Control
and Cybernetics
VOL. 1 (1972) No. 1/2

The optimization of regional service systems

by

STANISŁAW PIASECKI

Department of Operations Research
Military Technical Academy
Warszawa

In the paper the problem of choosing an optimal spatial structure of the service system, e.g. repair system, is considered. The problem consists of selecting service subregions corresponding to individual repair bases and in fixing optimal technology and the number of service staff in separate bases. Some simplifications are introduced to the mathematical model considered. In fact, stationary and homogeneous deterministic model is discussed. An performance function, whose extremum defines an optimal solution of the problem, is determined. The most important constraints are analyzed.

1. Description of the problem

Assume that on a certain territory (for example in a country) we are using N machines distributed among J points (production and service enterprise etc.) which will be marked with the index $j=1, 2, ..., J$. In every point there are N_j machines, so $\sum_{j=1}^{J} N_j=N$. Every fixed period of work θ the machines have to be renovated and, therefore, a net of repair bases has to be organized.

Every operated machine, while it is working, gives a defined income $\mathcal{A}$ in an unit of time. Its purchase price equals C.

Besides, total working time $\mathcal{T}$ of every machine (general resources) during which the machine may work since the time it has been purchased till its complete consumption, is known.

The number of machines and their lay-out do not change, and this is equivalent to the assumption that a new machine substitutes for every consumed and withdrawn machine. Every machine which is due for renovation is to be transported as a whole or in parts to the appropriate base and back. On the given territory the transport net is determined in such a way that we may define the time τ_{ij} and transportation cost k_{ij} of the machine from its working place to the repair base and back.

Besides, on the given territory we may establish the points in which it is possible to organize repair bases. The indices $i=1, 2, ..., I$ will denote these points. To every point there may be assigned the area G_i; every machine which is in the point described to the area G_i is renovated in the repair base i. The decision not to organize the repair base in the point i is tantamount to the determination of such an area G_i to which does not belong any point $j=1, 2, ..., J$.

To be more precise, we shall call the area G_i such a set of numbered points from which machines have to be renovated in the repair base i.

The sign G will denote a full set to which belong all points (in which machines are operated).

The division into areas $\hat{G}$ will be a series of sets $\hat{G}=\langle G_1, G_2, ..., G_I\rangle$ whose properties are that the summation of the sets is a full set and the intersection of any two sets is a void set:

$$\bigcap_{i=1}^{I} G_i = G,$$

$$G_s \cap G_i = \varnothing \qquad s, i=1, 2, ..., I \tag{1}$$

The division when all sets G_i except one are void satisfies the case when all repairs are concentrated in one base.

The case when all sets G_i are non-void is tantamount to full deconcentration of repairs.

If D denotes the set of such indices i whose corresponding sets G_i are void: $D=\{i\colon G_i=\varnothing\}$ then the case of full concentration is satisfied by the set D with $I-1$, elements, and full deconcentration by the void set $D=\varnothing$.

Every repair base may have a certain number of parallelly and independently working repair lines, so called production "strings". Repair lines — their work and organization-may be based on various repair techniques.

Every technique may differ in tools, working place lay-out and capacity in the degree of the mechanization of material handling etc.

We assume that to every repair technique corresponds an optimal work lay-out which secures the elimination of delays and equal distribution of workload among every workplace.

So, the choice of technology is determined by work organization. Every repair technique and connected with it work organization is plotted on PERT graph. Every possible technique will be marked with index $r_i=1, 2, ..., R$, and bigger value r_i will be assigned to the technique securing quicker repair $\vartheta(r_i)$. The production capacity $\mu(r_i)$ of one repair line will be the maximum number of repairs which may be performed with the defined technique r_i in an unit of time.

The production capacity $M(m_i, r_i)$ of i-th repair base will be the maximum number of repairs which may be performed on m_i particular repair lines: $M(m_i, r_i)=m_i\,\mu(r_i)$.

The cost $\beta(r_i)$ of one repair or the cost $\mathcal{B}(r_i)$ of maintaining in a unit of time one repair line at work, its production capacity being fully utilized, is determined for every repair technique.

We assume that the machine being out of work, i.e. being transported or repaired, does not yield income $\mathcal{A}$. Machine stoppages other then above mentioned, such for example as due to shortage of raw materials also reduce the profit measured with the income $\mathcal{A}$ given by the machine in a unit of time.

The problem is to determine such system of repairs which would guarantee the achievement of maximum profit of the exploited set of machines, taking into account their depreciation, transport and renovation.

The solution to the problem should, above all, determine the optimum organization structure of the system (number of repair bases, size of every base, techniques and organization of repair lines in every base), — optimal territorial structure of the system (division into areas $\hat{G}$ and territorial lay-out of repair bases), securing the achievement of maximum profit of the exploited set of machines.

It is to be noted that any service system may be formulated in a similar way, and the description of the problem given here represents only one of the many service systems, namely repair system. Another example is the system of supplies, where warehouses are considered as bases and production enterprises or customers as units served by the system.

Similarly it may betreated a production system whose output is supplied to other production enterprises.

Every of the mentioned examples may involve interesting problems of optimal production or service capacity, concentration, organization and territorial structure etc.

For the sake of more comprehensive explanation we shall continue, however, with the example of service system, namely repair system.

In general the set of operated machines does not to be homogeneous (may include various types of machines) as regards the values of $\mathcal{A}$, C, θ, $\mathcal{T}$, as well as μ, ϑ, β, or $\mathcal{B}$.

The same, repair bases may perform various types of repairs for example: current, medium, and major repairs. In this case for every machine we shall have three different magnitudes ϑ: the time of current, medium and major repair.

Besides none parameters may be of chance type (for example, time between repair θ, repairs time ϑ etc.) and may change in time (for example, the number of utilized machines N_j, the profit of machine $\mathcal{A}$, etc).

It is obvious that the solution to the problem subjected to such general assumptions meets with fundamental computation difficulties. Not to obscure the presentation of the problem with computation difficulties, we shall consider a much simplified problem.

We shall assume that no parameter of the problem is a time function and is not a random value. We shall, therefore, consider a stationary and determined mathematical model of the problem. Besides, we shall assume that the set of machines and for repairs performed is homogeneous.

So, the system performs one type of repairs (for example major repairs — renovations) of one type of machines, and the range of repairs does not increase as the machines age is growing.

The quantitative conclusions drawn from the solution to so simplified problem may be considered as adequate in real situation if in sufficiently long periods the unchangeability of the parameters may be assumed, and if it is possible, to convert a machine of one type into a corresponding number of standard machines, and every kind of repairs into equivalent repairs of standard labour intensity. If the magnitudes are in fact of random character then, considering those magnitudes as determined, the optimal production capacity of the base will be lowered.

As we shall see further, even the solution of so simplified problem is not an easy matter because of the uneven allocation of machines and the irregularity of transport facilities existing on the given territory.

2. Mathematical statement of the problem

On the base of the existing communication net, transport facilities and time of loading and discharging, transport time τ_{ij} of the machine from its workplace j to the base i (including time of return) may be fixed for every two integers i, j.

Next, for every machine belonging to the point j and the base i a cycle of machine's exploitation T_{ij} is established. The exploitation cycle comprises four successively repeating periods:

— working time θ,
— time of transportation to and from the repair base τ_{ij},
— waiting time for repairs in the base i: t_i,
— time of the repair $\vartheta(r_i)$.

So $T_{ij}(r_i, t_i) = \theta + \tau_{ij} + t_i + \vartheta(r_i)$.

Waiting time for the repair t_i equals zero when base production capacity $M(m_i, r_i) = m_i\, \mu(r_i)$ is larger of equals the demand for the repairs at the given time.

We determine the minimum production capacity $M_i^o(r_i)$ so that t_i is equal to zero.

The number of machines $n_i(r_i, t_i)$ which in a unit of time are due for renovation in the base i subject to area G_i being established is

$$n_i(r_i, t_i) = \sum_{j \in G_i} \frac{N_j}{T_{ij}(r_i, t_i)}$$

hence

$$n_i(r_i, t_i)|_{t_i=0} = M_i^o(r_i)$$

or

$$M_i^o(r_i) = \sum_{j \in G_i} \frac{N_j}{\theta + \tau_{ij} + \vartheta(r_i)}. \tag{2}$$

The magnitudes $n_i(r_i, t_i)$ — number of repairs which should be performed in the base i in a unit of time and $M_i^o(r_i)$ — balance production capacity of the base (comparison between requirements and possibilities of repairs) depend on the size of the area G_i. When $M(m_i, r_i) < M_i^o(r_i)$ the value t_i is bigger the zero. This value may be expressed by the equation

$$M(m_i, r_i) = \sum_{j \in G_i} \frac{N_j}{\theta + t_i + \tau_{ij} + \vartheta(r_i)} = 0 \tag{3}$$

as the function m_i, r_i.

When $M(m_i, r_i) \geqslant M_i^o(r_i)$ the value t_i is equal to zero.

If the number of repairs to be performed in a unit of time in the base i on the machines coming from the point j is multiplied by the cost of transport k_{ij}, then total transport costs of the machines from point j to point i in a unit of time will be written: $\dfrac{N_j k_{ij}}{T_{ij}(r_i, t_i)}$.

Summing up the costs of all points belonging to the area G_i we shall get the total transport cost κ_i for the base i

$$\kappa_i(r_i, t_i) = \sum_{j \in G_i} \frac{N_j k_{ij}}{T_{ij}(r_i, t_i)}$$

or

$$\kappa_i(r_i, t_i) = \sum_{j \in G_i} \frac{N_j k_{ij}}{\theta + t_i + \tau_{ij} + \vartheta(r_i)} \tag{4}$$

where t_i is expressed by the equation (3) for $m_i \, \mu(r_i) < M_i^o(r_i)$ or in an opposite case is equal to zero.

The magnitudes k_{ij} for the machine transported from the point j to the base i and back are defined on the base of the existing transport net, dispatch tariffs, costs of loading and discharging.

If the costs of maintaining of one repair line in working conditions $B(r_i)$ in an unit of time are defined, then the cost of maintaining the entire base B with m_i lines in an unit of time is

$$B(r_i, m_i) = m_i B(r_i) + B_0 \tag{5}$$

when, of course, the set G_i is not void.

B_0 denotes here the cost of maintaining in an unit of time those elements of the base which do not depend on the number of repair lines m_i.

As the machine during her lifetime undergoes $(T-\theta)/\theta$ repairs (total resources of the machine T and the resources θ between the repairs are chosen in such a way that the number $(T-\theta)/\theta$ is an integer) and with every repair there is connected a period of idleness which is $\tau_{ij} + t_i + \vartheta(r_i)$ so, the total idleness period due to the reapirs is

$$\frac{\mathcal{T} - \theta}{\theta} [\tau_{ij} + t_i + \vartheta(r_i)]$$

and the operation period which is the sum working time and periods of idleness is

$$\mathcal{T} + \frac{\mathcal{T} - \theta}{\theta} [\tau_{ij} + t_i + \vartheta(r_i)].$$

The income derived from the machine in the time of her operation less depreciation is

$$\mathcal{A}\mathcal{T}-C$$

Hence, the net income which one machine yields in an unit of time is

$$\frac{\mathcal{A}\mathcal{T}-C}{\mathcal{T}+\frac{\mathcal{T}-\theta}{\theta}[\tau_{ij}+t_i+\vartheta(r_i)]}$$

and the summation of the incomes $A_i(r_i, t_i)$, of all machines repaired in the base i in an unit of time is

$$A_i(r_i, t_i)=\sum_{j\in G_i}\frac{N_j(\mathcal{A}\mathcal{T}-C)}{\mathcal{T}+\frac{\mathcal{T}-\theta}{\theta}[\tau_{ij}+t_i+\vartheta(r_i)]}. \tag{6}$$

Finally, the net income Z_i from all machines due for repairs in the base i in a n unit of time, less depreciation, repair and transport costs may be written

$$Z_i=A_i-B_i-\kappa_i$$

or

$$Z_i(G_i, r_i, m_i)=\sum_{j\in G_i}N_j\left\{\frac{\mathcal{A}\mathcal{T}-C}{\mathcal{T}+\frac{\mathcal{T}-\theta}{\theta}[\tau_{ij}+t_i(m_i, r_i)+\vartheta(r_i)]}-\right.$$

$$\left.-\frac{k_{ij}}{\theta+\tau_{ij}+t_i(m_i, r_i)+\vartheta(r_i)}\right\}-m_i\,B(r_i)-B_0$$

if the set G_i is not void.

Of course, if the set G_i is void then $Z_i=0$.

The net income Z received in an unit of time from all machines operated on a given territory, subject to the division into areas G being established is $Z=\sum_{i\in D} Z_i$ or after substituting the values Z_i

$$Z(\hat{G}, \hat{r}, \hat{m})=\sum_{i\notin D}\left[\sum_{j\in G_i}N_j\left\{\frac{\mathcal{A}\mathcal{T}-C}{\mathcal{T}+\frac{\mathcal{T}-\theta}{\theta}[\tau_{ij}+t_i+\vartheta(r_i)]}-\right.\right.$$

$$\left.\left.-\frac{k_{ij}}{\theta+\tau_{ij}+t_i+\vartheta(r_i)}\right\}-m_i\,B(r_i)-B_0\right]$$

where:

$$\hat{r}=\langle r_1, r_2, ..., r_I\rangle,\ r_i=1, 2, ..., R,$$

$$\hat{m}=\langle m_1, m_2, ..., m_I\rangle,\ m_i=0, 1, 2, ...,$$

$$\hat{G}=\langle G_1, G_2, ..., G_I\rangle \text{ satisfies conditions (1)},$$

$$D=\{i\colon G_i=\varnothing\},$$

t_i is expressed by the equation (3) or equals zero if $m_i\,\mu(r_i) \geqslant M_i^o(r_i)$ and $M_i^o(r_i)$ is expressed by the formula (2).

Formally for $i \in D$ we accept $m_i = 0$ and $r_i = 0$.

When m_i are not integers then they may be considered as coefficients of workshops shifts. Similarly the integers $m_i > 1$ may be treated as the numbers of workshops or as coefficients of shifts of one or more workshops.

So, the problem may be mathematically formulated as the determination of the magnitudes $\hat{G}^*$, $\hat{m}^*$, $\hat{r}^*$ for which the function Z is to be maximized, i.e.

$$Z(\hat{G}^*, \hat{m}^*, \hat{r}^*) \geqslant Z(\hat{G}, \hat{m}, \hat{r})$$

for all possible $\hat{G}$, $\hat{m}$, $\hat{r}$ where

$$\hat{G}^* = \langle G_1^*, G_2^*, \ldots, G_I^* \rangle,$$

$$\hat{m}^* = \langle m_1^*, m_2, \ldots, m_I^* \rangle,$$

$$\hat{r}^* = \langle r_1^*, r_2^*, \ldots, r_I^* \rangle.$$

It may be seen that the solution to this problem is not a simple question. Especially if numbers I and J are great, then there are many possible divisions into areas $\hat{G}$, and the choice of the best division $\hat{G}^*$ by the comparison of value Z for various $\hat{G}$ becomes impossible, specially if we consider that for every division $\hat{G}$ also values $\hat{m}^*$ and $\hat{r}^*$ area to be determined.

For solution of the problem work out one program for $I, J \leqslant R$.

Optymalizacja struktury przestrzennej systemu obsługi

Zanalizowano problem wyboru optymalnej struktury przestrzennej systemu obsługi (np. systemu remontowego), tj. wyboru podobszarów obsługi przyporządkowanych poszczególnym bazom remontowym, a także wyboru optymalnej technologii i liczby obsługi w poszczególnych bazach. Wprowadzono szereg uproszczeń do rozważanego modelu matematycznego. Rozważono w zasadzie deterministyczny model stacjonarny i jednorodny. Określono funkcję celu, której ekstremum określa optymalne rozwiązanie zadania, oraz zanalizowano najważniejsze ograniczenia.

Оптимизация районой системы обслуживания

В статье представлена проблема выбора оптимальной пространственной структуры системы обслуживания например системы ремонтных станций, т.е. выбора под-областей обслуживания, приписанных даным ремонтным станциям, а также выьбора оптимальной технологии и численности обслуживающего персонала. Введено ряд упрощений в математическую модель. Главным образом рассмотрена только детерминическая, однородная у стационарная модель. Представлен критерии, экстремум которого является оптимальным решением задачи и розработано более важные ограничения.