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Aggregation in dynamic PERT systems

by

LECH KRUŚ

Polish Academy of Sciences
Institute of Applied Cybernetics

Systems of PERT type consisting of dynamic operations are considered. It is assumed that each operation can be described by means of Optimal Performance Characteristic (OPC) in the power form. The problem consists in determining OPC of an aggregated system and evaluating optimal values of individual operations. A theorem making possible to reduce the structure of every network to the series-parallel form is given. Using this theorem an advantage can be taken of solutions obtained for series-parallel networks. It is shown that OPC of an aggregated system has also the power form. Considerations are illustrated with an example of optimal control of mass shift under some constraints imposed upon the trajectory.

1. Introduction

There is an important class of optimum control problems, where complexes consisting of dynamic, independent operations are considered. The operations are related by common resources and common output gain expressed by the performance measure. Moreover, they should be performed in the determined order what can be usefuly described by a directed graph. Arcs of the graph mean operations, nodes — events of start or completion of single operations. An example of such a network is shown in Fig. 1.

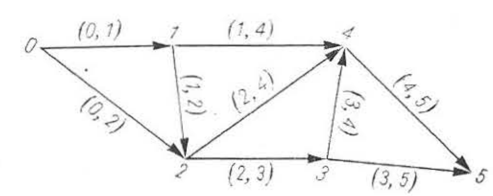

Fig. 1. The network of operations creating a PERT system

The optimum control by aggregation of the complexes was developed in papers by Kulikowski [1—5]. The conception so called Optimum Performance Characteristics (O.P.C.) was introduced there. The OPC is the expression representing the values of performance under optimum control, as a function of process parameters such as interval time, energy cost etc. The O.P.C. make it possible to reduce the dynamic problem to a static form.

There exists a large class of processes where dynamic independent operation can be described by the O.P.C. in the form as follows

$$A_{ij}=\frac{k_{ij}}{B_{ij}^{\beta}\,C_{ij}^{\gamma}\,...\,Z_{ij}^{\omega}}. \tag{1}$$

The network events are successively denoted by integers 0, 1, ..., n. A pair (i,j) denotes the operation that is realized from event i to event j. All parameters in (1) should be understood towards the operation (i,j):

A_{ij}, B_{ij} are minimized quantities, $\beta, \gamma, ... > 0$;

Z_{ij}, Y_{ij} are maximized quantities, $\omega, \psi, ... < 0$;

k_{ij} is a process performance index (P.P.I.).

The paper deals with a complex consisting operations described by the O.P.C. in the form (1). The way of finding both the O.P.C. of an aggregated system and optimum parameters of individual operations is presented. The introduced decomposition theorem makes it possible to reduce the optimization problem in complicated network to the problem of series-parallel networks structure. Then the solutions obtained for series and parallel connections of operations which are presented in [1] can be utilized. It is proved that the O.P.C. of the whole system is of the form (1). The P.P.I. value is calculated numerically.

2. Optimum Performance Characteristic of aggregated system

It is assumed that events are ordered in the network. Each operation is described by the O.P.C. in the form (1). The set of all operations (i,j) which belong to the network is denoted by S. It is useful to divide the set S into two subsets:

(i) S_u is the set of operations generating the ordering path:

$$(i,j)\in S_u \Leftrightarrow (i,j)\in S \cap i+1=j,$$

(ii) S_p is the set of the other operations:

$$(i,j)\in S_p \Leftrightarrow (i,j)\in S \cap i<j-1.$$

The optimization problem consist in finding such parameters of individual operations $\{B_{ij}, C_{ij}, ..., Z_{ij} \in i,j\}: (i,j)\in S$ that minimize the performance measure of the whole complex, subject to the constraints that are required by the network structure.

$$\min_{B_{ij}, C_{ij}, ..., Z_{ij}\in\Omega}\left\{A=\sum_{(i,j)\in S} A_{ij}=\sum_{(i,j)\in S}\frac{k_{ij}}{B_{ij}^{\beta}\,C_{ij}^{\gamma}\,...\,Z_{ij}^{\omega}}\right\} \tag{2}$$

where

$$\Omega=\Big\{B_{ij}, C_{ij}, ..., Z_{ij}: \sum_{l=0}^{n-1} B_{l,l+1}=B,\ \sum_{l=0}^{n-1} C_{l,l+1}=C,\ ...,\ \sum_{l=0}^{n-1} Z_{l,l+1}=Z;$$

$$\sum_{l=r}^{s-1} B_{l,l+1}=B_{rs},\ \sum_{l=r}^{s-1} C_{l,l+1}=C_{rs},\ ...\ \sum_{l=r}^{s-1} Z_{l,l+1}=Z_{rs} \quad \text{for } (r,s)\in S_p;$$

$$B_{ij}, C_{ij}, ...>0,\ Z_{ij}, ...\geqslant 0\Big\}. \tag{3}$$

The parameters of the whole complex are denoted by $A, B, ..., Z$.

Now we present the solutions of problems (2), (3) for series and parallel operation connections. They were obtained in [2]. Next, the decomposition theorem will be introduce. The main idea of the paper consist in applying that theorem in such a way, that complicated network could be replaced by series-parallel connections of operations.

Then, the solution (4), (5), (6) could be applied.

The series connection (Fig. 2)

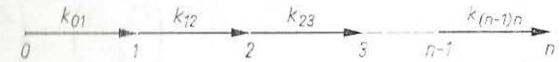

Fig. 2. Series connection of operations

$$\left\{\min_{B_{l,l+1}, C_{l,l+1}, ..., Z_{l,l+1}\in\Omega_1}\left[\sum_{l=0}^{n-1}\frac{k_{l,l+1}}{B_{l,l+1}^{\beta}\,C_{l,l+1}^{\gamma}\,...\,Z_{l,l+1}^{\omega}}\right]\right\}=\frac{k}{B^{\beta}C^{\gamma}\,...\,Z^{\omega}}, \tag{4}$$

where

$$\Omega_1=\Big\{B_{l,l+1}, C_{l,l+1}, ..., Z_{l,l+1}: \sum_{0}^{n-1} B_{l,l+1}=B,\ \sum_{0}^{n-1} C_{l,l+1}=$$

$$=C,\ ...,\ \sum_{0}^{n-1} Z_{l,l+1}=Z;\ B_{l,l+1}, C_{l,l+1}, ...>0;\ Z_{l,l+1}, ...\geqslant 0\Big\},$$

$$k=\left[\sum_{l=0}^{n-1}(k_{l,l+1})^{1/q}\right]^{q}, \quad q=1+\beta+\gamma+...+\omega.$$

The optimum values of parameters are:

$$\frac{B_{l,l+1}^{*}}{B}=\frac{C_{l,l+1}^{*}}{C}=...=\frac{Z_{l,l+1}^{*}}{Z}=\left[\frac{k_{l,l+1}}{k}\right]^{1/q}. \tag{5}$$

The parallel connection (Fig. 3)

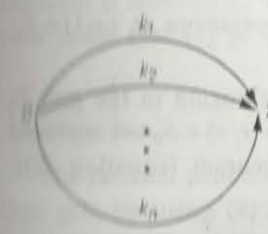

Fig. 3. Parallel connection of operations

$$\left\{\begin{matrix}\displaystyle\sum_{l=1}^{n}\frac{k_l}{B_l^{\beta}\,C_l^{\gamma}\,...\,Z_l^{\omega}}\\ \text{subject to } B_l=B,\\ C_l=C, ..., Z_l=Z\end{matrix}\right\}=\left[\sum_{l=1} k_l\right]\frac{1}{B^{\beta}\,C^{\gamma}\,...\,Z^{\omega}}. \tag{6}$$

Decomposition theorem

Let us assume that the operation described by the O.P.C. (1) is given. (It means, the values $A, B, C, \ldots, Z; \beta, \gamma, \ldots, \omega, k$ are known). Let the positive, real parameters $B_{i,i+1}, C_{i,i+1}, \ldots, Z_{i,i+1}$, $i=0, 1, \ldots, n-1$, satisfy

$$\sum_{i=0}^{n-1} B_{i,i+1}=B, \quad \sum_{i=0}^{n-1} C_{i,i+1}=C, \ldots, \sum_{i=0}^{n-1} Z_{i,i+1}=Z, \tag{7}$$

then

$$\min_{l_{i,i+1}\in W_n} \left\{\sum_{i=0}^{n-1} \frac{l_{i,i+1}}{B_{i,i+1}^{\beta}\, C_{i,i+1}^{\gamma} \ldots Z_{i,i+1}^{\omega}}\right\}=\frac{k}{B^{\beta}\, C^{\gamma} \ldots Z^{\omega}}, \tag{8}$$

where

$$W_n=\left\{l_{i,i+1}:\geqslant 0, \sum_{i=0}^{n-1}\left[\frac{l_{i,i+1}}{C_{i,i+1}^{\gamma} \ldots Z_{i,i+1}^{\omega}}\right]^{\frac{1}{\beta+1}}=\left[\frac{k}{C^{\gamma} \ldots Z^{\omega}}\right]\right\}. \tag{9}$$

The optimum values of $l_{i,i+1}^{*}$, $i=0, 1, \ldots, n-1$, are:

$$l_{i,i+1}^{*}=\left(\frac{B_{i,i+1}}{B}\right)^{\beta+1}\left(\frac{C_{i,i+1}}{C}\right)^{\gamma} \ldots \left(\frac{Z_{i,i+1}}{Z}\right)^{\omega} k. \tag{10}$$

The proof is presented in [6]. At first the theorem have been proved for $n=2$ (i.e. the operation division into two component operations). Then the induction method was utilized in order to show that the theorem was true for integer $n>2$.

The theorem permits to replace one operation by the optimization problem of series connection of operations. Moreover, the parameters of the component operations can take any positive, real values, which satisfy conditions (7).

Let us consider the problem (2). Each operation $(r, s) \in S_p$ will be divided into $(s-r)$ suboperations by using the decomposition theorem. The parameters of suboperations created by division of operation (r, s) are denoted by $l_{(r,s)i}, B_{(rs)i}, C_{(rs)i}, \ldots, Z_{(rs)i}$. The index i means, that the considered suboperation is parallel to the operation $(i, i+1) \in S_u$. The above parallelism is satisfied when the parameters of the suboperations take values as follows:

$$B_{(rs)i}=B_{i,i+1}, \quad C_{(rs)i}=C_{i,i+1}, \ldots, Z_{(rs)i}=Z_{i,i+1},$$

where $B_{i,i+1}, C_{i,i+1}, \ldots, Z_{i,i+1}$ are the parameters of the operations $(i, i+1) \in S_u$, $i=r, r+1, \ldots, s$.

Below, the problem (2) is rewritten but the operations belonging to the sets S_u and S_p are treated separately. The functions of the operations $(r, s) \in S_p$ are replaced by "minimizing problems" according to decomposition theorem (equation (8)). Sums

$$\sum_{i=r}^{s-1} B_{i,i+1}, \quad \sum_{i=r}^{s-1} C_{i,i+1}, \ldots, \sum_{i=r}^{s-1} Z_{i,i+1}; \quad (r, s) \in S_p$$

are put in places of the parameters: $B_{rs}, C_{rs}, \ldots, Z_{rs}$.

$$\min_{B_{i,i+1},\ldots,Z_{i,i+1}\in\Omega}\left[\sum_{i=0}^{n-1}\frac{k_{i,i+1}}{B_{i,i+1}^{\beta} \ldots Z_{i,i+1}^{\omega}}+\sum_{(r,s)\in S_p}\frac{k_{rs}}{B_{rs}^{\beta} \ldots Z_{rs}^{\omega}}\right]=$$

$$=\min_{B_{i,i+1},\ldots,Z_{i,i+1}\in\Omega_u}\left\{\sum_{i=1}^{n-1}\frac{k_{i,i+1}}{B_{i,i+1}^{\beta} \ldots Z_{i,i+1}^{\omega}}+\right.$$

$$\left.+\sum_{(r,s)\in S_p}\left[\min_{l(r,s)i\in V_{rs}}\sum_{i=r}^{s-1}\frac{l_{(is)i}}{B_{i,i+1}^{\beta} \ldots Z_{i,i+1}^{\omega}}\right]\right\}=$$

$$=\min_{B_{i,i+1},\ldots,Z_{i,i+1}\in\Omega_u}\left\{\min_{l(rs)i\in V_{rs}}\left[\sum_{i=0}^{n-1}\frac{k_{i,i+1}}{B_{i,i+1}^{\beta} \ldots Z_{i,i+1}^{\omega}}+\right.\right.$$

$$\left.\left.+\sum_{r,s\in S_p}\sum_{i=r}^{s-1}\frac{l_{(rs)i}}{B_{i,i+1}^{\beta} \ldots Z_{i,i+1}^{\omega}}\right]\right\},$$

where

$$\Omega_u=\left\{B_{i,i+1}, \ldots, Z_{i,i+1}: \sum_{i=1}^{n-1} B_{i,i+1}=B, \ldots, \sum_{i=1}^{n-1} Z_{i,i+1}=Z;\right.$$

$$\left.B_{i,i+1}, C_{i,i+1}, \ldots>0; Z_{i,i+1}, \ldots\geqslant 0\right\}.$$

$$V_{rs}=\left\{l_{(rs)i}:\geqslant 0, \sum_{i=r}^{s-1}\left[\frac{l_{(rs)i}}{C_{i,i+1}^{\gamma} \ldots Z_{i,i+1}^{\omega}}\right]^{\frac{1}{\beta+1}}=\left[\frac{k_{rs}}{C_{rs}^{\gamma} \ldots Z_{rs}^{\omega}}\right]^{\frac{1}{\beta+1}}\right\}.$$

The set Ω is not closed because the variables $B_{i,i+1}, C_{i,i+1}, \ldots$ can take only positive, real values. However the minimized function $f(\boldsymbol{B}, \boldsymbol{C}, \ldots, \boldsymbol{Z})\rightarrow\infty$. When any of the variables $B_{i,i+1}\rightarrow 0$, $C_{i,i+1}\rightarrow 0, \ldots$ (the vector $\boldsymbol{B}=[B_{0,1}, B_{1,2}, \ldots, B_{n-1,n}]$; similarly $\boldsymbol{C}, \ldots, \boldsymbol{Z}$). That fact permits to minimize the function f in closed, limited set

$$\Omega_2=\left\{B_{i,i+1}, C_{i,i+1}, \ldots, Z_{i,i+1}: \sum_{i=0}^{n-1} B_{i,i+1}=B, \ldots, \sum_{i=0}^{n-1} Z_{i,i+1}=\right.$$

$$\left.Z; B_{i,i+1}\geqslant b, C_{i,i+1}\geqslant c, \ldots, Z_{i,i+1}\geqslant z, \ldots\geqslant 0, i=0, 1, \ldots, n-1\right\},$$

where constant values $b, c, \ldots$ are small enough, positive, real numbers.

The variables $l_{(rs)i}$ belong to the closed and limited sets V_{rs}. The minimized function is continuous inside Ω_2 and V_{rs}. Then it is possible to change the minimization order (see [3] p. 203).

$$\min_{l(rs)i\in V_{rs}}\left\{\min_{B_{i,i+1},\ldots,Z_{i,i+1}\in\Omega_2}\left[\sum_{i=0}^{n-1}\frac{k_{i,i+1}}{B_{i,i+1}^{\beta} \ldots Z_{i,i+1}^{\omega}}+\sum_{(rs)\in S_p}\sum_{i=r}^{s-1}\frac{l_{(rs)i}}{B_{i,i+1}^{\beta} \ldots Z_{i,i+1}^{\omega}}\right]\right\}.$$

The internal problem concerns the series-parallel connections of the operations. One gets by using (4) and (5)

$$\min_{l(rs)i\in V_{rs}}\left\{\left[\sum_{i=0}^{n-1}\left(k_{i,i+1}+\sum_{(rs)\in S_i} l_{(rs)i}\right)^{1/q}\right]^{q}\frac{1}{B^{\beta}\, C^{\gamma} \ldots Z^{\omega}}\right\}, \tag{11}$$

where S_i is the set of operations $(r, s) \in S_p$ that are realized simultaneously with the operation $(i, i+1)$, i.e.

$$(r, s) \in S_i \Leftrightarrow (r, s) \in S_p \cap r \leqslant i \cap s \geqslant (i+1).$$

When the optimum values of $B_{i,i+1}, \ldots, Z_{i,i+1}$ are introduced to the conditions which define V_{rs} (eg. (11)), one gets the sets V_{rs} to be independent of the values $B, C, \ldots, Z$:

$$V_{rs}=\left\{l_{(rs),i}: \geqslant 0, \sum_{i=r}^{s-1}\left[\left(\frac{l_{(rs)i}}{K_{i,i+1}}\right)^{\frac{1}{\beta+1}} K_{i,i+1}^{1/q}\right]=\left(\frac{k_{rs}}{K_{rs}}\right)^{\frac{1}{\beta+1}} K_{rs}^{1/q}, \qquad (12)$$

where $K_{i,i+1}=k_{i,i+1} \sum_{(rs)\in S_i} l_{(rs)i}$,

$$K_{rs}=\left[\sum_{i=r}^{s-1} K_{i,i+1}^{1/q}\right]^q, \quad q=1+\beta+\gamma+\ldots+\omega.$$

Therefore the minimum value of the performance measure of the whole complex becomes

$$A=\frac{k_z}{B^\beta C^\gamma \ldots Z^\omega}, \qquad (13)$$

where

$$k_z=\min_{l(rs)i \in V_{rs}}\left[\sum_{i=1}^{n-1}\left(k_{i,i+1}+\sum_{(rs)\in S_i} l_{(rs)i}\right)^{1/q}\right]^q, \qquad (14)$$

V_{rs} is defined by (12)

The optimum values of parameters are as follows:

$$\frac{B^*_{i,i+1}}{B}=\frac{C^*_{i,i+1}}{C}=\ldots=\frac{Z^*_{i,i+1}}{Z}=\left[\frac{k_{i,i+1}+\sum_{(rs)\in S_i} l_{(rs)i}}{k_z}\right]^{1/q}.$$

The numerical methods can be applied to solve the problem (14). The values k_z and $l_{(rs)i}$ obtained in this way enable to calculate the optimum parameters of single operations. For a given network and values k_{ij}, the problem (14) as independent of the parameters $B, C, \ldots, Z$ should be solved only once. The obtained O.P.C. of the whole complex is of the same form as the O.P.C. of the individual operations.

3. An example

A transport process using three electrical motors which work in three perpendicular directions a, b, c (see Fig. 4) is considered. They should shift a massive body from point 0 to point 3 in the given time interval T, with minimum energy consumption. In addition, the movement trajectory must pass by the edges 1—1 and 2—2. Such a problem can appear when the work of a tool-machine or a crane is optimized. The network of activities that must be done by the electrical motors is shown in Fig. 5.

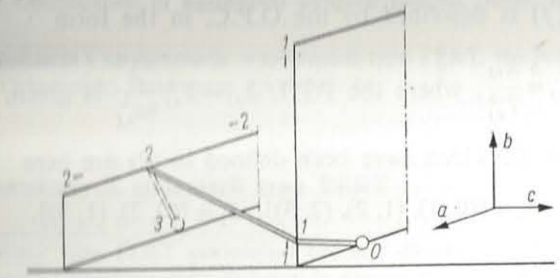

Fig. 4. Movement trajectory of a massive body displacement from point *0* to point *3*

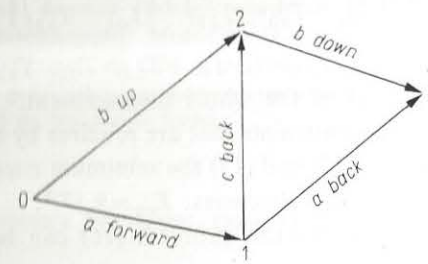

Fig. 5. The network of operations to be done by the motors in order to displace a body along the trajectory of Fig. 4

Let us consider a single operation. One electrical motor should shift a massive body to the given distance Y, in the given interval T_1. The position of the body $y(t)$ can be described by the equality:

$$y(t)=y(0)+a\int_0^t (t-\tau)\, x(\tau)\, d\tau$$

where a is a given coefficient, $x(t)$ is the current in the armature of the motor.

The problem consist in finding such a control current $x(t) \in L^2[0, T]$ which minimizes the energy cost

$$E(x)=\int_0^{T_1} [x(\tau)]^2\, d\tau$$

subject to the constraints

$$y(T_1)-y(0)=Y, \left.\frac{dy(t)}{dt}\right|_{t=T_1}=0, \left.\frac{dy(t)}{dt}\right|_{t=0}=0.$$

It can be shown that

$$x^*(t)=\frac{3Y}{aT^3}\left(\frac{T_1}{2}-t\right) \qquad (15)$$

and minimum energy cost

$$E=\frac{3}{4}\frac{Y^2}{a^2 T_1^3}.$$

The last relation is the O.P.C. of the single operation of a transport process.

Now we can consider again the problem which is shown in Figures 4 and 5. Each operation (i, j) is described by the O.P.C. in the form

$$E_{ij}=\frac{k_{ij}}{T_{ij}^3}, \text{ where the P.P.I. } k_{ij}=Y_{ij}^2\frac{3}{4a_{ij}^2} \text{ is given.}$$

The sets S_u and S_p, which have been defined in (2) are here

$$S_u=\{(0,1),(1,2),(2,3)\},\quad S_p=\{(0,2),(1,3)\}.$$

One looks for

$$\min_{T_{ij}\in\Omega}\left[\left(\frac{k_{01}}{T_{01}^3}+\frac{k_{12}}{T_{12}^3}+\frac{k_{23}}{T_{23}^3}\right)+\left(\frac{k_{02}}{T_{02}^3}+\frac{k_{13}}{T_{13}^3}\right)\right]$$

where $\Omega=\{T_{ij}:>0,\ T_{01}+T_{12}+T_{23}=T,\ T_{01}+T_{12}=T_{02},\ T_{12}+T_{23}=T_{13}\}$.

T denotes the time interval of the whole displacement.

The set Ω is defined as the constraints that are requires by the network structure.

As follows from the results (12) and (13) the minimum energy cost for the whole displacement in the time interval T becomes: $E_{ow}=k_z/T^3$.

The P.P.I. k_z being solution to the problem (16) can be found numerically

$$k_z=\min_{\substack{l_{(02)0},\,l_{(02)1}\in V_{02}\\ l_{(13)1},\,l_{(13)2}\in V_{13}}}\left[(k_{01}+l_{(02)0})^{1/4}+(k_{12}+l_{(02)1}+l_{(13)1})^{1/4}+(k_{23}+l_{(13)2})^{1/4}\right], \quad (16)$$

where

$$V_{02}=\{l_{(02)0},\,l_{(02)1}:\geqslant 0,\ l_{(02)0}^{1/4}+l_{(02)1}^{1/4}=k_{02}^{1/4}\},$$

$$V_{13}=\{l_{(13)1},\,l_{(13)2}:\geqslant 0,\ l_{(13)1}^{1/4}+l_{(13)2}^{1/4}=k_{13}^{1/4}\}.$$

The optimum values $l_{(02)0}^*$, $l_{(02)1}^*$, $l_{(13)1}^*$, $l_{(13)2}^*$ that can be obtained by solving the problem (15), allow to determine the optimum time intervals of the individual operations:

$$T_{01}=\left[\frac{k_{01}+l_{(01)0}^*}{k_z}\right]^{1/4}T,\quad T_{12}=\left[\frac{k_{12}+l_{(02)1}^*+l_{(13)1}^*}{k_z}\right]^{1/4}T,\quad T_{23}=\left[\frac{k_{23}+l_{(13)2}^*}{k_z}\right]^{1/4}T,$$

$T_{02}=T_{01}+T_{12}$, $T_{13}=T_{12}+T_{23}$; and the optimum currents (relation (15))

$$x_{ij}^*(t)=\frac{3Y_{ij}}{a_{ij}T_{ij}^3}\left[\frac{T_{ij}}{2}-1\right].$$

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Agregacja w dynamicznych systemach typu PERT

Rozważono systemy typu PERT zawierające operacje dynamiczne. Przyjmuje się, że każdą operację można opisać Charakterystyką Sterowania Optymalnego (CSO) o postaci potęgowej. Problem polega na znalezieniu CSO zagregowanego systemu i określeniu optymalnych wartości parametrów poszczególnych operacji. Podano twierdzenie umożliwiające sprowadzenie każdej sieci do postaci szeregowo-równoległej. Można wtedy wykorzystać rozwiązania zadań dla sieci szeregowo-równoległych. Wykazano, że CSO zagregowanego systemu ma także postać potęgową. Rozwiązania zilustrowano przykładem optymalnego sterowania przesuwaniem masy przy pewnych ograniczeniach nałożonych na trajektorię ruchu.

Агрегация в динамических системах типа ПЕРТ

Рассмотрены системы типа ПЕРТ, содержащие динамические операции. Предполагается, что каждая операция может быть описана Характеристикой Оптимального Управления (ХОУ) степенного вида. Задача состоит в нахождении ХОУ агрегированной системы и определении оптимальных значений параметров отдельных операций. Дана теорема, позволяющая свести каждую сеть к пареллельно-последовательному виду. Можно тогда использовать решения задач для параллельно-последовательных сетей. Показано, что ХОУ агрегированной системы имеет также степенной вид. Решения иллюстрируются примером оптимального управления передвижением массы при некоторых ограничесниях накладываемых на траекторию движения.