and Cybernetics
VOL. 2 (1973) No. 1/2

Extension of Operational Calculus

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Basing on algebraic properties of right invertible operators in arbitrary linear spaces (cf. [1], [2]) a method of solving of equations with scalar coefficients by a decomposition of a rational function into vulgar fractions is given. This method can be applied to differential and difference equations.

1. Introduction

In [1] the author studied algebraic properties of right invertible operators in linear spaces. There is introduced calculus of right inventible operators, for instance, there is proved Taylor Formula for such operators, also there are given definitions of indefinite and definite integrals for such operators. Using these properties, an initial value problem and a mixed boundary value problems for equations with operators being polynomials in right invertible operators (with arbitrary operator coefficients) are solved in an explicit form. There are also given applications to difference equations and hyperbolic equations. This theory is developped in [2]. Indeed, Lectures Notes [2] are prepared for students of the first year of studies in Department of Cybernetics of J. Dąbrowski Military Technical Academy in Warszawa.

In the present paper there is described the method of solving of equations with scalar coefficients. In particular, if we consider the operator $D=d/dt$ in some concrete spaces, we obtain results analogous to Operational Calculus in sense of Heaviside or Mikusiński. However this method is much more general and can be applied in cases, where the classical Operational Calculus is not working. The reason is very simple: we do not use the notion of convolution and Titchmarsh theorem.

2. Algebraic analysis

In this section we shall give fundamental definitions and theorems which will be usefull in our subsequent considerations. All theorems will be given without proofs which can be found either in [1] or in [2].

Let X be a linear space over an algebraically closed field $\mathcal{F}$ of scalars. In sequel we can admit that $\mathcal{F}$ is the field C of complexes. By $L(X)$ we denote the set of all linear (i.e. additive and homogeneous) operators A defined on a linear subset $\mathcal{D}_A$ of X, called the domain of A, and mapping $\mathcal{D}_A$ into X. The set $Z_A=\{x\in\mathcal{D}_A: Ax=0\}$ is called the kernel of an operator $A\in L(X)$.

DEFINITION 2.1. An operator $D\in L(X)$ is said to be right invertible if there is an operator $R\in L(X)$ such that $\mathcal{D}_R=X$, $RX\subset\mathcal{D}_D$ and $DR=I$, where I denotes the identity operator.

The set of all right invertible operators belonging to $L(X)$ will be denoted by $\mathbf{R}(X)$. The operator R is called a right inverse of D. The set of all right inverses of an operator $D\in\mathbf{R}(X)$ will be denoted by $\mathcal{R}_D$. Sometimes we will write also: $\mathcal{R}_D=\{R_\gamma\}_{\gamma\in\Gamma}$. The set $\mathcal{R}_D x=\{R_\gamma x\}_{\gamma\in\Gamma}$, where $x\in X$ is arbitrarily fixed, is called an indefinite integral of x. Each of elements $R_\gamma x$, where $\gamma\in\Gamma$, is called a primitive element for x, because, by definition, $D(R_\gamma x)=DR_\gamma x=x$.

The kernel Z_D of an operator $D\in\mathbf{R}(X)$ is called the space of constants for D and every element $z\in Z_D$ is called a constant. So we have to differ scalars and constants.

DEFINITION 2.2. An operator $A\in L(X)$ is said to be a Volterra operator, if the operator $I-\lambda A$ is invertible for every scalar λ.

The set of all Volterra operators belonging to $L(X)$ will be denoted by $V(X)$.

PROPOSITION 2.1. If $D\in\mathbf{R}(X)$ and $R\in\mathcal{R}_D$, then $D^k R^k=I$ for $k=1, 2, \ldots$.

DEFINITION 2.3. An operator $F\in L(X)$ is said to be an initial operator for $D\in\mathbf{R}(X)$ corresponding to an $R\in\mathcal{R}_D$ if

(i) F is a projection onto Z_D, i.e. $F^2=F$ and $FX=Z_D$;

(ii) $FR=0$ on X.

THEOREM 2.1. Suppose, that $\mathcal{R}_D=\{R_\gamma\}_{\gamma\in\Gamma}$ is a family of right inverses of an operator $D\in\mathbf{R}(X)$. Then this family induces uniquely a family $\mathcal{F}_D=\{F_\gamma\}_{\gamma\in\Gamma}$ of initial operators for D by means of the identity

$$F_\gamma=I-R_\gamma D \quad \text{on } \mathcal{D}_D \quad \text{for all } \gamma\in\Gamma. \tag{2.1}$$

It means that F_γ is an initial operator for D corresponding to R_γ if and only if the identity (2.1) holds on the domain of D.

Theorem 2.1 characterizes initial operators by right inverses. In all applications we are given on the begining an initial operator and we have to determine the corresponding right inverse. This is possible if we use.

THEOREM 2.2. Suppose, we are given the operators $D\in\mathbf{R}(X)$ and $F\in L(X)$ such that $F^2=F$ and $FX=Z_D$. Then F is an initial operator for D corresponding to the right inverse $R=\hat{R}-F\hat{R}$, where R is uniquely determined, independently of the choice of a right inverse $\hat{R}\in\mathcal{R}_D$.

THEOREM 2.3. (Taylor — Gontcharov Formula). Suppose that $D\in\mathbf{R}(X)$. Let $\mathcal{F}_D=\{F_\gamma\}_{\gamma\in\Gamma}$ denote the family of initial operators induced by $\mathcal{R}_D$ and let $\{\gamma_n\}\subset\Gamma$ be an arbitrary sequence of indices. Then for $N=1, 2, \ldots$ the following identity holds on the domain of D^N:

$$I=F_{\gamma_0}+\sum_{k=1}^{N-1} R_{\gamma_0}\ldots R_{\gamma_{k-1}}F_{\gamma_k}D^k+R_{\gamma_0}\ldots R_{\gamma_{N-1}}D^N. \tag{2.2}$$

In particular, if we put $F_{\gamma_n}=F$, $R_{\gamma_n}=R$ $\quad(n=0, 1, \ldots, N-1)$, we obtain a Taylor Formula:

$$I=\sum_{k=0}^{N-1} R^k FD^k+R^N D^N \text{ on } \mathcal{D}_{D^N} \quad (N=1, 2, \ldots). \tag{2.3}$$

Observe, that a difference of two primitive elements for an $x\in X$ is a constant. Indeed, if $z=R_\alpha x-R_\beta x$, where $R_\alpha, R_\beta\in\mathcal{R}_D$, then $z\in Z_D$. This implies, that the operator $I_\alpha^\beta=F_\beta R_\alpha$, where $F_\beta\in\mathcal{F}_D$, $R_\alpha\in\mathcal{R}_D$, $D\in\mathbf{R}(X)$ has the following properties:

(i) For arbitrary $x\in X$ the element $I_\alpha^\beta x$ is a constant;

(ii) $I_\beta^\alpha=-I_\alpha^\beta$ $\quad(\alpha, \beta\in\Gamma)$

(iii) $I_\alpha^\delta+I_\delta^\beta=I_\alpha^\beta$ $\quad(\alpha, \beta, \delta\in\Gamma)$

(iv) If y is a primitive element of an $x\in X$, then $I_\alpha^\beta x=F_\beta y-F_\alpha y$.

These properties permit us to call the operator I_α^β definite integral of an operator $D\in\mathbf{R}(X)$.

THEOREM 2.3. Suppose that $D\in\mathbf{R}(X)$, $R\in\mathcal{R}_D\cap\mathbf{V}(X)$, $Q(D)=\sum_{k=0}^{N} q_k D^k$, where $q_N=1$, $q_0, \ldots, q_{N-1}$ are scalars.

If $\lambda=0$ is not a root of the polynomial $Q(\lambda)$, then

(i) the operator $Q(I, R)=\sum_{k=0}^{N} q_k R^{N-k}$ is invertible;

(ii) $Q(D)\in\mathbf{R}(X)$ and $\hat{R}=R^N[Q(I, R)]^{-1}=[Q(I, R)]^{-1}R^N\in\mathcal{R}_{Q(D)}\cap\mathbf{V}(X)$;

(iii) $Z_{Q(D)}=[Q(I, R)]^{-1}Z_D=\{z\in X: z=[Q(I, R)]^{-1}\sum_{k=0}^{N-1} R^k z_k,\ z_k\in Z_D$ are arbitrary$\}$ and moreover $\dim Z_{Q(D)}=N\dim Z_D$;

(iv) all solutions of the equation

$$Q(D)x=y, \quad y\in X, \tag{2.4}$$

are of the form

$$x=[Q(I, R)]^{-1}\left[R^N y+\sum_{k=0}^{N-1} R^k z_k\right], \tag{2.5}$$

where $z_0, \ldots, z_{N-1}\in Z_D$ are arbitrary.

3. Exponential elements

DEFINITION 3.1. If λ is an eigenvalue of an operator $D\in\mathbf{R}(X)$, then every eigenvector of D corresponding to the value λ is called an exponential element.

THEOREM 4.1. Suppose, that $D \in \mathbf{R}(X)$, $R \in \mathcal{R}_D$ and that the operator $I-\lambda R$ is invertible for a scalar λ. Then

(i) λ is an eigenvalue of the operator D and the eigenspace X_λ of D corresponding to λ is

$X_\lambda = \{e_\lambda(z) : e_\lambda(z) = (I-\lambda R)^{-1} z, \ z \in Z_D \text{ is arbitrary}\}$.

Hence $\dim X_\lambda = \dim Z_D$.

(ii) If $\lambda \neq 0$ and $\dim Z_D \neq 0$, then there exist the exponential elements $e_\lambda(z) \neq 0$.

(iii) If F is an initial operator for D corresponding to R then the exponential elements $e_\lambda(z)$ are determined by their initial values

$$e_\lambda = (I-\lambda R)^{-1} F e_\lambda{}^{1}. \tag{3.1}$$

Proof. (i). By definition, $(I-\lambda R)\, e_\lambda(z) = (I-\lambda R)(I-\lambda R)^{-1} z = z$, where $z \in Z_D$. Hence $e_\lambda = z + \lambda R e_\lambda$, which implies $De_\lambda = Dz + \lambda DRe_\lambda = \lambda e_\lambda$. This proves, that $e_\lambda(z)$ is an eigenvector of D corresponding to the eigenvalue λ, provided $z \in Z_D$. Since, by our assumption, the operator $I-\lambda R$ is invertible, we conclude that $\dim X_\lambda = \dim Z_D$.

(ii) Suppose, that $\lambda \neq 0$ and $\dim Z_D \neq 0$. Since the operator $I-\lambda R$ is invertible, the equality $e_\lambda(z) = (I-\lambda R)^{-1} z = 0$, where $z \in Z_D$ is arbitrary, holds if and only if $z=0$, which contradicts our assumption, that $\dim Z_D \neq 0$.

(iii) Definition 3.1 and the point (i) of our theorem together imply, that $De_\lambda = \lambda e_\lambda$. Hence $Fe_\lambda = (I-RD)\, e_\lambda = (I-\lambda R)\, e_\lambda$ (by definition of the operator F). Since the operator $I-\lambda R$ is invertible, we obtain the required formula (3.1).

COROLLARY 3.1. Suppose, that $D \in \mathbf{R}(X)$ and that there exists $R \in \mathcal{R}_D \cap \mathbf{V}(X)$. Then

(i) Every scalar λ is an eigenvalue of D, i.e. for every λ there exist exponential elements (not all vanishing if $\dim Z_D \neq 0$);

(ii) There exist $x \in X$ such that $Dx = x$, namely $x = e_1(z)$, where $z \in Z_D$.

Proof. (i). The assumption, that the operator $I-\lambda R$ is invertible for every scalar λ and the point (i) of theorem 4.1 together imply, that every scalar λ is an eigenvalue of D. Moreover, to every scalar λ there correspond the eigenvectors $e_\lambda(z) = (I-\lambda R)^{-1} z$, where $z \in Z_D$ is arbitrary. This and the point (ii) of theorem 3.1 together imply that $e_\lambda(z) \neq 0$, provided that $z \neq 0$. Putting $\lambda = 1$ we obtain the point (ii) of our Corollary.

COROLLARY 3.2. Suppose, that $D \in \mathbf{R}(X)$, $\mathcal{R}_D = \{R_\gamma\}_{\gamma \in \Gamma}$ and that $\mathcal{F}_D$ denotes the induced family of initial operators. Then for every $R_\alpha \in \mathbf{V}(X)$, $F_\beta \in \mathcal{F}_D$, $\alpha, \beta \in \Gamma$

$$(F_\beta - F_\alpha)\, e_\lambda(z) = \lambda F_\beta R_\alpha e_\lambda(z) \tag{3.2}$$

for all $z \in Z_D$ and for all scalars λ.

Indeed, definition 3.1, corollary 3.1 and property (iv) of definite integrals (in Section 2) together imply that

$$\lambda F_\beta R_\alpha e_\lambda(z) = F_\beta R_\alpha [\lambda e_\lambda(z)] = F_\beta R_\alpha De_\lambda(z) = (F_\beta - F_\alpha)\, e_\lambda(z).$$

[1]) When it does not lead to any misunderstanding, we shall write briefly: e_λ instead of $e_\lambda(z)$.

Suppose now that X is a linear space over the field $\mathbf{R}$ of reals. Write: $Y = X \oplus iX$ [2]. The field of scalars for the space Y is obviously $\mathbf{C}$. For every $\eta \in Y$ we shall write $\eta = x_1 + ix_2$, $x_1, x_2 \in X$ and $\eta^* = x_1 - ix_2$. We extend the operators $D \in \mathbf{R}(X)$ and $R \in \mathcal{R}_D \cap \mathbf{V}(X)$ by means of the equalities:

$$D\eta = Dx_1 + iDx_2; \qquad R\eta = Rx_1 + iRx_2. \tag{3.3}$$

This definition implies that $D\zeta = 0$ if and only if $\zeta = z_1 + iz_2$, where $z_1, z_2 \in X$ and $Dz_1 = 0$, $Dz_2 = 0$, and moreover $D \in \mathbf{R}(Y)$. Write:

$$\begin{aligned} c_\lambda(\zeta) &= \frac{1}{2}\, [e_{i\lambda}(\zeta) + e_{-i\lambda}(\zeta^*)], \\ s_\lambda(\zeta) &= \frac{1}{2i}\, [e_{i\lambda}(\zeta) - e_{-i\lambda}(\zeta^*)], \qquad \zeta, \zeta^* \in Z_D, \ \lambda \in \mathbf{C}, \end{aligned} \tag{3.4}$$

and $e_\lambda(\zeta) = (I-\lambda R)^{-1} \zeta$, as before.

These definitions immediately imply the following equalities:

$$\begin{aligned} c_\lambda(z_1 + iz_2) &= (I + \lambda^2 R^2)^{-1} (z_1 + \lambda R z_2), \\ s_\lambda(z_1 - iz_2) &= (I + \lambda^2 R^2)^{-1} (\lambda R z_1 + z_2), \end{aligned} \tag{3.5}$$

where $z_1, z_2 \in Z_D \subset X$. Indeed,

$$\begin{aligned} c_\lambda(z_1 + iz_2) &= \tfrac{1}{2} [e_{i\lambda}(z_1 + iz_2) + e_{-i\lambda}(z_1 - iz_2)] = \\ &= \tfrac{1}{2} [(I - i\lambda R)^{-1} (z_1 + iz_2) + (I + i\lambda R)^{-1} (z_1 - iz_2)] = \\ &= \tfrac{1}{2} (I + \lambda^2 R^2)^{-1} (2z_1 + 2i\lambda R i z_2) = (I + \lambda^2 R^2)^{-1} (z_1 + \lambda R z_2). \end{aligned}$$

A similar proof is for the second of the equalities (3.5). In particular, if $\zeta = \zeta^*$, i.e. if $z_2 = 0$, we have

$$c_\lambda(z) = (I + \lambda^2 R^2)^{-1} z; \ s_\lambda(z) = \lambda R (I + \lambda^2 R^2)^{-1} z, \tag{3.6}$$

where $z \in Z_D \subset X$.

Assume now that X is not only a linear space over $\mathbf{R}$ but, a commutative linear ring and that the operator $D \in \mathbf{R}(X)$ satisfies the following equality:

$$D(xy) = xDy + yDx \quad \text{for all } x, y \in \mathcal{D}_D. \tag{3.7}$$

In this case the space $Y = X \oplus iX$ is a commutative linear ring over the field $\mathbf{C}$, if we define the multiplication by means of the equality:

$\eta\eta' = x_1 x_1' - x_2 x_2' + i(x_1 x_2' + x_1' x_2)$, where $\eta = x_1 + ix_2$, $\eta' = x_1' + ix_2'$, $x_1, x_2, x_1', x_2' \in X$. We shall show, that

$$e_{\lambda+\mu}(\zeta) = e_\lambda(\zeta)\, e_\mu(\zeta) \tag{3.8}$$

for all $\lambda, \mu \in \mathbf{C}$, $\zeta = z_1 + iz_2$, $z_1, z_2 \in Z_D \subset X$.

[2]) i.e. Y is a direct sum of X and $iX = \{ix : x \in X\}$.

Indeed, by definition, we have $De_{\lambda+\mu}(\zeta)=(\lambda+\mu)\,e_{\lambda+\mu}(\zeta)$. On the other hand, $D\,[e_\lambda(\zeta)\,e_\mu(\zeta)]=e_\lambda(\zeta)\,De_\mu(\zeta)+e_\mu(\zeta)\,De_\lambda(\zeta)=$

$$=\mu e_\lambda(\zeta)\,e_\mu(\zeta)+\lambda e_\lambda(\zeta)\,e_\mu(\zeta)=(\lambda+\mu)\,e_\lambda(\zeta)\,e_\mu(\zeta),$$

which implies the required formula (3.8).

Hence

$$e_{\mu_1+i\mu_2}=e_{\mu_1}\,e_{i\mu_2}. \tag{3.9}$$

COROLLARY 3.3. If X is a commutative linear ring over the field $\mathbf{R}$, $D\in\mathbf{R}(X)$ and satisfies the condition (3.7), $R\in\mathcal{R}_D\cap\mathbf{V}(X)$, then the operators c_λ and s_λ defined by Formulae (3.4) have the following properties:

(i) For every $z\in Z_D$ and $\lambda\in\mathbf{R}$

$$s_\lambda^2(z)+c_\lambda^2(z)=z.$$

In particular, if e is a unity of the ring X, then

$$s_\lambda^2(e)+c_\lambda^2(e)=e \text{ for every } \lambda\in\mathbf{R}. \tag{3.10}$$

(ii) For every $z\in Z_D$ and $\lambda\in\mathbf{R}$

$$Ds_\lambda(z)=\lambda c_\lambda(z),\ Dc_\lambda(z)=-\lambda s_\lambda(z).$$

Indeed,

$$s_\lambda^2(z)+c_\lambda^2(z)=\frac{1}{4}\,[e_{i\lambda}(z)+e_{-i\lambda}(z)]^2+\frac{1}{4i^2}\,[e_{i\lambda}(z)-e_{-i\lambda}(z)]^2=$$

$$=\frac{1}{4}\,[e_{i\lambda}^2(z)+2e_{i\lambda}(z)\,e_{-i\lambda}(z)+e_{-i\lambda}^2(z)+$$

$$-e_{i\lambda}^2(z)+2e_{i\lambda}(z)e_{-i\lambda}(z)-e_{-i\lambda}^2(z)]=\frac{1}{4}\cdot 4e_{i\lambda}(z)\,e_{-i\lambda}(z)=e_0(z)=z.$$

Putting $z=e$, we obtain formula (3.10). Moreover,

$$Ds_\lambda(z)=\lambda DR(I+\lambda^2R^2)^{-1}z=(I+\lambda^2R^2)^{-1}z=\lambda c_\lambda(z).$$

$$Dc_\lambda(z)=\tfrac{1}{2}\,[De_{i\lambda}(z)+De_{-i\lambda}(z)]=\tfrac{1}{2}\,i\lambda\,[e_{i\lambda}(z)-e_{-i\lambda}(z)]=-\lambda s_\lambda(z).$$

4. Solutions of equations with scalar coefficients

THEOREM 4.1. Suppose, that X is a linear space over the field $\mathbf{C}$ [3], $D\in\mathbf{R}(X)$, $\dim Z_D\neq 0$, $R\in\mathcal{R}_D\cap\mathbf{V}(X)$ and that

$$Q(\lambda)=\sum_{k=0}^{N}q_k\lambda^k=\lambda^M\sum_{j=1}^{n}(\lambda-\lambda_j)^{r_j}, \tag{4.1}$$

where $q_0,\ldots,q_{N-1}\in\mathbf{C}$, $q_N=1$, $\lambda_j\neq 0$, $\lambda_j\neq\lambda_k$ for $j\neq k$, $0\leqslant M<N$, $(j,k=1,2,\ldots,n)$, $r_1+\ldots+r_n=N-M$. Write:

[3]) As it has been mentioned in Section 2, this assumption is not essential.

$$Q(t,s)=t^M\prod_{j=1}^{n}(t-\lambda_j s^{r_j}), \tag{4.2}$$

$$\tilde{Q}(t,s)=\prod_{j=1}^{n}(t-\lambda_j s)^{r_j}=\sum_{k=0}^{N-M}\tilde{q}_k t^k s^{N-M-k}, \tag{4.3}$$

$(\tilde{q}_0,\ldots,\tilde{q}_{N-M-1}\in\mathbf{C},\ \tilde{q}_{N-M}=1)$. Decompose the rational function $[\tilde{Q}(1,s)]^{-1}$ into vulgar fractions. Since $Q(1,s)=\tilde{Q}(1,s)$, we have

$$[Q(1,s)]^{-1}=[\tilde{Q}(1,s)]^{-1}=\sum_{j=1}^{n}\sum_{m=1}^{r_j}d_{jm}(1-\lambda_j s)^{-m} \tag{4.4}$$

where d_{jm} are well-determined scalars $(j=1,\ldots,n;\ m=1,\ldots,r_j)$. Then every solution of the equation

$$Q(D)\,x=y,\qquad y\in X, \tag{4.5}$$

is of the form

$$x=\sum_{j=1}^{n}\sum_{m=1}^{r_j}d_{jm}(I-\lambda_j R)^{-m}\left(R^N y+\sum_{k=0}^{N-1}R^k z_k\right) \tag{4.6}$$

where $z_0,\ldots,z_{N-1}\in Z_D$ are arbitrary and $d_{11},\ldots,d_{nr_n}$ are scalars determined by the decomposition (4.4).

Proof. To begin with, we consider the case $M=0$. Then $\tilde{Q}(t,s)=Q(t,s)$ and the point (i) of theorem 2.4. implies that the operator $\tilde{Q}(I,R)$ is invertible. Thus the decomposition (4.4) and the point (ii) of theorem 2.4 together imply that

$$x=[Q(I,R)]^{-1}\left[R^N y+\sum_{k=0}^{N=1}R^k z_k\right]=$$

$$=\sum_{j=1}^{n}\sum_{m=1}^{r_j}d_{jm}(I-\lambda_j R)^{-m}\left[R^N y+\sum_{k=0}^{N=1}R^k z_k\right],$$

where $z_0,\ldots,z_{N-1}\in Z_D$ are arbitrary.

Suppose now that $M<N$ is a positive integer. According with the notations (4.3) and (4.4), if we put $\tilde{Q}(t)=\tilde{Q}(t,1)$, we can write the operator $Q(D)$ in the form

$$Q(D)=Q(D,I)=D^M\tilde{Q}(D,I)=D^M\tilde{Q}(D)$$

where the polynomial $\tilde{Q}(\lambda)$ has all roots non equal to zero. Hence the equation (4.5) can be rewritten in the form

$$D^M\tilde{Q}(D)\,x=y. \tag{4.7}$$

Put $u=\tilde{Q}(D)\,x$. Then all solutions of the equation $D^M u=y$ are of the form

$$\tilde{Q}(D)\,x=u=R^M y+\sum_{k=0}^{M}R^k z_k,$$

where $\tilde{z}_0, \ldots, \tilde{z}_{M-1} \in Z_D$ are arbitrary. Now we apply the first part of our theorem to the equation

$$\tilde{Q}(D)\, x = R^M y + \sum_{k=0}^{M-1} R^k \tilde{z}_k .$$

We conclude that

$$x = [\tilde{Q}(I, R)]^{-1} \left\{ R^{N-M} \left[R^M y + \sum_{k=0}^{M-1} R^k \tilde{z}_k \right] + \sum_{k=0}^{N-M-1} R^k z_k \right\},$$

where $z_0, \ldots, z_{N-M-1} \in Z_D$ are arbitrary. Write

$$z_m = \tilde{z}_{M+m-N} \quad \text{for } m = N-M, \ldots, N-1. \tag{4.8}$$

Then, according with formulae (4.4) and (4.8)

$$x = [\tilde{Q}(I, R)]^{-1} \left[R^N y + \sum_{k=0}^{M-1} R^{N-M+m} \tilde{z}_m + \sum_{k=0}^{N-M-1} R^k z_k \right] =$$

$$= \sum_{j=1}^{n} \sum_{m=1}^{r_j} d_{jm} (I - \lambda_j R)^{-m} \left[R^N y + \sum_{k=0}^{N-M-1} R^k z_k + \sum_{m=N-M}^{N-1} R^k \tilde{z}_{M+m-N} \right] =$$

$$= \sum_{j=1}^{n} \sum_{m=1}^{r_j} d_{jm} (I - \lambda_j R)^{-m} \left[R^N y + \sum_{k=0}^{N-M-1} R^k z_k + \sum_{k=N-M}^{N-1} R^k z_k \right] =$$

$$= \sum_{j=1}^{n} \sum_{m=1}^{r_j} d_{jm} (I - \lambda_j R)^{-m} \left[R^N + \sum_{k=0}^{N-1} R^k z_k \right],$$

where $z_0, \ldots, z_{N-1} \in Z_D$ are arbitrary. This is, which was to be proved.

We still assume that all assumptions of theorem 4.1 are satisfied. An **initial value problem** for the operator $Q(D)$ is to find all solutions of the equation (4.5) satysfying the **initial conditions**:

$$FD^k x = y_k, \; y_k \in Z_D, \quad k = 0, 1, \ldots, N-1, \tag{4.9}$$

where F is an initial operator for D corresponding to R. This problem is **well-posed** if has a unique solution for every $y \in X$, $y_0, \ldots, y_{N-1} \in Z_D$. By definition, a homogeneous well-posed initial value problem has only zero as a solution.

Corollary 4.1. Suppose, that all assumptions of theorem 4.1 are satisfied. Then the initial value problem (4.5)—(4.9) is well-posed and its unique solution is of the form.

$$x = \sum_{j=1}^{n} \sum_{m=1}^{r_j} d_{jm} (I - \lambda_j R)^{-m} \left[R^N y + \sum_{k=0}^{N-1} \sum_{m=0}^{k} q_{N+m-M-j} R^k y_m \right], \tag{4.10}$$

where the scalars d_{jm} are determined by the decomposition (4.4).

Proof. Formulae (4.3), (4.4) and (4.6) together imply that

$$\sum_{k=0}^{N-M} \tilde{q}_{N-M-k} R^k x = \sum_{k=0}^{N-M} \tilde{q}_k R^{N-M-k} x = \tilde{Q}(I, R)\, x = R^N y + \sum_{k=0}^{N-1} R^k z_k .$$

Acting on both sides of this equality by operators D^j ($j = 0, 1, \ldots, N-1$) we obtain

$$\sum_{k=0}^{N-M} \tilde{q}_{N-M-k} D^j R^k x = D^j R^N y + \sum_{k=0}^{N-1} D^j R^k z_k, \quad j = 0, 1, \ldots, N-1 .$$

Hence we have the equality

$$\sum_{k=0}^{j} \tilde{q}_{N-M-k} D^{j-k} x + \sum_{k=j+1}^{N-M} \tilde{q}_{N-M-k} R^{k-j} x = R^{N-j} y + \sum_{k=0}^{j} D^{j-k} z_k +$$

$$+ \sum_{k=j+1}^{N-j} R^{k-j} z_k = R^{N-j} y + z_j + \sum_{k=j+1}^{N-1} R^{k-j} z_k, \quad j = 0, 1, \ldots, N-1.$$

Since $FR = 0$ and $Fz_j = z_j$ for $j = 0, 1, \ldots, N-1$, acting on both sides of this equality by the operator F and applying the conditions (4.9) we obtain

$$\sum_{m=0}^{j} \tilde{q}_{N+m-M-j}\, y_m = \sum_{k=0}^{j} \tilde{q}_{N-M-k}\, y_{j-k} = \sum_{k=0}^{j} \tilde{q}_{N-M-k} FD^{j-k} x = Fz_j = z_j ,$$

$$j = 0, 1, \ldots, N-1,$$

which was to be proved.

Theorem 4.1 and corollary 4.1 show that it is enough to know all roots of the polynomial $Q(\lambda)$ to determine all solutions of the equation (4.5) and to solve the initial value problem (4.5)—(4.9). The polynomial $Q(\lambda)$ is called the **characteristic polynomial** of the operator $Q(D)$ and its roots are called **characteristic roots** of $Q(D)$. Moreover, by the definition of exponential elements

$$Q(D)\, e_\lambda(z) = \sum_{k=0}^{N} q_k D^k e_\lambda(z) = \sum_{k=0}^{N} q_k \lambda^k e_\lambda(z) = Q(\lambda)\, e_\lambda(z) \tag{4.11}$$

for all $\lambda \in \mathbf{C}$, $z \in Z_D$. Since all coefficients of the polynomial $Q(\lambda)$ are scalars, we conclude that $Q(\lambda) \in \mathbf{C}$ for $\lambda \in \mathbf{C}$.

Formula (4.11) implies that $e_\lambda(z)$ is an eigenvector of the operator $Q(D)$ corresponding to the eigenvalue $Q(\lambda)$. If we put $\lambda = \mu$, where either $\mu = \lambda_j$ for a $j = 1, \ldots, n$, or $\mu = 0$ (only in the case $M \geqslant 1$), we have $Q(\mu) = 0$ and $Q(D)\, e_\mu(z) = Q(\mu)\, e_\mu(z) = 0$ for every $z \in Z_D$. Hence $e_\mu(z)$ are solutions of the equation $Q(D)\, x = 0$. We shall consider now the case, when all coefficients of the operator $Q(D)$ are **real**, i.e. $q_0, \ldots, q_{N-1} \in \mathbf{R}$. The results obtained remain true, since $\mathbf{R} \subset \mathbf{C}$, but may happen that solutions have complex coefficients. Having these solutions we will try to construct solutions with real coefficients.

If all roots of the characteristic polynomial $Q(\lambda)$ are **real**, then we do not change the method of solving of the equation (4.5) and the corresponding homogeneous equation, because all solutions obtained have real coefficients.

Consider now the case, when the coefficients of the characteristic polynomial are **complex**. It is well-know, that every polynomial with real coefficients has complex roots pairwise conjugate. We shall apply this property. Examine two cases:

(A) The characteristic polynomial $Q(\lambda)$ has 2 imaginary roots: $i\mu$ and $-i\mu$ (where $\mu \neq 0$ is a real). In the same way, as in Section 3, we consider the space $Y = X \oplus iX$ and the operators c_μ and s_μ defined by Formulae (3.4). Then to two linearly independent solutions $e_{i\mu}(\zeta)$, $e_{-i\mu}(\zeta^*)$ in the space Y, where $\zeta = z_1 + iz_2$, $\zeta^* = z_1 - iz_2$, $z_1, z_2 \in Z_D$ in X, there correspond two linearly independent solutions $c_\mu(z_1)$ and $s_\mu(z_2)$ in the space X.

(B) The characteristic polynomial $Q(\lambda)$ has two complex conjugate roots: $\lambda = \mu_1 + i\mu_2$, $\bar{\lambda} = \mu_1 - i\mu_2$, where $\mu_1^2 + \mu_2^2 > 0$. This case cannot be solved in the space X over reals without additional assumptions. Namely, assume that X is a commutative linear ring and that $D \in \mathbf{R}(X)$ satisfies the condition (3.7). In this case, using formula (3.9) we conclude that to two linearly independent solutions $e_{\mu_1 + i\mu_2}(\zeta)$, $e_{\mu_1 - i\mu_2}(\zeta^*)$ of the equation (4.2) in the linear ring $Y = X \oplus iX$, where $\zeta = z_1 + iz_2$, $\zeta^* = z_1 - iz_2$, $z_1, z_2 \in Z_D$ in X, there correspond two linearly independent solutions in the space X obtained in the same way, as in the point (A):

$$e_{\mu_1}(z_1)\, c_{\mu_2}(z_1) \text{ and } e_{\mu_1}(z_2)\, s_{\mu_2}(z_2),$$

where c_μ and s_μ are defined by formulae (3.4).

References

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Обобщение операционного исчисления

Используя алгебраические свойства правостронных обратимых операторов в прощвольных линейных пространстранствах [1, 2] подан метод решения управлений со скаларными коэффициентами применяющий декомпозицию рациональной функции на элементарные дроби. Представленный метод может быть использован для случая дифференциальных и разностных уравнений.

Uogólnienie rachunku operatorowego

Wykorzystując własności algebraiczne operatorów prawostronnie odwracalnych w dowolnych przestrzeniach liniowych [1, 2] podano metodę rozwiązywania równań o współczynnikach skalarnych przez rozkład pewnej funkcji wymiernej na ułamki proste. Przedstawiona metoda może być stosowana do równań różniczkowych i różnicowych.