tilde{r}_{ij}, \tilde{r}_j^*\right)} + \frac{1}{m}\sum_{i=1}^{m} d\left(\tilde{r}_{ij}, \tilde{r}_j^*\right)\right] \\ s.t.\ \ \sum_{j=1}^{n} w_j^2 = 1, w_j \geqslant 0, j = 1, 2, \cdots, n \end{cases}$$

In order to solve this model, we construct the Lagrange function:

$$L(w, \lambda) = C(w) + \frac{\lambda}{2}\left(\sum_{j=1}^{n} w_j^2 - 1\right) \tag{12}$$

where $\lambda$is the Lagrange multiplier.

Differentiating (10) with respect to $w_j = (j = 1, 2, \cdots, n)$ and λ, and setting the partial derivatives to zero, the following set of equations is obtained:

$$\begin{cases} \frac{\partial L(w,\lambda)}{\partial w_j} = \\ \left[\sum_{i=1}^{m} \sum_{k=1}^{m} d\left(\tilde{r}_{ij}, \tilde{r}_{kj}\right) + \sqrt{\frac{1}{m} \sum_{i=1}^{m} d^2\left(\tilde{r}_{ij}, \tilde{r}_j^*\right)} + \frac{1}{m} \sum_{i=1}^{m} d\left(\tilde{r}_{ij}, \tilde{r}_j^*\right) \right] + \lambda w_j = 0 \\ \frac{\partial L(w,\lambda)}{\partial \lambda} = \sum_{j=1}^{n} w_j^2 - 1 = 0 \end{cases} \quad (13)$$

By solving the model (11), we get a simple and exact formula for determining the attribute weights as follows:

$$w_j^* = \frac{\sum_{i=1}^{m} \sum_{k=1}^{m} d\left(\tilde{r}_{ij}, \tilde{r}_{kj}\right) + \sqrt{\frac{1}{m} \sum_{i=1}^{m} d^2\left(\tilde{r}_{ij}, \tilde{r}_j^*\right)} + \frac{1}{m} \sum_{i=1}^{m} d\left(\tilde{r}_{ij}, \tilde{r}_j^*\right)}{\sqrt{\sum_{j=1}^{n} \left[\sum_{i=1}^{m} \sum_{k=1}^{m} d\left(\tilde{r}_{ij}, \tilde{r}_{kj}\right) + \sqrt{\frac{1}{m} \sum_{i=1}^{m} d^2\left(\tilde{r}_{ij}, \tilde{r}_j^*\right)} + \frac{1}{m} \sum_{i=1}^{m} d\left(\tilde{r}_{ij}, \tilde{r}_j^*\right) \right]^2}} \quad (14)$$

By normalizing w_j^* $(j = 1, 2, \cdots, n)$ by a unit, we have

$$w_j = \frac{\sum_{i=1}^{m} \sum_{k=1}^{m} d\left(\tilde{r}_{ij}, \tilde{r}_{kj}\right) + \sqrt{\frac{1}{m} \sum_{i=1}^{m} d^2\left(\tilde{r}_{ij}, \tilde{r}_j^*\right)} + \frac{1}{m} \sum_{i=1}^{m} d\left(\tilde{r}_{ij}, \tilde{r}_j^*\right)}{\sum_{j=1}^{n} \left[\sum_{i=1}^{m} \sum_{k=1}^{m} d\left(\tilde{r}_{ij}, \tilde{r}_{kj}\right) + \sqrt{\frac{1}{m} \sum_{i=1}^{m} d^2\left(\tilde{r}_{ij}, \tilde{r}_j^*\right)} + \frac{1}{m} \sum_{i=1}^{m} d\left(\tilde{r}_{ij}, \tilde{r}_j^*\right) \right]} \quad (15)$$

Based on the above models, we develop a practical method for solving the MADM problems, in which the information about attribute weights is incompletely known or completely unknown, and the attribute values take the form of interval-valued intuitionistic fuzzy information. The method involves the following steps:

Step 1. Let $\tilde{R} = (\tilde{r}_{ij})_{m\times n}$be an interval-valued intuitionistic fuzzy decision matrix, where $\tilde{r}_{ij} = ([a_{ij}, b_{ij}], [c_{ij}, d_{ij}])$, which is an attribute value, given by an expert, for the alternative $A_i.\in A$ with respect to the attribute $G_j.\in G$, $w = (w_1, w_2, \cdots, w_n)$ be the weight vector of attributes, where $w_j \in [0, 1]$, $j = 1, 2, \cdots, n$, H is a set of the known weight information, which can be constructed by the forms 1-5.

Step 2. If the information about the attribute weights is partly known, then we solve the model (M-1) to obtain the attribute weights. If the information about the attribute weights is completely unknown, then we can obtain the attribute weights by using Eq. (13).

Step 3. Utilize the weight vector $w = (w_1, w_2, \cdots, w_n)$ and by Eq. (7) obtain the overall values $\tilde{r}_i$ of the alternative $A_i\,(i = 1, 2, \cdots, m)$.

Step 4. Calculate the scores $S\left(\tilde{r}_i\right)$ of the overall interval-valued intuitionistic fuzzy preference value $\tilde{r}_i$ $(i = 1, 2, \cdots, m)$to rank all the alternatives A_i

($i = 1, 2,\ldots,m$) and then to select the best one(s) (if there is no difference between two scores $S(\tilde{r}_i)$ and $S(\tilde{r}_j)$, then we need to calculate the accuracy degrees $H(\tilde{r}_i)$ and $H(\tilde{r}_j)$ of the overall interval-valued intuitionistic fuzzy preference value $\tilde{r}_i$ and $\tilde{r}_j$, respectively, and then rank the alternatives A_i and A_j in accordance with the accuracy degrees $H(\tilde{r}_i)$ and $H(\tilde{r}_j)$).

Step 5. Rank all the alternatives A_i ($i = 1, 2,\ldots,m$) and select the best one(s) in accordance with $S(\tilde{r}_i)$ and$H(\tilde{r}_i)$ $(i = 1, 2, \cdots, m)$.

Step 6. End.

4. Illustrative example

Suppose there is an investment company, which wants to invest a sum of money in the best option (adapted from Herrera and Herrera-Viedma, 2000). There is a panel with five possible alternatives to invest the money: A1 is a car company; A_2 is a food company; A_3 is a computer company; A_4 is an arms company; A_5 is a TV company. The investment company must take a decision according to the following four attributes: G_1 is the risk analysis; G_2 is the growth analysis; G_3 is the social-political impact analysis; G_4 is the environmental impact analysis. The five alternatives A_i ($i = 1, 2, 3, 4, 5$) are to be evaluated using the interval-valued intuitionistic fuzzy information by the decision maker under the above four attributes, as listed in the following matrix.

$$\tilde{R} = \begin{bmatrix} ([0.4, 0.5], [0.3, 0.4]) & ([0.4, 0.6], [0.2, 0.4]) & ([0.1, 0.3], [0.5, 0.6]) & ([0.3, 0.4], [0.3, 0.5]) \\ ([0.6, 0.7], [0.2, 0.3]) & ([0.6, 0.7], [0.2, 0.3]) & ([0.4, 0.7], [0.1, 0.2]) & ([0.5, 0.6], [0.1, 0.3]) \\ ([0.3, 0.6], [0.3, 0.4]) & ([0.5, 0.6], [0.3, 0.4]) & ([0.5, 0.6], [0.1, 0.3]) & ([0.4, 0.5], [0.2, 0.4]) \\ ([0.7, 0.8], [0.1, 0.2]) & ([0.6, 0.7], [0.1, 0.3]) & ([0.3, 0.4], [0.1, 0.2]) & ([0.3, 0.7], [0.1, 0.2]) \\ ([0.3, 0.4], [0.2, 0.3]) & ([0.3, 0.5], [0.1, 0.3]) & ([0.2, 0.5], [0.4, 0.5]) & ([0.3, 0.4], [0.5, 0.6]) \end{bmatrix}$$

Then, we utilize the approach developed to get the most desirable alternative(s).

Case 1: Information about the attribute weights is partly known and is given as follows:

$$0.20 \leqslant w_1 \leqslant 0.25, 0.16 \leqslant w_2 \leqslant 0.18, 0.30 \leqslant w_3 \leqslant 0.35, 0.22 \leqslant w_4 \leqslant 0.28,$$

and $H = \left\{ w_j \geqslant 0, j = 1, 2, 3, 4, \sum_{j=1}^{4} w_j = 1 \right\}$.

Step 1 Utilize the model (M-1) to establish the following single-objective programming model:

$$\begin{cases} \max\ C(w) = 2.785w_1 + 1.821w_2 + 3.740w_3 + 2.794w_4 \\ s.t. \\ 0.20 \leqslant w_1 \leqslant 0.25, 0.16 \leqslant w_2 \leqslant 0.18, 0.30 \leqslant w_3 \leqslant 0.35, 0.22 \leqslant w_4 \leqslant 0.28 \\ w \in H \end{cases}$$

By solving this model, we get the weight vector of attributes: w =(0.21 0.16 0.35 0.28)^T.

Step 2 Utilize the weight vector $w = (w_1, w_2, \cdots, w_n)$ and by Eq. (7) obtain the overall values $\tilde{r}_i$ of the alternatives $A_i\,(i = 1, 2, \cdots, m)$.

$$\begin{aligned}
\tilde{r}_1 &= ([0.3548, 0.3780], [0.3764, 0.4591])\\
\tilde{r}_2 &= ([0.5178, 0.5871], [0.1889, 0.2035])\\
\tilde{r}_3 &= ([0.3801, 0.5409], [0.2951, 0.2456])\\
\tilde{r}_4 &= ([0.5012, 0.6307], [0.1859, 0.1668])\\
\tilde{r}_5 &= ([0.3482, 0.3569], [0.3166, 0.4014])
\end{aligned}$$

Step 3 Calculate the scores $S(\tilde{r}_i)$ of the overall interval-valued intuitionistic fuzzy preference values $\tilde{r}_i\ (i = 1, 2, \cdots, m)$

$$S(\tilde{r}_1) = -0.0513, S(\tilde{r}_2) = 0.3562, S(\tilde{r}_3) = 0.1902, S(\tilde{r}_4)$$
$$= 0.3896, S(\tilde{r}_5) = -0.0065$$

Step 4 Rank all the alternatives A_i (i = 1, 2, 3, 4, 5) in accordance with the scores$S(\tilde{r}_i)$ $(i = 1, 2, \cdots, 5)$of the overall interval-valued intuitionistic fuzzy preference values $\tilde{r}_i\ (i = 1, 2, \cdots, m)$: $A_4 \succ A_2 \succ A_3 \succ A_5 \succ A_1$, and thus the most desirable alternative is A_4.

Case 2: If the information about the attribute weights is completely unknown, we utilize another approach developed to get the most desirable alternative(s).

Step 1 Utilize Eq. (13) to get the attribute weight vector w =(0.250 0.163 0.336 0.251)^T.

Step 2 Utilize the weight vector $w = (w_1, w_2, \cdots, w_n)$ and, by Eq. (7), obtain the overall values $\tilde{r}_i$ of the alternatives $A_i\,(i = 1, 2, \cdots, m)$.

$$\begin{aligned}
\tilde{r}_1 &= ([0.3623, 0.3851], [0.3733, 0.4560])\\
\tilde{r}_2 &= ([0.5270, 0.5930], [0.1926, 0.2074])\\
\tilde{r}_3 &= ([0.3810, 0.5440], [0.2985, 0.2511])\\
\tilde{r}_4 &= ([0.5210, 0.6396], [0.1826, 0.1693])\\
\tilde{r}_5 &= ([0.3515, 0.3585], [0.3059, 0.3932])
\end{aligned}$$

Step 3 Calculate the scores $S(\tilde{r}_i)\ (i = 1, 2, \cdots, m)$of the overall interval-valued intuitionistic fuzzy preference values $\tilde{r}_i\ (i = 1, 2, \cdots, m)$,

$$S(\tilde{r}_1) = -0.0410, S(\tilde{r}_2) = 0.3600, S(\tilde{r}_3) = 0.1878, S(\tilde{r}_4)$$
$$= 0.4044, S(\tilde{r}_5) = 0.0054.$$

Step 4 Rank all the alternatives A_i (i = 1, 2, 3, 4, 5) in accordance with the scores$S(\tilde{r}_i)$: $A_4 \succ A_2 \succ A_3 \succ A_5 \succ A_1$, and thus the most desirable alternative is A_4.

Now, in the following, we utilize Xu's approach (Xu, 2007c) to get the most desirable alternative(s).

Case 1: Information about the attribute weights is partly known and given as follows:

$$0.20 \leqslant w_1 \leqslant 0.25, 0.16 \leqslant w_2 \leqslant 0.18, 0.30 \leqslant w_3 \leqslant 0.35, 0.22 \leqslant w_4 \leqslant 0.28,$$

and $H = \left\{ w_j \geqslant 0, j = 1, 2, 3, 4, \sum_{j=1}^{4} w_j = 1 \right\}$.

Step 1 Utilize the model (M-7) from Xu (2007c) to establish the following single-objective programming model:

$$\begin{cases} \min \ \bar{f}(w) = 1.850w_1 + 1.775w_2 + 2.250w_3 + 2.200w_4 \\ s.t. \\ 0.20 \leqslant w_1 \leqslant 0.25, 0.16 \leqslant w_2 \leqslant 0.18, 0.30 \leqslant w_3 \leqslant 0.35, 0.22 \leqslant w_4 \leqslant 0.28 \\ w \in H \end{cases}$$

By solving this model, we get the weight vector of attributes: $w = (0.20\ 0.16\ 0.30\ 0.22)^T$.

Step 2 Then, by formula (21) from Xu (2007c), we have

$$\begin{array}{l} \dot{d}(A_1, A^+) = 0.472, \dot{d}(A_2, A^+) = 0.270, \dot{d}(A_3, A^+) = 0.345 \\ \dot{d}(A_4, A^+) = 0.276, \dot{d}(A_5, A^+) = 0.451 \end{array}.$$

Since $\dot{d}(A_2, A^+) \prec \dot{d}(A_4, A^+) \prec \dot{d}(A_3, A^+) \prec \dot{d}(A_5, A^+) \prec \dot{d}(A_1, A^+)$,

then $A_2 \succ A_4 \succ A_3 \succ A_5 \succ A_1$, hence, the most desirable alternative is A_2.

Case 2: If the information about the attribute weights is completely unknown, we utilize another approach, developed to obtain the most desirable alternative(s).

Step 1 Utilize Eq. (24) from Xu (2007c) to get the following weight vector of attributes: $w = (0.287\ \ 0.322\ \ 0.185\ \ 0.206)^T$.

Step 2 Then, by (23), again from Xu (2007c), we get

$$\begin{array}{l} \dot{d}(A_1, A^+) = 0.2556, \dot{d}(A_2, A^+) = 0.1654, \dot{d}(A_3, A^+) = 0.2204 \\ \dot{d}(A_4, A^+) = 0.1648, \dot{d}(A_5, A^+) = 0.2619. \end{array}$$

Since $\dot{d}(A_4, A^+) \prec \dot{d}(A_2, A^+) \prec \dot{d}(A_3, A^+) \prec \dot{d}(A_5, A^+) \prec \dot{d}(A_1, A^+)$,

then $A_4 \succ A_2 \succ A_3 \succ A_5 \succ A_1$, hence, the most desirable alternative is A_4.

In the above example the rankings of the five alternatives derived by the two approaches are slightly different. For instance, the alternative A_1 is ranked last, and the rankings of the other four alternatives are reversed. This is mainly because the approach from Xu (2007c) derives the weight information and selects the most desirable alternative based on the intuitionistic fuzzy ideal solution (IFIS). However, our approach derives the weight information and selects the most desirable alternative based on the deviation value of all alternatives with respect to other alternatives. So, different approaches to deriving the weight information may result in different rankings.

5. Conclusions

In this paper, we have investigated the problem of MADM with incompletely known information on attribute weights, for which the attribute values are given in terms of interval-valued intuitionistic fuzzy numbers. To determine the attribute weights, a combined optimization model, based on the deviation method, by which the attribute weights can be determined, is established. For the situations, in which the information about attribute weights is completely unknown, we establish another combined optimization model. By solving this model, we get a simple and exact formula, which can be used to determine the attribute weights. We utilize the interval-valued intuitionistic fuzzy weighted averaging (IIFWA) operator to aggregate the interval-valued intuitionistic fuzzy information corresponding to each alternative, and then rank the alternatives and select the most desirable one(s) according to the score function and accuracy function. The proposed method is also compared to the existing methods to show common advantages and effectiveness. Finally, an illustrative example is given. In the future, we shall continue working in the application of the interval-valued intuitionistic fuzzy multiple attribute decision-making to other domains.

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References

Alonso, S., Herrera-Viedma, E., Chiclana, F., Herrera, F. (2009) Individual and Social Strategies to Deal With Ignorance Situations in Multi-Person Decision Making. *International Journal of Information Technology and Decision Making*, **8**(2), 313–333.

Alonso, S., Herrera-Viedma, E., Chiclana, F., Herrera, F. (2010) A Web Based Consensus Support System for Group Decision Making Problems and Incomplete Preferences. *Information Sciences*, **180** (23), 4477-4495.

Atanassov, K. (1986) Intuitionistic fuzzy sets. *Fuzzy Sets and Systems*, 20, 87-96.

Atanassov, K. and Gargov, G. (1989) Interval valued intuitionistic fuzzy sets. *Fuzzy Sets and Systems*, 31, 343-349.

Atanassov, K. (1994) Operators over interval-valued intuitionistic fuzzy sets. *Fuzzy Sets and Systems*, 64, 159-174.

Cabrerizo, F.J., Pérez, I.J. and Herrera-Viedma, E. (2010) Managing the Consensus in Group Decision Making in an Unbalanced Fuzzy Linguistic Context with Incomplete Information. *Knowledge-Based Systems* **23**(2), 169-181.

Cholewa, W. (1985) Aggregation of fuzzy opinions: An axiomatic approach. *Fuzzy Sets and Systems*, 17, 249–259.

Dubois, D. and Prade, H. (1985) A review of fuzzy set aggregation connectives. *Information Sciences*, 36, 85-121.

Dubois, D. and Prade, H. (1986) Weighted minimum and maximum operations in fuzzy set theory. *Information Sciences*, 39, 205–210.

Herrera, F., Herrera-Viedma, E. (1997) Aggregation Operators for Linguistic Weighted Information. *IEEE Transactions on Systems, Man and Cybernetics*, 27, 646-656.

Herrera, F., Herrera-Viedma, E., Verdegay, J.L. (1998) Choice Processes for Non-Homogeneous Group Decision Making in Linguistic Setting. *Fuzzy Sets and Systems*, 94, 287-308.

Herrera-Viedma, E., Alonso, S., Chiclana, F., Herrera, F. (2007) A Consensus Model for Group Decision Making with Incomplete Fuzzy Preference Relations. *IEEE Transactions on Fuzzy Systems*, **15**(5), 863-877.

Herrera, F. and Herrera-Viedma, E. (2000) Linguistic decision analysis: steps for solving decision problems under linguistic information. *Fuzzy Sets and Systems*, **115**(8), 67-82.

Kim, S.H., Choi, S.H. and Kim, J.K. (1999) An interactive procedure for multiple attribute group decision making with incomplete information: range based approach. *European Journal of Operational Research*, 118, 139-152.

Kim, S.H. and Ahn, B.S. (1999) Interactive group decision making procedure under incomplete information. *European Journal of Operational Research*, 116, 498-507.

Li, D.F. (2008) Extension of the LINMAP for multiattribute decision making under Atanassov's intuitionistic fuzzy environment. *Fuzzy Optimization and Decision Making*, **7** (1), 17-34.

Li, D.F., Wang, Y.C., Liu, S. and Shan, F. (2009) Fractional programming methodology for multi-attribute group decision making using IFS. *Applied Soft Computing*, 9(1), 219-225.

Li, D.F. (2010) Linear programming method for MADM with interval-valued intuitionistic fuzzy sets. *Expert Systems with Applications*, **37**(6), 5939-5945.

Lin, L., Yuan, X.H. and Xia, Z.Q. (2007) Multicriteria fuzzy decision-making methods based on intuitionistic fuzzy sets. *Journal of Computer and System Sciences*, 73, 84-88.

Liu, H.W. (2007) Multi-criteria decision-making methods based on intuitionistic fuzzy sets. *European Journal of Operational Research*, **179**(1), 220-233.

Liu, P.D. (2009) Multi-attribute decision-making method research based on interval vague set and TOPSIS method. *Technological and Economic Development of Economy* **15**(3), 453-463.

Park, P.S. and Kim, S.H. (1997) Tools for interactive multi-attribute decision making with incompletely identified information. *European Journal of Operational Research*, 98, 111-123.

Porcel, C., Herrera-Viedma, E. (2010) Dealing with Incomplete Information in a Fuzzy Linguistic Recommender System to Disseminate Information in University Digital Libraries. *Knowledge-Based Systems*, 23, 32-39.

Wang, Y.M. (1998) Using the method of maximizing deviations to make decision for multi-indices. *System Engineering and Electronics*, 7, 24-26, 31.

Wei, G.W. (2008a) Maximizing deviation method for multiple attribute decision making in intuitionistic fuzzy setting. *Knowledge-Based Systems*, **21**(6), 833-836.

Wei, G.W. (2008b) Induced intuitionistic fuzzy ordered weighted averaging operator and its application to multiple attribute group decision making. *Lecture Notes in Artificial Intelligence* **5009**, 124-131.

Wei, G.W. (2009a) Some geometric aggregation functions and their application to dynamic multiple attribute decision making in intuitionistic fuzzy setting. *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems*, **17**(2), 179-196.

Wei, G.W. (2009b) Uncertain linguistic hybrid geometric mean operator and its Application to group decision making under uncertain linguistic environment. *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems*, **17**(2), 251-267.

Wei, G.W.(2010a) Some induced geometric aggregation operators with intuitionistic fuzzy information and their application to group decision making. *Applied Soft Computing*, **10**(2), 423-431.

Wei, G.W. (2010b) GRA method for multiple attribute decision making with incomplete weight information in intuitionistic fuzzy setting. *Knowledge-Based Systems*, **23**(3), 243-247.

Wei, G.W. (2010c) A method for multiple attribute group decision making based on the ET-WG and ET-OWG operators with 2-tuple linguistic information. *Expert Systems with Applications*, **37**(10), 7895-7900.

Wei, G.W. (2011a) Gray relational analysis method for intuitionistic fuzzy multiple attribute decision making, *Expert Systems with Applications*, **38** (7), 11671-11677.

Wei, G.W. (2011b) A novel efficient approach for interval-valued intuitionistic fuzzy multiple attribute decision making with incomplete weight information. *Information: An International Interdisciplinary Journal*, **14**(1), 97-102.

WEI, G.W., ZHAO, X.F. and LIN, R. (2010a) Some induced aggregating operators with fuzzy number intuitionistic fuzzy information and their applications to group decision making. *International Journal of Computational Intelligence Systems*, **3**(1), 84-95.

WEI, G.W., ZHAO, X.F., WANG H.J. and LIN, R. (2010b) GRA Model for Selecting an ERP System in Trapezoidal Intuitionistic Fuzzy Setting. *Information: An International Interdisciplinary Journal* **13**(4), 1143-1148.

WEI, G.W., LIN R., ZHAO, X.F. and WANG, H.J. (2010c) Models for Multiple Attribute Group Decision Making with 2-tuple Linguistic Assessment Information. *International Journal of Computational Intelligence Systems*, **3**(3) 315-324.

WEI, G.W., LIN, R., ZHAO, X.F. and WANG H.J. (2010d) TOPSIS-Based Linear-Programming Methodology for Multiple Attribute Decision Making with Incomplete Weight Information in Intuitionistic Fuzzy Setting. *Information: An International Interdisciplinary Journal* **13**(5), 1249-1257.

WEI, G.W., WANG, H.J. and LIN, R.(2011e) Application of correlation coefficient to interval-valued intuitionistic fuzzy multiple attribute decision making with incomplete weight information. *Knowledge and Information Systems*. **26**(2), 337-349.

XU, Z.S. (2007a) Intuitionistic fuzzy aggregation operators. *IEEE Transactions on Fuzzy Systems*, 15, 1179–1187.

XU, Z.S. (2007b) Methods for aggregating interval-valued intuitionistic fuzzy information and their application to decision making. *Control and Decision*, **22**(2), 215-219.

XU, Z.S. (2007c) Models for multiple attribute decision-making with intuitionistic fuzzy information. *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems*, **15**(3), 285-297.

XU, Z.S. and CHEN, J. (2007) An approach to group decision making based on interval-valued intuitionistic fuzzy judgment matrices. *System Engineer-Theory and Practice*, **27**(4), 126-133.

XU, Z.S. and YAGER, R. R. (2006) Some geometric aggregation operators based on intuitionistic fuzzy sets. *International Journal of General Systems*, 35, 417-433.

XU, Z.S. (2008) Dynamic intuitionistic fuzzy multi-attribute decision making. *International Journal of Approximate Reasoning* **48**(1) (2008) 246-262.

YE, J. (2009a) Multicriteria fuzzy decision-making method based on a novel accuracy function under interval-valued intuitionistic fuzzy environment. *Expert Systems with Applications* **36**(3), 6899-6902.

YE, J. (2009b) Using an improved measure function of vague sets for multicriteria fuzzy decision-making. *Expert Systems with Applications* **37**(6), 4706-4709.

ZADEH, L.A. (1965) Fuzzy sets. *Information and Control*, 8, 338–353.

Zadeh, L.A. (2005) Toward a generalized theory of uncertainty (GTU) - an outline. *Information Sciences*, 172, 1-40.

Zhang X. and Liu P.D. (2010) Method for aggregating triangular fuzzy intuitionistic fuzzy information and its application to decision making. *Technological and Economic Development of Economy* **16**(2), 280-290.