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Optimum control of a decentralized development planning model

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The optimum control of a development planning model is presented. The model includes m exogeneous and n endogeneous factors with intensities related by a system of linear differential equations. The macro-model represents an aggregation of micro-models. The optimum development strategy can be realized in the decentralized form with the exchange of information between the global (macro) and local (micro) controllers. The decentralized development planning of the integrated power system is studied as a typical example. The optimum development strategy consists in allocation of investments which minimize the global maintenance cost.

1. Introduction

The classical system theory deals with the models of systems which consist of fixed number of physical objects united by some kind of interaction and fixed spatial structure.

There are known, however, examples of systems, called here the development systems, such as e.g. ecological, social, demographic or economic systems, urban development or transports systems etc., which change their structure in time as a result of growth, evolution, development, investments etc.

The models of development systems can be devided into two general classes: the micro- and macro-models. The macro-models deal usually with the aggregated quantitative aspects of the development systems such as the increase of population, national product investments etc., whereas the micro-models emphasize the local and structural properties of the system, such as the technological parameters, interconnection links, organizational structure, environment influences etc.

The main interest represents the controlled class of development systems, such as the planned economic growth, planned investment program etc., and will be studied here from the point of view of the optimum control theory. The model which is being used in the present paper can be treated as an extension of a model of multi-sector economic growth. It has, however, an important property. The macro-model represents an aggregated system of the collection of micro-models. As a result, the decentralization of control actions at the macro-level (e.g. government-level) and micro-level (e.g. the production plant-level) follows. The overall optimum planning strategy can be achieved by exchange of information between macro- and micro-levels.

2. The macro-model of development system

Consider the system of n endgeneous and m exogeneous interrelated factors or processes. A typical example of that system is an ecological system which consists of n different species (endogeneous factors) struggling for existance by eating weaker specimen and competing for m kinds of food (exogeneous factors). Another example is the multi-sector economy which exchanges m final products, produced by the sectors (endogeneous factors) and utilizes m primary resources (exogeneous factors).

One of the most important characteristic, which determines the change of factors intensity: $x_i(t)$, $i=1, ..., n$; $y_i(t)$, $i=1, ..., m$, is the relative growth coefficient

$$r_i(t)=\frac{\mathrm{d}x_i}{\mathrm{d}t} : x_i, \quad i=1, ..., n, \tag{1}$$

$$p_{ij}(t)=\frac{\mathrm{d}y_i}{\mathrm{d}t} : y_{ij}, \quad i=1, ..., m, \tag{2}$$

where x_i, r_i — endogeneous factors intensity and growth coefficients; y_i, p_i — exogeneous factors intensity and growth coefficients.

In the present paper we shall deal with the class of development systems characterized by linear interrelations, i.e.

$$\sum_{i=1}^{n} \alpha_{ij}\, r_i(t) - \sum_{i=1}^{n} \beta_{ij}\, p_j(t) - \alpha_j(t) = 0, \quad j=1, ..., n, \tag{3}$$

where α_{ij}, β_{ij} — given real numbers; $\alpha_j(t)$, $p_i(t)$ — given continuous functions.

We shall call the system described by (3) with $D=\mathrm{Det}\,|\alpha_{ij}|\neq 0$ the linear regular development system (LRD system). An example of LRD system is the ecological system which consists of two species N_1, N_2 with intensities x_1, x_2. The first (N_1) feeds upon the environment and would grow with the rate coefficient $\alpha_1+p_1(t)$ if N_2 was not growing, i.e. $r_2=0$. N_2 feeds mostly on N_1 and would diminish with the rate $p_2(t)-\alpha_2$ if N_1 was not growing, i.e. $r_1=0$. The equations of the system become:

$$\begin{array}{l} r_1=\alpha_1-\alpha_{21}\, r_2+p_1(t), \\ r_2=-\alpha_2+\alpha_{12}\, r_1+p_2(t), \end{array} \quad \begin{vmatrix} 1, & \alpha_{21} \\ -\alpha_{12}, & 1 \end{vmatrix} \neq 0,$$

where $\alpha_1, \alpha_2, \alpha_{21}, \alpha_{12}$ — given, positive constants; $p_1(t)$, $p_2(t)$ — given functions.

Theorem 1. The LRD system, which at $t=0$ is in the given state $x_i(0)=A_i$, $i=1, \dots, n$, develops in time according to the formula

$$x_i(t)=A_i \exp\left[\int_0^t \gamma_i(\tau)\,\mathrm{d}\tau\right]\prod_{j=1}^{m}[y_j(t)/y_j(0)]^{\gamma_{ij}}, \quad i=1, \dots, n, \tag{4}$$

where the numbers γ_i, γ_{ij} are defined by the relations

$$\sum_{j=1}^{m} \gamma_{ij}\, p_j(t)+\gamma_i(t) \overset{\mathrm{df}}{=} D_i(t)/D, \quad i=1, \dots, n,$$

$D_i(t)$ — the determinant obtained by setting

$$\left[\sum_{j=1}^{m} \beta_{j1}\, p_j(t)+\alpha_1(t), \dots, \sum_{j=1}^{m} \beta_{jn}\, p_j(t)+\alpha_n(t)\right]^T$$

in the place of i-th column of D.

Proof. Solving (3) with respect to r_i one gets

$$r_i(t)=\gamma_i(t)+\sum_{j=1}^{m} \gamma_{ij}\, p_j=\frac{D_i(t)}{D}.$$

Keeping in mind that the solution of the differential equation $\dot{x}=f(t)\,x$ (which for $t=0$ is $x(0)=A$) becomes

$$x(t)=A \exp\left[\int_0^t f(\tau)\,\mathrm{d}\tau\right],$$

one gets

$$\ln\frac{x_i}{A_i}=\int_0^t \left[\gamma_i(\tau)+\sum_{j=1}^{m}\frac{\gamma_{ij}}{y_j}\frac{\mathrm{d}y_j}{\mathrm{d}\tau}\right]\mathrm{d}\tau=\int_0^t \gamma_i(\tau)\,\mathrm{d}\tau+\sum_{j=1}^{m}\gamma_{ij}\ln[y_j(t)/y_j(0)].$$

Then

$$x_i(t)=A_i\left[\exp\int_0^t \gamma_i(\tau)\,\mathrm{d}\tau\right]\prod_{j=1}^{m}[y_j(t)/y_j(0)]^{\gamma_{ij}}, \quad i=1, \dots, n, \qquad \text{Q.E.D.}$$

As stated by Theorem 1 the growth intensities of the endogeneous factors in a LRD system are completely specified by the exogeneous factors intensities which can in turn be controlled by an inteligent controller according to his goals.

Several optimum control problems can be formulated for the LRD models. Begin with the optimum allocation of resources. Assume the global amounts of resources of each endogeneous factor to be limited, i.e.

$$\sum_{i=1}^{n} y_{ij}(t)\leqslant Y_j(t), \quad j=1, \dots, m, \tag{5}$$

where $Y_j(t)$ — given functions.

Find the nonnegative functions $y_{ij}(t)=y^o_{ij}(t)$ which maximize the value of integrated system production

$$P(y_{ij})=\int\limits_0^\infty \sum_{i=1}^n w_i(\tau)\, x_i(\tau)\, d\tau, \qquad x_i(t)=A_i \exp \int\limits_0^t \gamma_i(\tau)\, d\tau \prod_{j=1}^n [y_{ij}(t)/y_{ij}(0)]^{\gamma_{ij}},$$

where $w_i(\tau)$ — given positive monotically decreasing weighting functions, subject to the constraints (5). The present problem is a nonlinear programming problem. In the particular case when $\gamma_{ij}=\delta_j$, $i=1, \ldots, n$, the solution of the problem can be stated in the form of the following:

Theorem 2. There exists a unique optimum allocation of resources strategy for the LRD system (with $\gamma_{vj}=\delta_j$, $v=1, \ldots, n$):

$$y^o_{ij}(t)=\frac{k_i(t)}{k(t)}\, Y_j(t), \quad j=1, \ldots, m, \tag{6}$$

where

$$k_i(t)=\left\{A_i \exp\left[\int\limits_0^t \gamma_i(\tau)\, d\tau\right] w_i(t)\right\}^{1/q},$$

$$k(t)=\sum_{i=1}^n k_i(t), \quad q=\sum_{j=1}^m \delta_j-1<0, \tag{7}$$

and the corresponding maximum value of integrated production

$$P(y^o_{ij})=\int\limits_0^\infty [k(t)]^q \prod_{j=1}^m [Y_j(t)]^{\delta_j}\, dt.$$

Proof. In the paper [3] the following optimization problem has been solved: find the values $B_i=B^o_i$, $C_i=C^o_i$, ..., $Z_i=Z^o_i$, $i=1, \ldots, n$, which minimize

$$A=\sum_{i=1}^n A_i=\sum_{i=1}^n \{k^q_i B_i^{-\beta} C_i^{-\gamma} \ldots Z_i^{-\omega}\}^{1/\alpha}, \quad q=\alpha+\beta+\ldots+\omega>0, \tag{8}$$

subject to the constraints

$$\sum_{i=1}^n B_i \leqslant B, \ \sum_{i=1}^n C_i \leqslant C, \ldots, \sum_{i=1}^n Z_i \geqslant Z,$$

$$B_i \geqslant 0, \ C_i \geqslant 0, \ldots, Z_i > 0, \quad i=1, \ldots, n,$$

$B, C, \ldots, Z$ — given numbers; $\alpha, \beta, \ldots > 0$, $\omega, \ldots < 0$.

The optimum solution becomes

$$B^o_i/B=C^o_i/C=\ldots=Z^o_i/Z=k_i/k, \quad i=1, \ldots, n,$$

and

$$A=\{k^q B^{-\beta} C^{-\gamma} \ldots Z^{-\omega}\}^{1/\alpha}, \quad k=\sum_{i=1}^n k_i. \tag{9}$$

The same result is obtained when A is a maximalized quantity and $q= -(\alpha+\beta+\gamma+...+\omega)$. When $t \in [0, \infty]$ is a fixed number the problem of maximization of $\sum_{i=1}^{n} w_i(t)\, x_i(t)$, where $x_i(t)$ is described by (4), with $\gamma_{ij}=\delta_j$, $i=1, ..., n$, is equivalent to the problem (8). Observe that

$$P=\int_0^{\infty} \sum_{i=1}^{i} w_i(t)\, x_i(t)\, \mathrm{d}t$$

is maximum when

$$\sum_{i=1}^{n} w_i(t)\, x_i(t)= \sum_{i=1}^{n} \bar{x}_i(t)$$

is maximum for each $t \in [0, \infty)$.

Indeed, $x_i(t)$ is a parametric function of t of the variables $y_{ij}(t)$ and

$$\left.\frac{\mathrm{d}P(y_{ij}+\gamma h_{ij})}{\mathrm{d}\gamma}\right|_{\gamma=0} = \int_0^{\infty} h_{ij}(t) \frac{\mathrm{d}\bar{x}_i(t)}{\mathrm{d}y_{ij}(t)}\mathrm{d}t$$

is zero for all variations $h_{ij}(t)$ only if for all $t \in [0, \infty)$ one gets

$$\frac{\mathrm{d}\bar{x}_i(t)}{\mathrm{d}y_{ij}(t)}=0, \quad i=1, ..., n, \quad j=1, ..., m.$$

These conditions are also sufficient for optimality. Q.E.D.

It should be observed that the aggregated system characteristic (9) has the same analytic form as the subsystem characteristics but with the performance index $k=\sum_{i=1}^{n} k_i$. The smaller the value of k the better the properties of the aggregated system, in the sense that A is greater ($q<0$), when $B, C, ..., Z$ are kept constant.

It should be noted that a closely related class of optimization problems can also be formulated assuming the production intensity $x_i(t)$ to be not less than the given demand function $X_i(t)$, i.e. $x_i(t) \geqslant X_i(t)$, $t \in [0, \infty)$, and minimizing the cost of the most expensive resources. Such a situation happens, for example in the electric power systems. Keeping in mind the proof of theorem 2 one can observe that it can be easily reformulated to satisfy the present assumptions. The formulae (6), (7) describing the distribution of the remaining resources preserve their previous form.

The next important class of optimization problems for the LRD systems is the optimization of investment or reproduction strategy. Before the general problem is formulated consider a simple single sector economy described by the equation [2, 9]:

$$x(t)=A \exp(\gamma t)\,[L(t)]^{\delta}\,[K(t)]^{\delta_1}, \tag{10}$$

where: $x(t)$ is the production income or national product intensity; $L(t)$ is the intensity of labor employed, or employment; $K(t)$ is the intensity of capital used; A is a given positive number; γ is a positive number called the annual rise in efficiency; δ, δ_1 are positive numbers, the elasticities of production with regard to labor and capital, respectively.

It is also assumed that $\delta_1 = 1-\delta$, i.e. the national product increases at the same rate as the combined increase of $L(t)$ and $K(t)$.

The formation of capital is assumed to be linked with national product or income by the simple relation:

$$\mathrm{d}\bar{K}(t)/\mathrm{d}t = bz(t) \tag{11}$$

where b is a given coefficient and $z(t)$ is the control variable which may vary between 0 and $x(t)$, i.e.

$$0 \leqslant z(t) \leqslant x(t), \quad t \in [0, \infty). \tag{12}$$

It is assumed that the capital used $K(t)$ in the economy does not exceed the capital available $\bar{K}(t)$, i.e.

$$K(t) \leqslant \bar{K}(t) = b\int_0^t z(\tau)\,\mathrm{d}\tau + c, \tag{13}$$

where $c = K(0)$.

In the similar way the labor used $L(t)$ should not exceed the supply of labor $\bar{L}(t)$ which is specified by the demographic growth, i.e.

$$L(t) \leqslant \bar{L}(t) = L_0 \exp(\lambda t), \tag{14}$$

L_0, λ — given positive numbers.

The system performance measure is assumed to be

$$P(z) = \int_0^\infty w(t)\,[x(t) - z(t)]\,\mathrm{d}t \tag{15}$$

where $w(t)$ — given positive monotonously decreasing weighting function. A typical example of $w(t)$ is $w(t) = (1+\varepsilon)^{-t}$ where ε is the discount rate. According to (15) $P(z)$ represents the net income in the economy. The optimum control problem consists in finding such a nonnegative strategy $z(t) = z^o(t)$ which maximizes

$$P(z) = \int_0^\infty w(t)\left\{AL_0^\delta \exp[t(\gamma+\lambda\delta)] \times \left(b\int_0^t z(\tau)\,\mathrm{d}\tau + c\right)^{1-\delta} - z(t)\right\}\mathrm{d}t \tag{16}$$

Subject to the constraint

$$G(z) = AL_0^\delta \exp[t(\gamma+\lambda\delta)]\left(b\int_0^t z(\tau)\,\mathrm{d}\tau + c\right)^{1-\delta} - z(t) \geqslant 0, \quad t \in [0, \infty). \tag{17}$$

Since no harm can be done by using the full employment and the capital available the upper bounds for $L(t)$, $K(t)$ has been set in (16), (17).

We shall solve the problem (16) and (17) for the particular case $w(t) = \begin{cases} 1, & t < T \\ 0, & t \geqslant T \end{cases}$ using the method described in [4]. Denote $AL_0^\delta n^{1-\delta}$ by a, $\gamma + \lambda\delta$ by α, and c/b by c_1. The Lagrangean for the present problem becomes

$$\Phi(z, \lambda) = P(z) + \int_0^T \lambda(t)\,G(z)\,\mathrm{d}\tau = \int_0^T [1+\lambda(t)]\,\alpha \exp(\alpha t)\left[\int_0^t (z(\tau)\,\mathrm{d}\tau + c_1)^{1-\delta} - z(t)\right]\mathrm{d}t. \tag{18}$$

Find the differential $\frac{d}{d\gamma}\Phi(z+\gamma h;\lambda)|_{\gamma=0}=d\Phi(z,\lambda;h)$ [1]) and the gradient $\Phi'_z(z,\lambda)$ of (13)

$$d\Phi(z,\lambda;h)=\int_0^T[1+\lambda(t)]\left[(1-\delta)\,a\times\exp(\alpha t)\left(\int_0^t z(\tau)\,d\tau+c_1\right)^{-\delta}\int_0^t h(\tau)\,d\tau+\right.$$

$$\left.-h(t)\right]dt=\int_0^T h(\tau)\,dt\left\{\int_\tau^T a(1-\delta)\,[1+\lambda(t)\exp(\alpha t)\times\left(\int_0^t z(\tau)\,d\tau+\right.\right.$$

$$\left.\left.+c_1\right)^{-\delta}dt-1-\lambda(\tau)\right\}.$$

$$\Phi'_z(z,\lambda)=\int_0^T a(1-\delta)\,[1+\lambda(t)]\exp(\alpha t)\times\left(\int_0^t(z(\tau)\,d\tau+c_1\right)^{-\delta}dt-1-\lambda(\tau).$$

We shall show that when T is long enough the following optimum "bang-bang" strategy $z=z^o$ exists:

$$z^o(t)=\begin{cases}x(t), & t\in[0,T_1]\\ 0, & t\in[T_1,T]\end{cases},\tag{19}$$

where the switching time $T_1\leqslant T$.

Since Φ is a concave functional in z the necessary and sufficient conditions of optimality require [4] that the nonnegative Lagrange function $\lambda(t)=\lambda^o(t)$, exists such that

$$\lambda^o(t)\begin{cases}\geqslant 0, & t\in[0,T_1]\\ =0, & t\in[T_1,T]\end{cases},\tag{20}$$

and the gradient is zero for $t\in[0,T_1]$ and negative for $t\in[T_1,T]$, i.e.

$$\Phi'_z(z^o,\lambda^o)\begin{cases}=0, & t\in[0,T_1]\\ <0, & t\in[T_1,T]\end{cases},\tag{21}$$

These conditions have simple interpretation: when the constraint (17) is active the Lagrange function is nonnegative and it vanishes when (17) is non active. At the same time the gradient becomes zero (for $z^o>0$) and negative for $z^o=0$.

The optimum strategy in the subinterval $[0,T_1]$ can be derived by solving the equation

$$z^o(t)=a\exp(\alpha t)\left(\int_0^t z^o(\tau)\,d\tau+c_1\right)^{1-\delta},$$

which yields

$$z^o(t)=a\exp(\alpha t)\left\{\frac{\alpha\delta}{\alpha}[\exp(\alpha t)-1]+c_1^o\right\}^{\frac{1}{\delta}-1},\qquad t\in[0,T_1].\tag{22}$$

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[1]) $d\Phi(z,\lambda;h)$ should be treated as variation of the functional Φ at the point $\{z,\lambda\}$. It is a linear functional of the variation h and can be written $d\Phi(z,\lambda;h)=(\Phi',h)$, where Φ' is called the gradient of Φ [4].

Then from the equation

$$\Phi_z'(z^o, \lambda^o) = \int_{\tau}^{T} a(1-\delta)\exp(\alpha t)\left\{\frac{a\delta}{\alpha}[\exp(\alpha t)-1]+c_1\right\}^{\delta-1}[1+\lambda^o(t)]\,dt - \\ -1-\lambda^o(\tau)=0,$$

one finds

$$\lambda^o(t) = C\exp\left\{-\varepsilon\left\{t+\frac{1}{\alpha}\ln[\exp(-\alpha t)-d]\right\}\right\}-1, \qquad t\in[0, T_1],$$

where the constant C is chosen in such a way that $\lambda^o(T_1)=0$, i.e.

$$C=\exp\left\{\varepsilon\left\{T_1+\frac{1}{\alpha}\ln[\exp(-\alpha T_1)-d]\right\}\right\}$$

$$d=\frac{1}{1-c_1^\delta\dfrac{\alpha}{a\delta}}, \qquad \varepsilon=a^2(1-\delta)\frac{\delta}{\alpha}.$$

The value of T_1 can be derived by the condition $\Phi_z'(z^o, \lambda^o)|_{t=T_1}=0$, i.e.

$$\int_{T_1}^{T} a(1-\delta)\exp(\alpha t)\left\{\frac{a\delta}{\alpha}[\exp(\alpha t)-1]+c_1^\delta\right\}^{-1} dt=1$$

which can be written

$$\ln\frac{\exp(-\alpha T)-d}{\exp(-\alpha T_1)-d}\alpha\left[\frac{(a\delta/\alpha)-c_1^\delta}{da(1-\delta)}-(T-T_1)\right]. \tag{23}$$

The analysis of (23) shows that when T is not long enough there is no reproduction period ($T_1\to 0$) and $z^o(t)=0$, $t\in[0, T]$.

The optimum investment problem can be easily extended to the multi-sector economy, described by (4), and

$$0\leqslant\sum_{j=1}^{m} z_{ij}(t)\leqslant x_i(t), \ t\in[0,\infty), \qquad i=1,\dots,n,$$

$$y_{ij}(t)\leqslant Y_{ij}(t)=b_{ij}\int_0^t z_{ij}(\tau)\,d\tau+c_{ij}, \qquad j=1,\dots,m,$$

$$P(\underline{z})=\int_0^\infty\sum_{i=1}^n w_i(t)\left[x_i(t)-\sum_{j=1}^m z_{ij}(t)\right]dt,$$

where $w_i(t)$ — given weighting function; b_{ij}, c_{ij} — given numbers.

The optimum control problem consists in finding such a nonnegative strategy $\underline{z}(t)\equiv\{z_{ij}(t)\}$ which maximizes

$$P(\underline{z})=\int_0^\infty\sum_{i=1}^n w_i(t)\left\{A_i\exp(\gamma_i t)\prod_{j=1}^m\left[b_{ij}\int_0^t z_{ij}(\tau)\,d\tau\right]^{\gamma_{ij}}-\sum_{j=1}^m z_{ij}(t)\right\}dt \tag{24}$$

subject to the constraints

$$G_i(z) = A_i \exp(\gamma_i t) \prod_{j=1}^{m} \left[b_{ij} \int_0^t z_{ij}(\tau)\, d\tau \right]^{\gamma_{ij}} - \sum_{j=1}^{m} z_{ij}(t) \geq 0, \qquad i = 1, \ldots, n. \tag{25}$$

The general solution of the present problem is much more difficult and in order to solve it effectively special decomposition algorithms can be employed [5].

Another generalization of the model (10) is obtained by replacing the linear rise in efficiency γ_t by $\int_0^t \gamma(\tau)\, d\tau$, where

$$\gamma(t) \leq \gamma_0 + f\left[\int_0^t y(\tau)\, d\tau \right], \tag{26}$$

and

$$0 \leq y(t) + z(t) \leq x(t), \qquad t \in [0, \infty),$$

f is a given monotonously increasing function, $y(t)$ — part of the income used for the purpose of increasing production efficiency by research, innovations, new technology etc., γ_0 — positive number.

The present optimization problem can be formulated as follows: find the non-negative functions $y(t) = y^o(t)$, $z(t) = z^o(t)$ such that

$$P(y, z) = \int_0^\infty w(\tau)\,[x(\tau) - z(\tau) - y(\tau)]\, d\tau \tag{27}$$

is maximum, subject to the constraints (13), (14), (26) and

$$G(y, z) = x(t) - z(t) - y(t) \geq 0, \qquad t \in [0, \infty), \tag{28}$$

where

$$x(t) = A \exp\left(\int_0^t \gamma(\tau)\, d\tau \right) [L(t)]^{\delta}\, [K(t)]^{\delta_1}.$$

It should be observed that the optimum investment strategy (such as (22)) depends on parameters $A, \gamma, \delta, \beta, \lambda$ which at the beginning of planning period T are not known to the controller. The best what can be done is to extrapolate these parameters using the econometric methods based on the past observations of real economy. However, the A, γ, δ, β will depend on the size of investments within the planning period. In other words there is a feedback loop existing between the macro-model parameters and the optimum investment strategy. Besides, the macro-parameters represent the aggregated effect of development processes which take place in micro-models. A possible approach to the optimization of development processes would be the formulation of the general optimization problem for the complete set of micro-models. Since that approach requires the solution of complex variational problems of a great number of variables it is not feasible. Besides, it is necessary to implement the solution in the form suitable to the existing decentralized management and administration.

The optimization of development processes in the decentralized form requires the investigation of relations among the macro- and aggregated micro-parameters. That is a goal which will be pursued in the next sections.

3. The micro-model of the development system

In the general model considered in the present paper the macro-model should be regarded as the aggregation of a number N of micro-models within the particular sector i having a hierarchic structure, which shows the streams of resources circulating among the individual subsystems. An example of such a sector with global production intensity $x(t)$ (the sector index i has been dropped for the purpose of abreviation of notation) and global resources intensities $y_j(t), j=1, ..., m.$, which

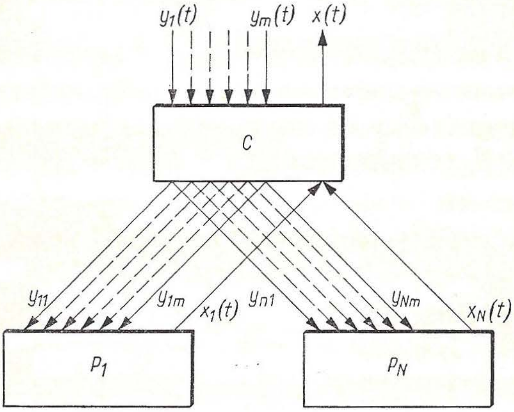

Fig. 1

are utilized (consumed) by the sector, has been shown in Fig. 1. Each subsystem P_ν, $\nu=1, ..., N$, consumes a part $y_{j\nu}(t)$, of $y_j(t)$, i.e.

$$\sum_{\nu=1}^{N} y_{j\nu}(t)\,\lambda_{j\nu}(t) \leqslant y_j(t), \quad j=1, ..., m, \tag{29}$$

where $\lambda_{j\nu}(t)$ are given functions representing the transmission or transportation losses.

It is assumed that $\lambda_{j\nu}(t) \geqslant 1$, $j=1, ..., m$, $\nu=1, ..., N$, $t \in [0, \infty)$.

The global production is assumed to be the sum of local productions $x_\nu(t)$, i.e.

$$\sum_{\nu=1}^{N} x_\nu(t)\,\lambda_\nu(t) = x(t), \tag{30}$$

where $\lambda_\nu(t)$ — given loss functions: $0 \leqslant \lambda_\nu(t) \leqslant 1$, $\nu=1, ..., N$, $t \in [0, \infty)$.

The controller C allocates the available recources $y_j(t), j=1, ..., m$, which satisfy (29), among the processes P_ν, $\nu=1, ..., N$, in such a way that for each time instant $t \in [0, \infty)$ the total production (30) attains the maximum value.

In other words the optimization problem for the controller consists in finding nonnegative functions $y_{j\nu}(t)=y^o_{j\nu}(t)$, $j=1, \dots, m$, $\nu=1, \dots, N$, which maximize (30) subject to the constraints (29). Assuming that each process can be described by the equation (compare (4)):

$$x_\nu(t)=[k_\nu(t)]^q \prod_{j=1}^{m} [y_{j\nu}(t)]^{\delta_j}, \quad \nu=1, \dots, N \tag{31}$$

$$k_\nu(t)=\left\{A_\nu \exp\left[\int_0^t \gamma_\nu(\tau)\,d\tau\right]\right\}^{-1/q}, \quad q=\sum_{j=1}^{m}\delta_j-1<0$$

one finds out (using similar reasoning as in the proof of theorem 2) that

$$x(t)=x^o(t)=\max \sum_{\nu=1}^{N} x_\nu(t)\,\lambda_\nu(t)=[k(t)]^q \prod_{j=1}^{m}[y_j(t)]^{\delta_j}, \tag{32}$$

and the optimum values of $y_{j\nu}(t)$ become

$$y^o_{j\nu}(t)=\frac{k^*_\nu(t)\,\lambda^{-1}_{j\nu}(t)}{k(t)}\,y_j(t), \quad j=1, \dots, m, \quad \nu=1, \dots, N$$

where

$$k^*_\nu(t)=k_\nu(t)\left\{\lambda_\nu(t)\prod_{j=1}^{m}[\lambda_{j\nu}(t)]^{-\delta_j}\right\}^{-1/q}, \quad k(t)=\sum_{\nu=1}^{N}k^*_\nu(t), \quad t\in[0,\infty). \tag{33}$$

It should be observed that the aggregated system characteristic (32) has the same analytic form as the sub-systems characteristics (31) but the aggregated performance index becomes

$$k(t)=\sum_{\nu=1}^{N}k_\nu(t)\,l_\nu(t) \tag{34}$$

where

$$l_\nu(t)=\left\{\lambda_\nu(t)\prod_{j=1}^{m}[\lambda_{j\nu}(t)]^{-\delta_j}\right\}^{-1/q}, \quad \nu=1, \dots, N$$

can be called the loss index. The loss indices satisfy the condition $l_\nu(t)\geqslant 1$, $t\in[0,\infty)$, $\nu=1, \dots, N$, and the aggregated performance index $k(t)$ increases after each aggregation.

As shown in the paper [6] for the case $k_\nu(t)=k_\nu=$const., $l_\nu(t)=l_\nu=$const., $\nu=1, \dots, N$, it is possible to solve the problem of synthesis of the best organizational structure of the hierarchic system. In the simple case when the subsystems P_ν, $\nu=1, \dots, N$, can be interchanged, i.e. when to the given transmission system, characterized by the numbers $l_1\leqslant l_2\leqslant \dots \leqslant l_N$, one can assign the processes, characterized by the set of performance indices $\{k\}^N_1$, the minimum value of the resulting performance $k'=\sum_{\nu=1}^{N}k_\nu l_\nu$ follows when $k_1\geqslant k_2\geqslant \dots \geqslant k_N$.

That result can be easily extended to the present case where k_ν and l_ν are given functions of time, and it can be formulated in the form of the following:

Theorem 3. The assignment process for the system with given $\{k_\nu(t)\}_1^N$, $\{l_\nu(t)\}_1^N$, $t \in [0, \infty)$ is optimum when for each $t \in [0, \infty)$ the following relations

$$k_1(t) \leqslant k_2(t) \leqslant \ldots \leqslant k_N(t), \; l_1(t) \geqslant l_2(t) \geqslant \ldots \geqslant l_N(t) \tag{35}$$

or

$$k_1(t) \geqslant k_2(t) \geqslant \ldots \geqslant k_N(t), \; l_1(t) \leqslant l_2(t) \leqslant \ldots \leqslant l_N(t) \tag{36}$$

hold. The conditions (35), (36) become also necessary for optimality in the case of strict inequalities in (35), (36).

Using theorem 3 it is possible to solve the synthesis problem for multi-level hierarchic structure with processes described by (31) in the similar way as it has been done, for the simpler case of $k_\nu(t) = k_\nu = \text{const.}$, $\nu = 1, \ldots, N$, in [6]. The resulting structure has the property that the aggregated performance index is minimum.

It should be also observed that when the values of $k_\nu(t)$ change the structure of the best organization should change as well (in order to yield the minimum value of the resulting performance measure). That requires the intervention of the additional controller which reorganizes the system structure if necessary.

In practical situations it happens that the amount of resources used by the subsystems is bounded from below or from above. In other words there are additional constraints imposed on subsystem characteristic (31):

$$\underline{Y}_{j\nu} \leqslant y_{j\nu}(t) \leqslant \overline{Y}_{j\nu}, \quad j = 1, \ldots, m, \quad \nu = 1, \ldots, N, \tag{37}$$

where $\underline{Y}_{j\nu}$, $\overline{Y}_{j\nu}$ — given numbers.

When the global amounts of resources $y_j(t)$, $j = 1, \ldots, m$, distributed among the subsystems, increase (decrease) the upper (lower) subsystem constraints become saturated, i.e. active. The saturation effect, which fixes the amount of resources consumed by a particular subsystem P_i, is equivalent to such a change of $k_i(t)$ that the amount of resources received by P_i is $\overline{Y}_{ji}$ or $\underline{Y}_{ji}$, $j = 1, \ldots, m$. Since the subsystems production capacities determine the saturation limits $\overline{Y}_{j\nu}$, the assumption that the saturations are reached at the same time for all j is not very restrictive. It should be also observed that the price being paid for small plant capacity and operation on the upper saturation bounds is the increase of the performance index $k_i(t)$. When all the subsystem are saturated there is no optimization possible and the global performance may be very poor.

In the cases when the subsystem characteristics do not posses the required analytic form of (31), or when they are given in graphical form, as shown e.g. in Fig. 2a, for $N = 2$, $m = 1$, the approximation should be used. In the case of Fig. 2a the subsystem characteristic can be approximated by the function of the form $F_i(z) = k_i(z + z_i^*)^\gamma$. For that purpose it is convenient to construct the plots of $[f_i'(z)]^{1/\gamma}$, $i = 1, 2$, as shown in Fig. 2b and approximate them by linear functions of the form $\alpha_i(z + z_i^*)$, $i = 1, 2$.

Then the functions $F_i(z)$, $i = 1, 2$, which approximate $f_i(z)$ become

$$F_i(z) = k_i(z + z_i^*)^\gamma + c_i, \quad \begin{array}{l} i = 1, \; z \in [\underline{Z}_1, \overline{Z}_1] \\ i = 2, \; z \in [\underline{Z}_2, \overline{Z}_2] \end{array} \tag{38}$$

where $k_i = \alpha_i^\beta/(1+\beta)$, $\gamma = 1 + \beta$, c_i — numbers determined by the plot of $f_i(z)$.

Denoting $z_i+z_i^*$ by y_i one can observe that the problem of maximization of

$$\sum_{i=1}^{n} F_i(z_i)=k_i(z_i+z_i^*)^{\gamma}+c_i$$

subject to the constraints $\sum_{i=1}^{n} z_i \leqslant Z$, $z_i \in [\underline{Z}_i, \bar{Z}_i]$, $i=1, \ldots, n$, is equivalent to

$$\max \sum_{i=1}^{n} [k_i y_i^{\gamma}+c_i], \qquad \sum_{i=1}^{n} y_i \leqslant Y=\sum_{i=1}^{n} z_i^*+Z,$$

$$y_i \in [\underline{Z}_i-z_i^*, \bar{Z}_i-z_i^*], \qquad i=1, \ldots, n.$$

When the intervals $[\underline{Z}_i, \bar{Z}_i]$ are small the approximation can be exact enough for practical purposes. It is also possible to extend the approximation domain by

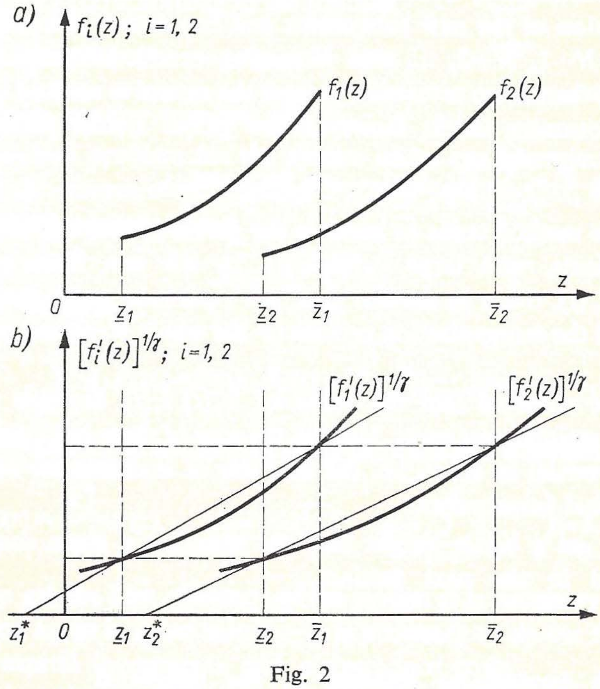

Fig. 2

choosing another set of subintervals $[\underline{\underline{Z}}_i, \bar{\bar{Z}}_i]$, $i=1, 2, \ldots, n$, with the same or different γ. Since the optimum solutions z_i^o satisfy the condition $f_1'(z_1^o)=f_2'(z_2^o)=\ldots=f_n'(z_n^o)$ it is convenient to choose $\underline{Z}_i, \bar{Z}_i, \underline{\underline{Z}}_i, \bar{\bar{Z}}_i, \ldots,$ in such a way that

$$f_1'(\underline{Z}_1)=f_2'(\underline{Z}_2)=\ldots=f_n'(\underline{Z}_n),$$

$$f_1'(\bar{Z}_1)=f_2'(\bar{Z}_2)=\ldots=f_n'(\bar{Z}_n),$$

$$f_1'(\underline{\underline{Z}}_1)=f_2'(\underline{\underline{Z}}_2)=\ldots=f_n'(\underline{\underline{Z}}_n),$$

$$f_1'(\bar{\bar{Z}}_1)=f_2'(\bar{\bar{Z}}_2)=\ldots=f_n'(\bar{\bar{Z}}_n),$$

. . . .

. . . .

The approximation can be extended to the multidimensional case.

After these preliminary remarks and observations we can return to the main subject of the present section which is the optimum control of the development processes in the hierarchic systems with optimum allocation of resources.

It is assumed that the demand for the global system production is less than the subsystems production capacities and the saturations is not present (at least in certain subintervals of the planning interval). The decision at the macro-level determines the sum of the global investments which should be spent for the development of each particular sector. That sum should be allocated among the subsystems (by the sector controller) in an optimum manner. There are two main ways or methods of subsystem development (called extensive and intensive respectively):

(i) extension of the saturation bounds $\bar{Y}_{j\nu}$, which can be also called the plant renewal or reconstruction, extension of bottlenecks etc.;

(ii) construction of the new subsystems and connection links, characterized by $k_\nu(t)$, $l_\nu(t)$, $\nu=N+1, N+2, \ldots$, and increasing production efficiency.

In case when there is no saturation present the extensive method does not change the global performance index. However, the global system production capacity characterized by $\bar{Y}_j=\sum_{\nu=1}^{N} \bar{Y}_{j\nu}$, $j=1, \ldots, m$, increases.

The intensive method tends to increase the global system efficiency by replacing the obsolete plants, with large $k_\nu(t)\, l_\nu(t)$, by the new factories with smaller value of $k_\nu(t)\, l_\nu(t)$. Since in the last case the optimum assignment process may become violated the reorganization process of the whole structure may prove to be necessary.

It should be observed that the value of $k_\nu(t)\, l_\nu(t)$ depends on the capital investment and maintenance costs. Usually the sum of these two costs is limited and they are controversial in effect, i.e. the decrease of maintenance costs requires an increase of capital investments.

In order to get a better insight into the situation a more concreate example will be studied in the next section.

4. Decentralized development planning of the integrated power system

Consider the integrated electric power system consisting of N units S_j (power plants or generators) generating $P_j(t)$, $j=1, \ldots, N$, units of power, shown in Fig. 3. The transmission lines with resistances R_j (including also the internal resistances of the generators) link S_j, $j=1, \ldots, N$, with the common load which demands $P(t)$ units of power. For the sake of simplicity in calculations it is assumed that the load resistance R is small as compared to R_j. In that case the part of power generated by S_j and delivered to the load becomes $P_j\lambda_j=P_j R/(R+R_j)$, $j=1, \ldots, N$. Then $P=\sum_{j=1}^{N} \lambda_j P_j$.

The power demand $P(t)$ is a given function of time which changes almost periodically in each day and each year of the planning interval [1]. Besides, the mean value of $P(t)$ increases slowly in time.

The integrated power production can be considered as a sector of the economy. The production income within T is $I=\int_0^T cP(t)\,dt$, where c — the price of a 1 kWh of electrical energy. The net income of the sector is $I_n=I-C_i-C_m$, where C_i — investments and C_m — is the maintenance cost (mostly the cost of fuel burned in the power stations and transmission losses).

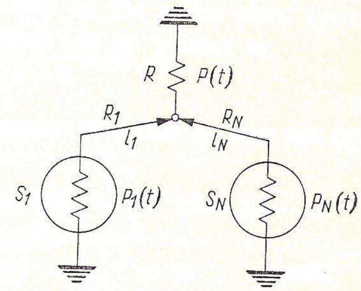

Fig. 3

It is assumed that the fuel-cost C^j_{mp} consumed by S_j depends on $P_j(t)$ according to the formula

$$C^j_{mp}=F_j(P_j)=k_j^{\delta-1}[P_j(t)]^\delta, \quad j=1,2,\ldots,N \tag{39}$$

where k_j — given positive numbers, $\delta>1$.

An optimization problem for the system under consideration can be stated as follows. Find the nonnegative functions $P_j(t)=P_j^o(t)$, $j=1,\ldots,N$, which minimize the global fuel consumption cost

$$C_{mp}=\sum_{j=1}^{N} C^j_{mp}, \tag{40}$$

subject to the constraints

$$\sum_{j=1}^{N} \lambda_j P_j(t)=P(t), \quad P_j(t)\in[\underline{P}_j, \bar{P}_j], \quad j=1,\ldots,N. \tag{41}$$

As shown in sec. 3 (formulae (32), (33)) the optimum strategies for $P_j(t)$ become

$$P_j^o(t)=\frac{k_j^*}{\lambda_j k}P(t), \quad k_j^*=k_j\lambda_j^{\frac{\delta}{1-\delta}}, \quad k=\sum_{j=1}^{N} k_j^*, \quad j=1,\ldots,N. \tag{42}$$

In order to get the best efficiency of the system it is necessary to choose proper values of λ_j and k_j, or in other words — it is necessary to choose the proper relation between the capital investments and maintenance costs.

The capital investment C_i consists of two parts; the cost of the new transmission lines C_{il}, which is approximately proportional to the volume of copper used for transmission lines $C_{il}=\sum_{j=1}^{N} C^j_{il}$, $C^j_{il}=\alpha d_j s_j$ (where d_j — distance, s_j — crossection, $\alpha=$const.), and the cost of the new power units C_{ip}, depending on the maximum power production capacities $\bar{P}_j=\max_{t\in[0,T]} P_j^o(t)$, and on the performance index k_j. A good approximation of the last relation is

$$C_{ip}=\sum_{j=1}^{N} C^j_{ip}, \quad C^j_{ip}=\vartheta(\bar{P}_j)^\varepsilon k_j^{-\kappa}, \tag{43}$$

where ϑ, ε, κ — given positive numbers.

The maintenance cost C_m consists as well of two parts: the cost of energy lost in the transmission lines

$$C_{ml}=\sum_{j=1}^{N} C_{ml}^{j}=\sum_{j=1}^{N} \beta \int_{0}^{T} P_{lj}(t)\,dt,$$

where $\beta=\text{const.}$, P_{lj} — the power lost in the j-th transmission line equal

$$P_{lj}(t)=\frac{R_j}{R+R_j}P_j^o(t)=\lambda_j\frac{R_j}{R}P_j^o(t), \quad j=1, \dots, N,$$

and the cost of fuel (40), where

$$C_{mp}^{j}=k_j^{\delta-1}\int_{0}^{T}[P_j^o(t)]^{\delta}\,d\tau=k_j^{\delta-1}(k_j^*/k)^{\delta}\lambda_j^{-\delta}B, \qquad B=\int_{0}^{T}[P(\tau)]^{\delta}\,d\tau. \tag{44}$$

Since

$$C_{ml}^{j}=\beta\frac{R_j}{R}\frac{k_j^*}{k}\int_{0}^{T}P(\tau)\,d\tau$$

where $R_j=\rho\,(d_j/s_j)$ (ρ — specific resistance of the transmission line) and $s_j=C_{il}^{j}/\alpha d_j$, one obtains

$$kC_{ml}^{j}=A\frac{\alpha\beta\rho}{R}d_j^2k_j^*=A\frac{\alpha\beta\rho}{R}d_j^2\,k_j\,\lambda_j^{\frac{\delta}{1-\delta}}, \qquad A=\int_{0}^{T}p(\tau)\,d\tau.$$

One finds also

$$\lambda_j\cong\frac{R}{R_j}=\frac{R}{\alpha\rho d_j^2}C_{il}^{j}, \quad j=1, \dots, N. \tag{45}$$

Then

$$kC_{ml}^{j}=K_j^{1-\delta_1}[C_{il}^{j}]^{\delta} \tag{46}$$

where

$$\delta_1=\frac{\delta}{1-\delta}, \qquad K_j=[A\beta k_j]^{\delta_2}\frac{\rho\alpha}{R}d_j^2, \qquad \delta_2=\frac{1}{1-\delta_1}.$$

Then the following problem of the development of transmission system can be formulated. Find the best allocation of investments: C_{il}^{oj}, $j=1, \dots, N$, which minimize the maintenance cost

$$kC_{ml}=\sum_{j=1}^{N}kC_{ml}^{j}=\sum_{j=1}^{N}K_j^{1-\delta_1}[C_{il}^{j}]^{\delta_1}. \tag{47}$$

Subject to the constraints

$$\sum_{j=1}^{N}C_{il}^{j}\leqslant C_{il}, \quad C_{il}^{j}\geqslant 0, \quad j=1, \dots, N. \tag{48}$$

The indices k_j, $j=1, \dots, N$, are treated as fixed and given.

Using the theorem 2 one obtains

$$C_{il}^{oj}=\frac{K_j}{K}C_{il}, \qquad C_{ml}^{oj}=\frac{K_j}{K}kC_{ml}, \quad j=1, \dots, N \tag{49}$$

where

$$K=\sum_{j=1}^{N} K_j$$

When the values of C_{il}^{oj} are being derived the unknown parameters of the transmission lines $s_j=C_{il}^{oj}/\alpha d_j$, $j=1, \ldots, N$, can be also determined.

In a similar way the problem of the development of the power-plant system can be solved. Eliminating k_j from (43), (44), which can be written as

$$C_{ip}^{j}=\vartheta C\lambda_j^{(\delta_1-1)\varepsilon}\, k^{-\varepsilon}\, k_j^{\varepsilon-\kappa} \tag{50}$$

$$C_{mp}^{j}=B\lambda_j^{(\delta_1-1)\delta}\, k^{-\delta}\, k_j^{2\delta-1} \tag{51}$$

one gets

$$k_j=[(\vartheta C)^{-1}\, C_{ip}^{j}\, \lambda_j^{(1-\delta_1)\varepsilon}\, k^{\varepsilon}]^{\frac{1}{\varepsilon-\kappa}} \tag{52}$$

and

$$k^{\delta_3}\, C_{mp}^{j}=(\bar{K}_j)^{1-\delta_4}\, [C_{ip}^{j}]^{\delta_4} \tag{53}$$

where

$$\bar{K}_j=[B(\vartheta C)^{-\delta_4}\, \lambda_j^{(1-\delta_1)\delta_3}]^{\frac{1}{1-\delta_4}},$$

$$\delta_3=\delta_4\,\varepsilon-\delta, \qquad \delta_4=\frac{2\delta-1}{\varepsilon-\kappa}.$$

Then the following development optimization problem can be formulated. Find the nonnegative numbers C_{ip}^{oj} (investments) $j=1, \ldots, N$, which minimize the maintenance cost

$$k^{\delta_3}\, C_{mp}=\sum_{j=3}^{N} (\bar{K})^{1-\delta_4}\, [C_{ip}^{j}]^{\delta_4} \tag{54}$$

subject to the constraint

$$\sum_{j=1}^{N} C_{ip}^{j}\leqslant C_{ip} \tag{55}$$

The $\lambda_j, j=1, \ldots, N$, are considered as fixed and given. Using the theorem 2 one obtains

$$C_{ip}^{oj}=\frac{\bar{K}_j}{\bar{K}}\, C_{ip}, \qquad C_{mp}^{oj}=\frac{\bar{K}_j}{\bar{K}}\, k^{\delta_3}\, C_{mp}, \qquad \bar{K}=\sum_{j=1}^{N} \bar{K}_j, \qquad j=1, \ldots, N. \tag{56}$$

It should be observed that by elimination of k_j and λ_j (by (45), (52)) one can express C_{mp}, C_{ml} as a power-type function of C_{il}^{j}, C_{ip}^{j} and solve (by theorem 2) the optimization problem for the case when k_j or λ_j are not fixed. It is also possible to minimize the global cost $C_{mp}+C_{ml}$ under assumption $C_{ip}+C_{il}\leqslant C_i$, where C_i = given.

A number of further generalizations of the development planning is possible. First of all it is possible to consider the power distribution and power consumption system. In that case the subsystems $S_1, ..., S_N$ of Fig. 3 should be treated as loads with the internal resistances and transmission resistances equal R_j, $j=1, ..., N$, $P(t)$ should be treated as available power supply. The direction of the arrows indicating the stream of energy should be reversed. The performance functions $F_j(P_j)$ should be treated as the production values versus the power consumption. In the present model it is necessary to maximize the global production

$$\sum_{j=1}^{N} \int_{0}^{T} F_j(P_j)\,\mathrm{d}t$$

subject to the constraints $\sum_{j=1}^{N} \lambda_j P_j \leqslant P(t)$.

Obviously the optimization of the present development model, including the transmission losses characterized by λ_j, and subprocess parameters (k_j) can be done along the same lines as in the previous case of the power-generation model.

In real integrated power systems these two models should be treated in the combined form, including the multi-level hierarchic structure, corresponding to the regional subcenters. In that last case it is also necessary to take into account the voltage transformers, hydro-power stations etc. It may be also necessary to introduce the saturation bounds and use the approximation technique of sec. 3, for description of the plots of $F_j(P_j)$ (see Fig. 2a, b) in the required analytic form.

It should be also noted that the model of the integrated power system can be readjusted to serve as a model for other systems dealing with generation and distribution of resources, such as: integrated natural and industrial gas system, integrated water system etc. However, in these models the relation between the investment cost C_{il} and the hydraulic resistance of the transmission pipes is much more comlicated (compare Ref. [7]).

Now we can discuss the relation between the macro- and micro-models from the point of view of optimum planning strategies. The investment and maintenance costs, connected with transmission lines, will be neglected for the sake of simplicity, i.e. $C_m = C_{mp}$, $C_i = C_{ip}$.

It should be noticed that when the development process occurs gradually, within the planning interval, the aggregated performance can not be treated as constant factor. The development process results in the change of aggregated performance index, which becomes a function of time, i.e. $k(t)$.

Since the aggregated characteristic $C_m(t) = F[P(t)]$ has the same analytic form as (39) one can write

$$C_m(t) = [k(t)]^{\delta-1} [P(t)]^{\delta}, \quad t \in [0, T].$$

The aggregated investment can be approximated by the function analogous to (43), i.e.

$$C_i(t) = \tilde{\vartheta}(\bar{P})^{\tilde{\varepsilon}} [k(t)]^{-\tilde{\kappa}}, \quad \bar{P} = \max_{t \in [0, T]} P(t).$$

The aggregated investment strategy $z(t)$ creates the capital $\int_0^t z(\tau)\,d\tau$, which is spent on the construction of new power plants, i.e. $\int_0^t z(\tau)\,d\tau = C_i(t)$.

Then an optimization problem for the macro-model can be formulated:

Find the nonnegative investment strategy $z(t) = z^o(t)$, which maximizes the net income:

$$I_n(z) = \int_0^T G(z)\,dt, \quad G(z) = cP(t) - C_m(t) - C_i(t),$$

$$C_m(t) = [k(t)]^{\delta-1}[P(t)]^{\delta},$$

$$C_i(t) = \tilde{\vartheta}\bar{P}^{\tilde{\varepsilon}}[k(t)]^{-\tilde{\kappa}} = \int_0^t z(\tau)\,d\tau,$$

subject to the constraint

$$G(z) \geqslant 0, \quad t \in [0, T],$$

where $P(t)$, $\tilde{\vartheta}$, $\bar{P}^{\tilde{\varepsilon}}$, $\tilde{\kappa}$, δ are given.

Eliminating $k(t)$ one gets

$$\begin{gathered} G(z) = cP(t) - \left[a\int_0^t z(\tau)\,d\tau\right]^{\beta}[P(t)]^{\delta} - \int_0^t z(\tau)\,d\tau \geqslant 0, \\ a = (\tilde{\vartheta}\bar{P}^{\tilde{\varepsilon}})^{-1}, \quad \beta = \frac{1-\delta}{\tilde{\kappa}}, \quad (\beta < 0). \end{gathered} \tag{57}$$

Assume that the constraint (57) is notactive. The necessary condition of optimality requires that $I_n'(z^o) = 0$. The variation of I_n becomes

$$dI_n(z, \lambda, h) = \int_0^T h(\tau)\,d\tau\left\{-\int_\tau^T \beta\left[a\int_0^t z(\tau)\,d\tau\right]^{\beta-1} \times aP(t)^{\delta}\,dt - \int_\tau^T dt\right\}.$$

Then

$$I_n'(z) = \int_\tau^T \left\{\frac{\beta a[P(t)]^{\delta}}{\left[a\int_0^t z(\tau)\,d\tau\right]^{1-\beta}} + 1\right\}dt = 0,$$

and

$$z^o(t) = \frac{\delta\beta}{\beta-1}(-\beta a)^{\frac{\beta}{1-\beta}}[P(t)]^{\frac{\delta+\beta-1}{1-\beta}}, \quad t \in [0, T]. \tag{58}$$

The constraint (57) will be not active if

$$G(z^o) = P(t)\left\{c - d\,[P(t)]^{\frac{\delta}{1-\beta}-1}\right\} > 0, \quad t \in [0, T]$$

where

$$d = (1-\beta^{-1})(-\beta)^{\frac{1}{1-\beta}}a^{\frac{\beta}{1-\beta}}$$

i.e. if the price c is set large enough with respect to $d\,[P(t)]^{\frac{\delta}{1-\beta}-1}$.

When the optimum macro-strategy of investment $C_i(t) = \int_0^t z^o(\tau)\, d\tau$, is known it is possible to derive the best investment strategies at the micro-levels, using formulae (51)—(56). At the same time the aggregated performance index $k(t)$ can be determined.

As a result the decentralized two-level optimum development planning follows. The global controller derives an investment strategy based on the macro-model with the estimated (extrapolated) parameters $(a, \tilde{\kappa})$.

The local controllers dealing with micro-models derive the best local development strategies and determine the corresponding performance indices. When the estimated and derived (global) performances do not coincide the values of macro-parameters should be modified. Consequently the decentralized development planning requires an exchange of information between the local and global controllers.

The decentralized development planning can be extended to the multi-sector and multi-level models which correspond to the existing administrative organizations.

It should be also noted that the decentralized approach to the planning and development of economic models helps to surmount the most difficult obstacle, which is the gap between the macro- and micro-models. In the literature dealing with the macro-models of economic growth [2, 8, 9], the annual rise in efficiency is believed to be a function of the technical progress, research, innovations and inventions etc. These notions remain, however, to be pure abstracts if they are seperated from the concreate micro-models and the investment and maintenance costs.

References

1. KIRCHMAYER, L. K., Economic operation of power systems. New York, 1958.
2. KORNAI, J.: Zastosowanie programowania w planowaniu. Warszawa, 1969.
3. KULIKOWSKI R., Optimization of aggregated dynamic systems. Proc. 3rd IFAC Congr. London 1966.
4. KULIKOWSKI R., Sterowanie w wielkich systemach. Warszawa, 1970.
5. KULIKOWSKI R., Decomposition of complex environment control problems. *Bull. Acad. Polon. Sci. Ser. Sci. Techn.* **18**, 9 (1970).
6. KULIKOWSKI R., Synthesis and optimum control of organization in large-scale systems. *Arch. Autom. i Telemech.* **18**, 3 (1970).
7. NEKRASOVA O. A., KHASILEV V. I., Optimum tree of pipe system. *Economics and Mathematical Methods* **6**, 3 (1970) (in Russian).
8. PHELPS E. S., Models of technical progress and the golden rule of research. *Review of Economic Studies* **33**, 2 (1966).
9. TINBERGEN I., BOS H. C., Mathematical models of economic growth. New York, 1962.

Sterowanie optymalne zdecentralizowanym modelem planowania

Przedstawiono zagadnienia sterowania optymalnego modelem planowania rozwoju. Model ten obejmuje m czynników egzogenicznych i n czynników endogenicznych. Ich intensywności są związane układem liniowych równań różniczkowych. Makromodel uzyskuje się w wyniku agregacji mikromodeli. Optymalną strategię rozwoju można realizować w postaci zdecentralizowanej z wymianą informacji między globalnym (makro) a lokalnymi (mikro) regulatorami. Jako typowy przykład rozpatrzono zdecentralizowane planowanie rozwoju zintegrowanego systemu energetycznego. Strategia rozwoju optymalnego polega na takim rozdziale inwestycji, który minimalizuje globalne koszty utrzymania.

Оптимальное управление моделью децентрализованного планирования развития

В статье представлено оптимальное управление моделью децентрализованного планирования развития. Модель содержит m эгзогенных и n эндогенных переменных, интенсивности которых описываются системой линейных дифференциальных уравнений. Макромодель получена путем синтеза микро-моделев. Оптимальная стратегия развития может быть реализована в децентрализированной форме с обменом информации между глобальными (мактро-) и локальными (микро-) регуляторами. В качестве характерного примера рассмотрено децентрализованное планирование развития синтегрированной энергетической системы. Оптимальная стратегия развития заключается в распределении капиталовложений с целью минимизирования общих расходов.