

On application of a quadratic programming procedure to optimal control problems in systems described by parabolic equations

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An optimal control problem of a system described by linear partial differential equation of parabolic type is considered. Both boundary and distributed controls are investigated. The performance index is a quadratic one.

It is shown that this problem can be reduced to some quadratic programming problem in a Hilbert space.

An iterative procedure of solving this problem is proposed. One dimensional heat conduction system is considered as an example. Some numerical results obtained using a digital computer are presented.

1. Introduction

The systems described by second order linear partial differential equation of parabolic type with mixed boundary conditions are considered. Both boundary and distributed control subject to amplitude constraints are investigated.

The initial state is given and the time of control T is fixed. Our problem of optimization is to minimize a quadratic functional depending on the state of the system at the time T .

It is shown that this problem can be reduced to the problem of minimizing a quadratic functional on a closed, convex and bounded set in a Hilbert space.

To solve this last problem the application of iterative procedure of a quadratic programming given in [3] and [9] is proposed.

The process of optimal one sided heating in one dimensional heat conduction system is considered as an example of application.

For this case the numerical results obtained using a digital computer are presented.

2. Problem statement

Let Ω be an open bounded set in n -dimensional Euclidean space R^n . We assume that the boundary Γ of Ω is of the class C^∞ .

Let $H^1(\Omega)$ denote a Sobolev space of functions $\varphi \in L^2(\Omega)$ such, that their derivatives (in the sense of distribution) $\partial\varphi/\partial x_i$, $i=1, 2, \dots, n$, belong to $L^2(\Omega)$. It is a Hilbert space [6] with the norm given by

$$\|\varphi\|_{H^1(\Omega)} = \left[\int_{\Omega} \varphi^2(x) dx \right]^{1/2} + \sum_{i=1}^n \left[\int_{\Omega} \left(\frac{\partial \varphi(x)}{\partial x_i} \right)^2 dx \right]^{1/2}. \quad (1)$$

By $\varphi|_{\Gamma}$ we denote the trace of the function φ on the boundary Γ of the set Ω . It follows from Sobolev theorem for fractional spaces [6, 8] that

$$\varphi|_{\Gamma} \in L^2(\Gamma). \quad (2)$$

Let $T > 0$ be a fixed number. We introduce the space F of the function φ from $(0, T)$ into $H^1(\Omega)$ satisfying the following condition

$$\|\varphi\| = \left[\int_0^T \|\varphi(t)\|_{H^1(\Omega)}^2 dt \right]^{1/2} + \left[\int_0^T \int_{\Gamma} \varphi^2(x, t) dx dt \right]^{1/2} < \infty. \quad (3)$$

By F' we denote the conjugate space of F .

In the space F we introduce the bilinear continuous form given by

$$\begin{aligned} \langle Af, \varphi \rangle = & \int_0^T \int_{\Omega} \sum_{i,j=1}^n a_{ij}(x, t) \frac{\partial f(x, t)}{\partial x_i} \frac{\partial \varphi(x, t)}{\partial x_j} dx dt + \\ & + \int_0^T \int_{\Gamma} q(x, t) f(x, t) \varphi(x, t) dx dt, \end{aligned} \quad (4)$$

where $0 < q_0 \leq q(x, t) \leq q_1 < \infty$ almost everywhere,

$$a_{ij}(x, t) \in L^\infty([0, T] \times \Omega). \quad (5)$$

Moreover we assume that for all $\xi \in R^n$, the following ellipticity condition is satisfied

$$\sum_{i,j}^n a_{ij}(x, t) \xi_i \xi_j \geq \alpha \sum_{i=1}^n \xi_i^2, \quad \alpha > 0. \quad (5b)$$

It follows [8] from (5a) and (5b) that the form $\langle Af, \varphi \rangle$ is coercive [6]. We also introduce in F a linear continuous form defined as follows

$$\begin{aligned} \langle w+z, \varphi \rangle = & \int_0^T \int_{\Omega} z(x, t) \varphi(x, t) dx dt + \int_0^T \int_{\Omega} v(x, t) \varphi(x, t) dx dt + \\ & + \int_0^T \int_{\Gamma} u(x, t) \varphi(x, t) dx dt \end{aligned} \quad (6)$$

where

$$\begin{aligned} z(x, t) &\in L^2([0, T] \times \Omega), \\ v(x, t) &\in L^2([0, T] \times \Omega), \\ u(x, t) &\in L^2([0, T] \times \Gamma), \\ w(x, t) &= [v(x, t), u(x, t)]. \end{aligned} \quad (7)$$

We consider the following Cauchy problem: find the function $f \in F$, such that $f' \in F'$, which satisfies the equation:

$$\langle f', \varphi \rangle + \langle Af, \varphi \rangle = \langle w + z, \varphi \rangle \quad \text{for every } \varphi \in F \quad (8)$$

along with the initial condition

$$f(x, 0) = f_0(x) \in L^2(\Omega). \quad (9)$$

$f' = (\partial f / \partial t) \in F'$ denotes here the time derivative of $f(x, t)$ taken in the sense of distributions, and

$$\langle f', \varphi \rangle = \int_0^T \int_{\Omega} \frac{\partial f(x, t)}{\partial t} \varphi(x, t) dx dt. \quad (10)$$

The above stated problem has a unique solution [6, 8]. Using Green's formula [6, 8] we find that the solution of the problem (8), (9) corresponds to the weak solution of the following boundary value problem

$$\frac{\partial f(x, t)}{\partial t} - \sum_{i,j=1}^n \frac{\partial}{\partial x_j} \left[a_{ij}(x, t) \frac{\partial f(x, t)}{\partial x_i} \right] = z(x, t) + v(x, t) \quad \text{in } \Omega \times (0, T), \quad (11)$$

$$\frac{\partial f(x, t)}{\partial \eta} + q(x, t) f(x, t) = u(x, t) \quad \text{in } \Gamma \times (0, T) \quad (12)$$

$$f(x, 0) = f_0(x) \quad (13)$$

where

$$\frac{\partial f(x, t)}{\partial \eta} = \sum_{i,j=1}^n a_{ij}(x, t) \frac{\partial f(x, t)}{\partial x_i} \cos(\eta, x_j)$$

where η is the unit vector of the outward normal to the surface Γ .

The vector function $w(x, t) = [v(x, t), u(x, t)]$ is called an admissible control if it satisfies the following conditions:

$$|v(x, t)| \leq v < \infty \quad \text{almost everywhere in } \Omega \times (0, T), \quad (14a)$$

$$|u(x, t)| \leq u < \infty \quad \text{almost everywhere in } \Gamma \times (0, T). \quad (14b)$$

Our problem of optimization is to find such an admissible control $w_{opt}(x, t) = [v_{opt}(x, t), u_{opt}(x, t)]$, which is also called an optimal control, for which the functional

$$J(w) = (g(x) - f(x, T), g(x) - f(x, T))^{1/2} = \left\{ \int_{\Omega} [g(x) - f(x, T)]^2 dx \right\}^{1/2} \quad (15)$$

assumes the minimal value.

$g(x)$ is here a given element of $L^2(\Omega)$, $f(x, T)$ is the value of a solution of (8), (9) at T corresponding to some admissible control, and (\cdot, \cdot) denotes the scalar product in $L^2(\Omega)$.

Note that the expression $f(x, T)$ is well defined as it follows from Theorem 1.1 on page 116 in [6].

To solve the problem of optimization we introduce the set $D \subset L^2(\Omega)$ of the values of the solutions of (8), (9) at T corresponding to all admissible controls, i.e.

$$D = \{f(x, T) : f(x, 0) = f_0(x); \langle f', \varphi \rangle + \langle Af, \varphi \rangle = \langle w + z, \varphi \rangle; |v(x, t)| \leq v, |u(x, t)| \leq u\}. \quad (16)$$

It is obvious that the set D is convex. On the other hand it follows from the results of [8] that D is also bounded and weakly closed in $L^2(\Omega)$; therefore it is weakly compact [3].

Thus our problem is reduced to finding a minimum of the quadratic functional (15) on the convex and weakly compact set D .

The functional (15) as a convex and differentiable one is weakly lower semi-continuous; hence it assumes its minimum on a weakly compact set [10].

The strict convexity of the functional (15) with respect to $f(x, T)$ and the convexity of D imply [10] that there is a unique function $f_{opt}(x, T)$ which assumes the minimum of the functional.

3. An iterative method of solving the optimization problem

To solve the problem of optimization we use a method of quadratic programming, which was introduced in [4] and generalized for the case of Hilbert space in [9]. In those papers one can find proofs of the convergence of the method.

In the method a sequence $\{f^i(x, T)\} \subset D$ which approximates the optimal solution $f_{opt}(x, T)$ is constructed.

Let us assume that we already know the j -th element of the sequence $f^i(x, T)$ and let us construct the $(j+1)$ -th element.

To this purpose we find an element $f^{j+1}(x, T) \in D$ satisfying the condition $(g(x) - f^j(x, T), \bar{f}^{j+1}(x, T)) \geq (g(x) - f^j(x, T), f(x, T))$ for every $f(x, T) \in D$. (17)

If there is more than one element $\bar{f}^{j+1}(x, T)$ satisfying (17) we take any arbitrary of them.

Furthermore we find the value

$$\bar{\alpha}^j = \frac{(g(x) - f^j(x, T), \bar{f}^{j+1}(x, T) - f^j(x, T))}{(\bar{f}^{j+1}(x, T) - f^j(x, T), \bar{f}^{j+1}(x, T) - f^j(x, T))} \quad (18)$$

and we fix the number $c \in (0, 1]$. We choose α^j as any number belonging to the interval

$$[\min \{1, c\bar{\alpha}^j\}, \min \{1, (2-c)\bar{\alpha}^j\}]. \quad (19)$$

In particular for $c=1$ we have $\alpha^j = \min \{1, \bar{\alpha}^j\}$. Finally we put

$$f^{j+1}(x, T) = f^j(x, T) + \alpha^j [f^{j+1}(x, T) - f^j(x, T)]. \quad (20)$$

It was shown in [4] and [9] that for any arbitrary initial element $f^0(x, T) \in D$ the sequence $\{f^j(x, T)\}$ constructed in this manner is convergent to $f_{opt}(x, T)$ strongly in $L^2(\Omega)$. Moreover for the j -th step of the iterative procedure the following estimation takes place

$$\max \left\{ 0, \frac{(g(x) - f^{j+1}(x, T), g(x) - f^j(x, T))}{(g(x) - f^j(x, T), g(x) - f^j(x, T))^{1/2}} \right\} \leq (g(x) - f_{opt}(x, T), g(x) - f_{opt}(x, T))^{1/2} \leq (g(x) - f^j(x, T), g(x) - f^j(x, T))^{1/2}. \quad (21)$$

In the application of the presented procedure the crucial point is the determination of an element $f^{j+1}(x, T)$ satisfying (17). To this purpose we introduce the following transformation.

Taking into consideration (10) and integrating by parts with respect to t the first element of the left hand side of the equation (8) we rewrite (8) in the following form

$$\int_{\Omega} f(x, T) \varphi(x, T) dx - \int_{\Omega} f(x, 0) \varphi(x, 0) dx - \langle f, \varphi' \rangle + \langle f, A^* \varphi \rangle = \langle w + z, \varphi \rangle. \quad (22)$$

The equation (22) has to be satisfied for any arbitrary function $\varphi \in F$. Let us choose [6], [7] the function $\varphi = \psi$ satisfying the equation adjoint to (8)

$$\langle \psi', f \rangle - \langle A^* \psi, f \rangle = 0 \quad (23)$$

along with the terminal condition

$$\psi(x, T) = g(x) - f^j(x, T). \quad (24)$$

In other words the function ψ is the weak solution of the following boundary value problem:

$$\frac{\partial \psi(x, t)}{\partial t} + \sum_{i,j=1}^n \frac{\partial}{\partial x_i} \left[a_{ij}(x, t) \frac{\partial \psi(x, t)}{\partial x_j} \right] = 0 \quad \text{in } \Omega \times (0, T), \quad (25)$$

$$\frac{\partial \psi(x, t)}{\partial \eta^*} + q(x, t) \psi(x, t) = 0 \quad \text{in } \Gamma \times (0, T), \quad (26)$$

along with the terminal condition (24).

Here we have

$$\frac{\partial \psi(x, t)}{\partial \eta^*} = \sum_{i,j=1}^n a_{ij}(x, t) \frac{\partial \psi(x, t)}{\partial x_j} \cos(\eta, x_i).$$

Substituting (23) and (24) to (22) we get

$$\int_{\Omega} [g(x) - f^j(x, T)] f(x, T) dx = \int_{\Omega} f(x, 0) \psi(x, 0) dx + \langle w + z, \psi \rangle. \quad (27)$$

Taking into considerations (6) and (27), we rewrite the condition (17) in the following form

$$\begin{aligned} \int_0^T \int_{\Omega} \bar{v}^{j+1}(x, t) \psi(x, t) dx dt + \int_0^T \int_{\Gamma} \bar{u}^{j+1}(x, t) \psi(x, t) dx dt \geq \\ \geq \int_0^T \int_{\Omega} v(x, t) \psi(x, t) dx dt + \int_0^T \int_{\Omega} u(x, t) \psi(x, t) dx dt, \end{aligned} \quad (28)$$

for all $|v(x, t)| \leq v$, $|u(x, t)| \leq u$.

$\bar{w}^{j+1}(x, t) = [\bar{u}^{j+1}(x, t), \bar{v}^{j+1}(x, t)]$ is here a control corresponding to $\bar{f}^{j+1}(x, t)$.

From (28) we obtain

$$\bar{v}^{j+1}(x, t) = \text{sgn } \psi(x, t) \text{ almost everywhere in } \Omega \times (0, T),$$

$$\bar{u}^{j+1}(x, t) = \text{sgn } \psi(x, t) \text{ almost everywhere in } \Gamma \times (0, T). \quad (29)$$

Note that in the case where the function $\psi(x, t)$ is equal to zero on a set of positive measure the appropriate control on this set can be chosen arbitrarily except that the conditions (14) have to be satisfied.

Having the control $\bar{w}^{j+1}(x, t)$ we find the corresponding functions $\bar{f}^{j+1}(x, t)$.

Further we find $\bar{\alpha}_j$ from (18) and we choose α_j from the interval (19).

Finally according to (20), we have

$$f^{j+1}(x, T) = f^j(x, T) + \alpha^j [f^{j+1}(x, T) - f^j(x, T)], \quad (30)$$

$$u^{j+1}(x, t) = u^j(x, t) + \alpha^j [\bar{u}^{j+1}(x, t) - u^j(x, t)], \quad (30a)$$

$$v^{j+1}(x, t) = v^j(x, t) + \alpha^j [\bar{v}^{j+1}(x, t) - v^j(x, t)]. \quad (30b)$$

The rest of the procedure is repeated in exactly the same way. The estimation (2) can be used to stop the iterative process when the required accuracy is achieved.

4. Example of an application — heat equation

As an example of an application we consider the process of one-sided heating in a one-dimensional heat conduction system [2].

Mathematically this problem can be formulated [2], [5] as follows:

There is given the system described in $(0, 1) \times (0, T)$ by the heat equation

$$\frac{\partial f(x, t)}{\partial t} - \alpha^2 \frac{\partial^2 f(x, t)}{\partial x^2} = 0 \quad (31)$$

along with the initial condition

$$f(x, 0) = f_0(x) \quad (32)$$

and the boundary conditions

$$\frac{\partial f(0, t)}{\partial x} = 0, \quad \frac{\partial f(1, t)}{\partial x} = \beta [u(t) - f(1, t)]. \quad (33)$$

Our aim is to minimize the functional

$$J(u) = \left\{ \int_0^1 [g(x) - f(x, T)]^2 dx \right\}^{1/2} \quad (34)$$

under the assumption that the admissible controls are all measurable functions u satisfying the condition

$$|u(t)| \leq 1 \text{ for almost all } t \in [0, T], \quad (35)$$

where: T is a given time, $f(x, T)$ is the value of the weak solution of (31)—(33) at T , $g(x) \in L^2[0, 1]$ is a given function, α^2 is the thermal diffusivity coefficient of the heated material, $\beta > 0$ is the heat transfer coefficient between the heated material and the environment.

In the sequel we shall assume that

$$\alpha^2 = 1, \quad \beta = 1, \quad f_0(x) \equiv 0.$$

The above problem is a particular case of the problem considered in Section 2 and to solve it we shall apply the method presented in Section 3.

We are interested in the weak solution of (31)—(33) i.e. we are looking for the function $f(x, t)$, which satisfies the equation

$$\int_0^1 \frac{\partial f(x, t)}{\partial t} \varphi(x) dx + \int_0^1 \frac{\partial f(x, t)}{\partial x} \frac{\partial \varphi(x)}{\partial x} dx + [u(t) - f(1, t)] \varphi(1) = 0 \quad (36)$$

for every $\varphi \in H^1(0, 1)$.

We shall find the approximation of the solution of (36) discretizing the function $f(x, t)$ in the domain $[0, 1] \times [0, T]$. We are looking only for discrete values of the function $f(x, t)$ as well as $u(t)$.

To this purpose we introduce in $[0, 1] \times [0, T]$ a mesh dividing the domain of x and t into M and N equal intervals $\Delta x = 1/M$ and $\Delta t = T/N$ respectively.

Let us denote

$$u(n \cdot \Delta t) = u_n, \quad (37a)$$

$$f(m \cdot \Delta x, n \cdot \Delta t) = f_n^m, \quad (37b)$$

$$\frac{\partial f(m \cdot \Delta x, n \cdot \Delta t)}{\partial t} = \frac{f_n^m - f_{n-1}^m}{\Delta t} \quad (37c)$$

$$\frac{\partial f(m \cdot \Delta x, n \cdot \Delta t)}{\partial x} = \frac{f_n^m - f_n^{m-1}}{\Delta x}. \quad (37d)$$

Using notations (37) we rewrite (36) in the following discrete form

$$\sum_{m=1}^M \frac{f_n^m - f_{n-1}^m}{\Delta t} \varphi_n^m \cdot \Delta x + \sum_{m=1}^M \frac{f_n^m - f_n^{m-1}}{\Delta x} \frac{\varphi_n^m - \varphi_n^{m-1}}{\Delta x} \Delta x - (u_n - f_n^M) \varphi_n^M = 0. \quad (38)$$

This equation has to be satisfied for any arbitrary numbers φ^m . Putting the coefficients of φ^m equal to zero we obtain respectively

for $m=0$

$$\frac{f_n^1 - f_n^0}{\Delta x} = 0, \text{ hence } f^0 = f^1, \quad (39a)$$

for $0 < m < M$

$$(f_n^m - f_{n-1}^m) \frac{\Delta x}{\Delta t} + (f_n^m - f_n^{m-1}) \frac{1}{\Delta x} - (f_n^{m+1} - f_n^m) \frac{1}{\Delta x} = 0, \quad (39b)$$

for $m=M$

$$(f_n^M - f_{n-1}^M) \frac{\Delta x}{\Delta t} + (f_n^M - f_n^{M-1}) \frac{1}{\Delta x} - (u_n - f_n^M) = 0. \quad (39c)$$

Denoting

$$\kappa = \frac{\Delta t}{(\Delta x)^2}, \quad f_n = \begin{bmatrix} f_n^1 \\ f_n^2 \\ \vdots \\ f_n^M \end{bmatrix}, \quad k = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ \kappa \cdot \Delta x \end{bmatrix} \quad (40)$$

we rewrite (39) in the form of the matrix equation

$$Gf_n = f_{n-1} + ku_n, \quad (41)$$

where the M -dimensional matrix G is given by

$$G = \begin{bmatrix} 1+\kappa, & -\kappa, & 0, & 0, & \dots, & 0, & 0, & 0 \\ -\kappa, & 1+2\kappa, & -\kappa, & 0, & \dots, & 0, & 0, & 0 \\ 0, & -\kappa, & 1+2\kappa, & -\kappa, & \dots, & 0, & 0, & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0, & 0, & 0, & 0, & \dots, & -\kappa, & 1+2\kappa, & -\kappa \\ 0, & 0, & 0, & 0, & \dots, & 0, & -\kappa, & 1+\kappa(1+\Delta x) \end{bmatrix} \quad (41a)$$

The equation (41) can be rearranged as follows

$$f_n = Gf_{n-1} + G^{-1}ku_n. \quad (42)$$

It is shown in Appendix that the matrix G^{-1} is stable. Taking into consideration that $f_0(x)=0$ we obtain from (42)

$$f_N = \sum_{n=1}^N G^{-(N-n+1)} ku_n. \quad (43)$$

Now let us change over to the iterative procedure of solving the problem of optimization.

Let us assume that using this procedure we have already found the function $f^j(x, T)$ (i.e. the corresponding vector f_N^j of discrete values) and the control function $u^j(x, T)$, u_n^j , $n=1, 2, \dots, N$, and we shall find f_N^{j+1} and u_n^{j+1} , $n=1, 2, \dots, N$.

To this purpose we find \bar{f}_N^{j+1} . For the discrete case the condition (17) takes on the form

$$(g - f_N^j, \bar{f}_N^{j+1}) \geq (g - f_N^j, f_N)$$

where (\cdot, \cdot) denotes here the inner product in M -dimensional Euclidean space R^M , and g is M -dimensional vector of the values of the function $g(x)$ at the points $m \cdot \Delta x$, $m=1, \dots, M$.

Substituting (43) to (44) and taking into consideration (35) we obtain

$$\bar{u}_n^{j+1} = \text{sgn}(g - f_N^j, G^{-(N-n-1)} k), \quad n=1, 2, \dots, N. \quad (45)$$

The vector \bar{f}_N^{j+1} we find substituting (45) into (43).

Further we find the constant $\bar{\alpha}^j$ from (18) where the appropriate inner products are taken in R^M space.

We choose the constant α^j from the interval (19) and according to (30) we put

$$f_N^{j+1} = f_N^j + \alpha^j (\bar{f}_N^{j+1} - f_N^j), \quad (46)$$

$$u_n^{j+1} = u_n^j + \alpha^j (\bar{u}_n^{j+1} - u_n^j), \quad n=1, 2, \dots, N. \quad (46a)$$

The rest of the procedure is repeated in exactly the same way. The iterative process is stopped if the error of approximating of $J(u_{opt}) = [(g - f_{N,opt}, g - f_{N,opt}) \Delta x]^{1/2}$ is not larger than the given number i.e. according to (21) if

$$\delta^j = [(g - f_N^j, g - f_N^j) \Delta x]^{1/2} - \max \left\{ 0, \frac{(g - \bar{f}_N^{j+1}, g - f_N^j) \Delta x^{1/2}}{(g - f_N^j, g - f_N^j)^{1/2}} \right\} \leq \varepsilon. \quad (47)$$

5. Numerical results

In numerical computations the following data were taken $T=1$, $M=20$, and $N=100$.

Moreover it was assumed that the function $g(x)$ is constant: $g(x) \equiv \text{const}$.

Table 1. The results of computations for $g(x) \equiv 0.5$

Number of iteration	Value of the functional $J(u^j)$	Estimation of the error δ^j
1	0.50000	0.50000
2	0.05230	0.05230
3	0.04552	0.04552
4	0.03934	0.03944
5	0.03047	0.02157
6	0.02420	0.01491
7	0.02359	0.02359
8	0.02316	0.01119
9	0.02080	0.02080
10	0.02005	0.01206
11	0.01993	0.00578
12	0.01857	0.01867
13	0.01815	0.01050
14	0.01804	0.00380

The computations were performed for $g(x) \equiv 0.6$ and $g(x) \equiv 0.5$. In both case $f^0(x, T) \equiv 0$ ($u^0(t) \equiv 0$) was taken as the initial state.

For the successive iteration the values of $\bar{u}_n^j, u_n^j, f_N^j, f_N^j$ were computed and printed.

Moreover for each iteration the values of the functional and the estimation δ^j of the error were computed and printed. The iterative procedure was stopped when the value δ^j was less than the given constant.

Table 2. The results of computations for $g(x) \equiv 0.6$

Number of iteration	Value of the functional $J(u^j)$	Estimation of error δ^j
1	0.60000	0.60000
2	0.08607	0.00745
3	0.08559	0.00112
4	0.08554	0.00031

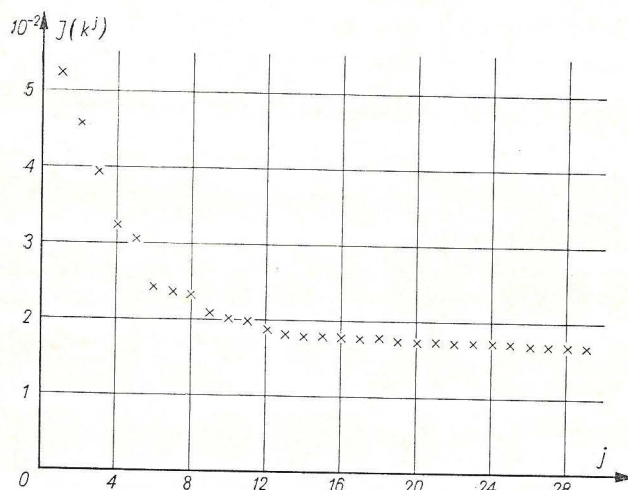


Fig. 1. The value of functional $J(u^j)$ v.s. the number of iterations j for $g(x) \equiv 0.5$

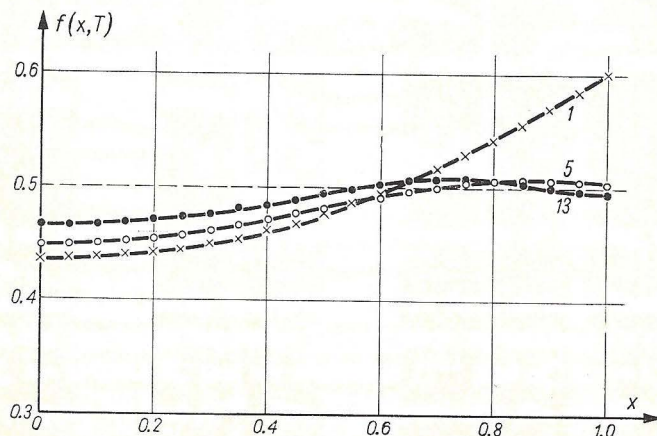


Fig. 2. The distribution of temperature for $g(x) \equiv 0.5$

The forms of control functions $u^j(t)$		
Number of iteration j	Switching points t_r	Value of $u^j(t)$ for $t \in [t_{r-1}, t_r]$
1	1.00	0.925
	0.81	0.965
5	0.91	0.874
	0.94	0.733
	1.00	0.456
	0.81	0.995
	0.87	0.983
	0.88	0.954
	0.89	0.910
	0.91	0.890
	0.93	0.871
	0.94	0.850
	0.95	0.871
	0.96	0.347
	0.97	-0.460
	0.98	-0.036
	0.99	0.476
	1.00	0.496

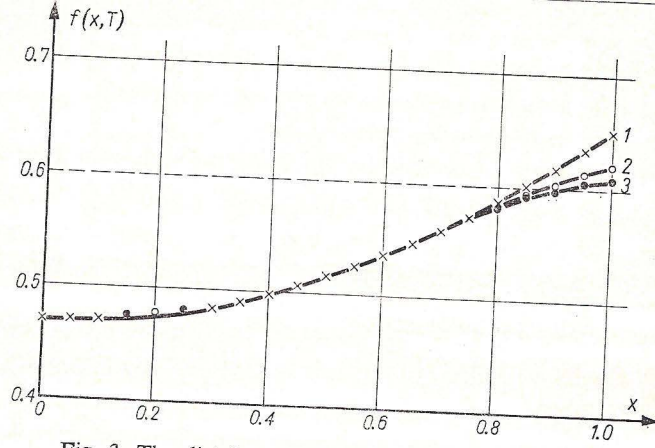


Fig. 3. The distribution of temperature for $g(x) \equiv 0.6$

The forms of control functions $u^j(t)$		
Number of iterations j	Switching points t_r	Value of $u^j(t)$ for $t \in [t_{r-1}, t_r]$
1	1.00	1.000
2	0.98	1.000
3	1.00	0.745
	0.98	1.000
	0.99	0.769
	1.00	0.582

The obtained results are presented in Tables 1 and 2.

Additionally the plot of values of the functional vs. the number of iterations for $g(x) \equiv 0.5$ is presented in Figure 1.

In Figure 2 and 3 the distribution of temperature obtained in some chosen iterations is shown. The shape of the control functions corresponding to these distributions is also given in Figure 2 and 3.

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Appendix

Proof of stability of the matrix G^{-1}

To prove the stability of the matrix G^{-1} we have to show, that the magnitude of all eigenvalues of this matrix is less than one. It is equivalent to the condition that the magnitude of all eigenvalues of the matrix G is larger than one.

The matrix G can be written in the form

$$G = I + \kappa P \quad (\text{A.1})$$

where

$$P = \begin{bmatrix} 1, & -1, & 0, & 0, & \cdot, & 0, & 0, & 0 \\ -1, & 2, & -1, & 0, & \cdot, & 0, & 0, & 0 \\ 0, & -1, & 2, & -1, & \cdot, & 0, & 0, & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0, & 0, & 0, & 0, & \cdot, & 2, & -1, & 0 \\ 0, & 0, & 0, & 0, & \cdot, & -1, & 2, & -1 \\ 0, & 0, & 0, & 0, & \cdot, & 0, & -1, & 1 + \Delta x \end{bmatrix} \quad (\text{A.2})$$

Let us denote by λ the eigenvalues of the matrix G , and by μ the eigenvalues of P . The following relation takes place

$$\lambda = 1 + \kappa\mu. \quad (\text{A.3})$$

We shall show that

$$\mu > 0. \quad (\text{A.4})$$

The stability of G^{-1} will follow from (A.3) and (A.4). To prove (A.4) we shall show that

$$|P - \mu I| > 0 \quad \text{for } \mu \leq 0. \quad (\text{A.5})$$

Let us denote by K_m , the m -dimensional Jacobi determinant [1]

$$K_m = \begin{bmatrix} 2 - \mu, & -1, & 0, & \dots, & 0, & 0 \\ -1, & 2 - \mu, & -1, & \dots, & 0, & 0 \\ 0, & -1, & 2 - \mu, & \dots, & 0, & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0, & 0, & 0, & \dots, & -1, & 2 - \mu \end{bmatrix} \quad (\text{A.6})$$

Expanding $|P - \mu I|$ successively with respect to the first, second and the last row we obtain

$$|P - \mu I| = [(1 - \mu)(2 - \mu)(1 + \Delta x - \mu) - (2 + \Delta x - 2\mu)] K_{M-3} + \\ - [(1 - \mu)(1 + \Delta x - \mu) - 1] K_{M-4}. \quad (\text{A.7})$$

We shall show that

$$0 < K_m < K_{m+1} \quad \text{for } \mu \leq 0. \quad (\text{A.8})$$

The proof will be performed by induction.

For $m=2, 3$ we have

$$K_2 = (2 - \mu)^2 - 1 > 0, \\ K_3 = (2 - \mu) [(2 - \mu)^2 - 2], \\ K_3 - K_2 = (1 - \mu) [(2 - \mu)^2 - 2] - 1 \geq 1 \quad \text{for } \mu \leq 0.$$

Let us assume that

$$K_{m-1} < K_m \quad (\text{A.9})$$

and we shall prove that (1.8) takes place.

Expanding K_{m+1} with respect to the first row and taking into consideration (A.9) we obtain

$$K_{m+1} = (2 - \mu) K_m - K_{m+1} > (1 - \mu) K_m \geq K_m$$

which proves (A.8).

From (A.7) and (A.8) we get

$$|P - \mu I| > [(1 - \mu)(2 - \mu)(1 + \Delta x - \mu) - (2 + \Delta x - 2\mu) + (1 - \mu)(1 + \Delta x - \mu) + \\ - 1] K_{M-3} = -\mu [(2 - \mu)(1 + \Delta x - \mu) - 1] K_{M-3} \geq 0 \quad \text{for } \mu \leq 0 \text{ q.e.d.}$$

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Zastosowanie pewnej metody programowania kwadratowego do problemów sterowania optymalnego układów opisywanych równaniami parabolicznymi

Rozważono problem sterowania optymalnego układu opisanego liniowym równaniem różniczkowym typu parabolicznego. Dopuszcza się możliwość sterowania brzegowego i sterowania wewnątrz obszaru. Funkcjonał jakości jest kwadratowy. Wykazano, że problem ten sprowadza się do pewnego zadania programowania kwadratowego w przestrzeni Hilberta. Dla rozwiązania tego zadania zaproponowano zastosowanie pewnej metody iteracyjnej. Jako przykład rozważono jednowymiarowe równanie ciepłoprzewodnictwa. Podano wyniki numeryczne uzyskane przy użyciu maszyny cyfrowej.

Применение одного метода квадратического программирования в задачах оптимального управления систем описываемых параболическими уравнениями

Рассматривается задача оптимального управления системой описанной линейным дифференциальным уравнением параболического типа. Допускается возможность краевого управления и управления внутри области. Задача рассматривается для квадратических функционалов качества.

Доказывается что вопрос можно свести к некоторой задаче квадратического программирования в Гильбертовом пространстве. Для решения задачи предлагается применение некоторого итерационного метода. В качестве примера приводится одномерное уравнение теплопроводности.

В статье приведены численные результаты полученные при применении цифровой машины.