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Algebraic criteria of controllability to zero function for linear constant time-lag systems

by

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A linear time-lag system described by the equation $\dot{x}(t)=Ax(t)+Bx(t-1)+Cu(t)$, where $x(t)\in R^n$, the control $u(t)\in R^m$; A, B, C=constant matrices, is considered. A general solution to the problem of controllability to zero function for such type of systems is obtained. The solution is given in the form of testable algebraic criteria expressed in terms of some special functions of the matrices A, B, C. The criteria are obtained thanks to a new method of transforming control problems in time-lag system into equivalent control problems stated for some non-delayed system with additional equality constraints on initial and terminal states. To solve the reformulated controllability problem some lemmas on maintainability of trajectory and compensation of disturbances for linear non-delayed systems are given.

1. Introduction

The purpose of the paper is to obtain testable algebraic criteria for controllability to zero function in a linear constant time-lag system. The class of systems considered is given by the following equation [1]

$$\dot{x}(t)=A\,x(t)+B\,x(t-1)+C\,u(t),\quad t\in[0,t_1],\ t_1>1, \tag{1.1}$$

with initial conditions

$$x(0)=x_0\in R^n,\quad x(t-1)=f(t),\quad t\in[0,1], \tag{1.2}$$

where $f\colon[0,1)\to R^n$, $u\colon[0,t_1]\to R^m$ are bounded measurable functions, A, B, C are constant matrices of suitable dimensions.

We shall further use the terminology of controllability and relative controllability for controllability to zero function on $[t_1-1, t_1]$ and to origin at t_1 respectively, where t_1 is assumed to be fixed.

The problem under consideration was investigated by many authors. Banks [11] gave a thorough review of main results obtained up to 1972 in the field of time-lag

[1]) Systems with delay $h\neq 1$ can be easily rewritten in the form of (1.1) after a suitable transformation of time.

systems controllability, so we shall summarize only the previous works. Gabasov, Kirillova, Churakova obtained some algebraic criteria but for special cases only. They proved that if one of the following holds (1) rank $[B, C]=\operatorname{rank} C$, (2) $A=0$ and $\det B\neq 0$, (3) $A=0$ and rank $C=1$, then controllability is equivalent to relative controllability. It may be interesting that algebraic criteria for relative controllability given in [1], [2] were obtained by a conjecture that every system of the form (1.1) is pointwise complete, that is, at $u=0$ for any $t>0$ and any $x_1 \in R^n$ there exist initial conditions (x_0, f) such that the solution to (1.1), (1.2) $x(t)=x_1$ (see Weiss [3]). The conjecture, however, is not true in general, as it was shown by Popov [7] and Zverkin [13] on a counterexample. In order to avoid the problem of pointwise completeness Manitius and Olbrot [5] defined controllability as reachability of any vector in R^n at t_1 and obtained controllability criteria in the case of many delays in "state" x and control u. Weiss gave in [4] a sufficient condition for relative controllability in the form similar to [1], [2], which is also a necessary one if pointwise complete systems are considered. For this class of systems there is a criterion for controllability of (1.1) obtained by Weiss [3], unfortunately not in a computable form.

The most difficult case is that of pointwise degenerate (not complete) systems. In this paper a method used in [6] for examining pointwise degeneracy is applied to obtain general solution to controllability problem for time-lag systems. This method is presented in Section 3 and has, in fact, a point in common with method of Popov [5] who used the same equivalent system without delay. In Section 2 we introduce the concept of controlled invariant by Basile and Marro [9] and further develop some theorems on maintainability, reachability and compensation of disturbances for linear systems. The results of Section 2 have direct application in the proof of controllability criterion (Sections 4, 5). Two numerical examples given in Section 6 illustrate the text.

Throught the paper the following notation will be used. By R_A, N_A we denote the range and the null space respectively of the operator A. $R^{n\times m}$ is the space of all real $n\times m$ matrices, $R^{n\times 1}$ will be written as R^n. For a subspace $Z\subset R^n$ and for $A\in R^{n\times n}$ denote $\mathscr{A}(Z)=Z+AZ+\ldots+A^{n-1}Z$ and $\mathscr{A}(B)$ instead of $\mathscr{A}(R_B)$. Identity operator will be denoted by I and the Moore-Penrose generalized inverse of A by A^+. $A^{-1}X$ is the set $\{y|\ Ay\in X\}=A^+(X\cap R_A)+N_A$ provided that $X\cap R_A\neq$ $\neq\emptyset$. $Y^\perp$ is an orthogonal complement to Y. $\{x_1, \ldots, x_r\}$ is the subspace spanned by vectors $x_1, \ldots, x_r$. We shall sometimes write $\{A\}$ for R_A. A^T is the transpose of A.

2. Preliminary results on controllability, maintainability and compensation of disturbances for linear systems

Let us start with definition of controlled invariant due to Basile and Marro [9, 10].

DEFINITION 2.1. Let $A\in R^{n\times n}$ and let S be a subspace in R^n. A subspace $X\subset R^n$ is said to be (A, S) — controlled invariant (or briefly (A, S) — invariant) if $AX\subset X+S$.

In linear system theory the concept of maximal (with respect to ordering generated by inclusion) controlled invariant contained in a given subspace plays an important role. Two algorithms for computing maximal controlled invariants are cited below. The first due to Wonham and Morse [8] is as follows.

LEMMA 2.1. There exist a unique maximal (A, S)-invariant contained in a given subspace $X\subset R^n$. Furthermore it equals to X_p where

$$X_0=X,\quad X_i=X_{i-1}\cap A^{-1}(S+X_{i-1})\ \text{and}\ \dim X=p.$$

We shall further write $Mic(A, S, X)$ for the maximal (A, S)-invariant in X and $Mic(A, B, X)$ instead of $Mic(A, R_B, X)$ or simply $Mic\,X$ if A, B are understood.

The computational technique offered by Basile and Marro [9] is more conveniant for it does not use the inverse A^{-1}. The algorithm may be introduced in the form of

LEMMA 2.2. The maximal (A, S)-controlled invariant contained in X is equal to $X_p^\perp$ where

$$X_0=X^\perp,\quad X_i=X^\perp+A^T(X_{i-1}\cap S)\ \text{and}\ p=\dim X.$$

Given a system

$$\begin{gathered}\dot{x}(t)=Ax(t)+Bu(t),\ t\geqslant 0,\\ x(t)\in R^n,\ u(t)\in R^m,\ A, B\text{-constant matrices}\end{gathered}\tag{2.1}$$

and a subspace $X\subset R^n$, define the notion of X-maintainability and X-reachability as follows

DEFINITION 2.2. A vector $x_0\in X$ is said to be X-maintainable if starting from x_0 a trajectory of (2.1) completely belonging to X can be obtained by means of a bounded control u.

DEFINITION 2.3. A vector $x_1\in X$ is said to be X-reachable from an $x_0\in X$ at time t_1 if there exists a trajectory of (2.1) such that $x(0)=x_0$, $x(t_1)=x_1$ and $x(t)\in X$ on $[0, t_1]$.

The main result of [9] is the following criterion of X-maintainability

LEMMA 2.3. A vector $x_0\in X$ is X-maintainable iff

$$x_0\in Mic(A, B, X).$$

Since, by definition, $Mic(Mic\,X)=Mic\,X$ it follows from Lemma 2.3.

COROLLARY 2.1. A vector x_0 is X-maintainable if and only if it is Mic X-maintainable.

In the sequel we need an algorithm for finding all X-reachable states at $t=1$ for the system (2.1). This problem was solved in [8] and [12] for the special case $x_0=0$.

LEMMA 2.4. The set of all vectors X-reachable from the origin at time $t_1 > 0$ is of the form of a subspace $Mcs(A, B, X)$ (maximal controllability subspace) and can be computed by the sequence

$$X_0 = 0, \quad X_i = Mic\, X \cap (AX_{i-1} + R_B)$$

$$\text{taking } Mcs\, X = X_q, \quad q = \dim Mic\, X.$$

In this section the notation $Mcs\, X$ for $Mcs(A, B, X)$ will be used.

Remark. The algorithm above was not explicitly stated by the authors. One can, however, easily deduce it from an algorithm given in [8], Theorem 4.1, taking into account Lemma 4.3 and Theorem 4.3 of [8] and Theorem 6.1 of [12].

In [8], Theorem 4.3, a different way of computing $Mcs\, X$ is presented. Let us quote this result in the form of:

LEMMA 2.5. The maximal controllability subspace has the form of a controllable subspace $\mathscr{A}(R_B \cap Mic\, X)$ where $\tilde{A} = A + BC$ for some $C \in R^{m \times n}$ such that $\tilde{A}\, Mic\, X \subset Mic\, X$.

The existance of C is proved in [8].

To extend the result of Lemma 2.4. we prove the following:

THEOREM 2.1. The set of all states X-reachable at $t = 1$ from an $x_0 \in Mic\, X$ is an element of the quotient space $Mic\, X / Mcs\, X$ and is of the form $x_1 + Mcs\, X$ where $x_1 \in Mic\, X$ is an arbitrary vector X-reachable at $t = 1$ from x_0.

Proof. If $x_1(t), x_2(t)$, $t \in [0, 1]$, are trajectories in X starting from $x_0 \in Mic\, X$ and corresponding to controls u_1, u_2 respectively then, by linearity, $x_2(t) - x_1(t)$ corresponds to $u_2 - u_1$, starts from origin and belongs to X. Hence, by Lemma 2.4, $x_2(1) - x_1(1) \in Mcs\, X$ that is $x_2(1) \in x_1(1) + Mcs\, X$ for an arbitrary X-reachable vector $x_2(1)$. Conversely, it is readily seen that any vector belonging to $x_1 + Mcs\, X$ can be reached from x_0 by a trajectory in X if x_1 is X-reachable from x_0.

Now consider the system (2.1) with additive disturbance z

$$\dot{x}(t) = A\, x(t) + B\, u(t) + D\, z(t), \quad t \geq 0, \tag{2.2}$$

where $z(t) \in R^p$, D is a constant matrix, u, z are assumed to be bounded, measurable defined over finite or infinite interval.

We state the following problem for the system (2.2): "Under what condition the state x_0 is X-maintainable for each disturbance z?"

When $x_0 = 0$ and u is chosen in the form of a constant feedback the solution to this problem is that $R_D \subset Mic\, X$ (see [8]). In general the following theorem is valid.

THEOREM 2.2. Given a subspace $X \subset R^n$, an initial state $x_0 \in X$ of (2.2) is X-maintainable for each z iff $x_0 \in Mic\, X$ and

$$R_D \subset Mic\, X + R_B. \tag{2.3}$$

Moreover, the above conditions are necessary and sufficient for $Mic\, X$-maintainability.

Proof.

Sufficiency. If (2.3) holds then there exist matrices D_1, D_2 such that $D_1 + D_2 = D$ and $R_{D_1} \subset R_B$, $R_{D_2} \subset Mic\, X$. Clearly this representation is unique only if $R_B \cap Mic\, X = 0$. The term $D_1 z$ may be neutralized to zero setting e.g. $u_1 = -B^+ D_1 z$. By a result of Wonham and Morse [8] there exists a control $u_2 = Fx$ for some $F \in R^{m \times n}$ such that the effect of $D_2 z$ on the trajectory of (2.2) is localized to $Mic\, X$. By Lemma 2.3 if $x_0 \in Mic\, X$ then one can find u_3 such that $e^{At} x_0 + \int_0^t e^{A(t-s)} B\, u(s)\, ds$ is in $Mic\, X$ for all $t \geq 0$. By linearity $u = u_1 + u_2 + u_3$ is a properly chosen control.

Necessity. Suppose x_0 is X-maintainable for each z. Take $z = 0$. By Lemma 2.3 it is seen that $x_0 \in Mic\, X$. Hence x_0 is $Mic\, X$-maintainable for each z. In fact, if on the contrary for some z' and some $t' > 0$ the state $x(t')$ cannot be in $Mic\, X$ then, by Lemma 2.3, $x(t')$ is not X-maintainable in the system (2.1) and this implies that x_0 is not X-maintainable for $z = z'$ on $[0, t']$, $z(t) = 0$ for $t > t'$ which is a contradiction. Hence for any z there exists a control u such that $x(t) \in Mic\, X$ and simultanously $\dot{x}(t) \in R_B + Mic\, X$. The latter follows by Definition 2.1. This and equation (2.2) imply that for any z: $D\, z(t) = \dot{x}(t) - A\, x(t) - B\, u(t) \in Mic\, X + R_B$ almost everywhere. Hence (2.3) follows immediately.

COROLLARY 2.2. Let a subspace $Y \subset R^n$ begiven. The property that for any disturbance z there exist an $x_0 \in Y$ such that x_0 is X-maintainable is valid for the system (2.2) iff

$$R_D \subset R_B + Mic\, X.$$

Proof. Sufficiency follows from Theorem 2.2 setting $x_0 = 0$. For necessity it can be proved, as in Theorem 2.2, that if the considered property holds the requirement of X-maintainability may be replaced by $Mic\, X$-maintainability. We complete the proof as in Theorem 2.2.

The result of Corollary 2.2 means that the possibility of choosing the initial state gives no advantage as far as the compensation of disturbances is considered.

COROLLARY 2.3. Assume that (2.3) holds. Given an arbitrary disturbance z, an initial point x_0 of (2.2) is X-maintainable iff $x_0 \in Mic\, X$.

Proof. In view of Theorem 2.2 sufficiency is obvious. By (2.3) and Theorem 2.3 it is possible to choose a control u_1 such that $\int_0^t e^{A(t-s)} (B\, u_1(s) + D\, z(s))\, ds$ is in X for all $t \geq 0$. Now set $u = u_1 + u_2$ where u_2 is arbitrary. If $x(t)$ is in X for some $u = u_1 + u_3$ then it follows from above that also $e^{At} x_0 + \int_0^t e^{A(t-s)} B u_2(s)\, ds$ is in X for some u_3. This implies, by Lemma 2.3, that $x_0 \in Mic\, X$.

In the sequel the knowledge of a set of all X-reachable states at $t = 1$, provided that $x(0)$, z are given, is desired. The solution to this problem can be easily obtained by argumentation of Theorem 2.1 and the result has similar form.

THEOREM 2.3. The set of all vectors X-reachable at $t=1$ from an $x_0 \in X$ while a given diturbance z affects on the system (2.2) is of the form $x_1 + Mcs\, X$ where x_1 is an arbitrary vector X-reachable at $t=1$ from x_0 by a given z.

3. An equivalent non-delayed system and reformulation of the controllability problem

Let us consider now the system described by the equation (1.1) with initial conditions (1.2) and the controllability property on a given interval $[0, t_1]$, $t_1 > 1$, that is the property that

$$\forall x_0 \in R^n\ \forall f \in \mathcal{F}\ \exists u \in \mathcal{U}: x(t)=0 \text{ on } [t_1-1, t_1] \tag{3.1}$$

where $\mathcal{F}=BM([-1, 0), R^n)$, $\mathcal{U}=BM([0, t_1], R^m)$ are classes of bounded measurable functions.

Let k be the integer such that $k-1<t_1\leqslant k$. Denote

$$x(i+t)=x_{i+1}(t),\ u(i+t)=u_{i+1}(t) \quad \text{for } i=0, \ldots, k-1,\ t \in [0, 1]. \tag{3.2}$$

Then we have by (1.1) and (1.2)

$$\begin{aligned}
\dot{x}_1(t)&=A x_1(t)+Bf(t)+C u_1(t), && t \in [0, 1],\\
\dot{x}_2(t)&=A x_2(t)+B x_1(t)+C u_2(t), && t \in [0, 1],\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
\dot{x}_k(t)&=A x_k(t)+B x_{k-1}(t)+C u_k(t), && t \in [0, 1],
\end{aligned}$$

where the k-th equation is to be extended on the whole interval $[0, 1]$ by defining $u(t)$ on $(t_1, k]$ in an arbitrary way. The equation above can be written in the form

$$\dot{y}(t)=\hat{A}\, y(t)+\hat{B} f(t)+\hat{C} v(t),\ t \in [0, 1] \tag{3.3}$$

where $y(t) \in R^{kn}$, $v \in \mathcal{V}=BM([0, 1], R^{km})$, $A \in R^{kn\times kn}$, $B \in R^{kn\times n}$ and $C \in R^{kn\times km}$ are as follows

$$y(t)=\begin{bmatrix} x_1(t)\\ x_2(t)\\ \vdots\\ x_k(t)\end{bmatrix}, \quad v(t)=\begin{bmatrix} u_1(t)\\ u_2(t)\\ \vdots\\ u_k(t)\end{bmatrix}, \tag{3.4}$$

$$\hat{A}=\begin{bmatrix} A & \ldots & 0\\ BA & & \\ \vdots & \ddots & \\ 0 & & BA\end{bmatrix}, \quad \hat{B}=\begin{bmatrix} B\\ 0\\ \vdots\\ 0\end{bmatrix}, \quad \hat{C}=\begin{bmatrix} C & \ldots & 0\\ & C & \\ & & \ddots\\ 0 & \ldots & C\end{bmatrix}. \tag{3.5}$$

By the continuity of a trajectory $x(t)$ and from (3.2)

$$x_i(1)=x_{i+1}(0)=x(i) \text{ for } i=1, \ldots, k-1.$$

Taking into account that $x_1(0)=x_0$ this implies

$$y(0)=y_0+\hat{K}\, y(1) \tag{3.6}$$

where

$$y_0=\begin{bmatrix} x_0\\ 0\\ \vdots\\ 0\end{bmatrix}, \quad \hat{K}=\begin{bmatrix} 0 & & 0\\ I & 0 & \\ \vdots & \ddots & \\ 0 & & I\ 0\end{bmatrix}. \tag{3.7}$$

For given x_0, v, f every solution to (3.3) satisfying (3.6) can be transformed, in accordance with (3.2), into a solution to (1.1) satisfying (1.2). This follows immediately by the method of continuation and uniqueness theorem for differential-difference equations.

Thus the equivalence between n-dimensional time-lag system (1.1) with (1.2) defined on $[0, t_1]$ and nk-dimensional non-delayed system (3.3) with (3.6) is established. This makes possible to replace the problem (3.1) by a certain problem for the system (3.3), (3.6) and to obtain then computable algebraic conditions for controllability.

4. Controllability conditions in case $t_1=k$

In this section we consider the case $t_1=k$, an integer. Denote

$$Y_0=R^n\times\{0\}^{n(k-1)}, \quad \bar{Y}=R^{n(k-1)}\times\{0\}^n, \quad Y_1=R^{n(k-2)}\times\{0\}^{2n}. \tag{4.1}$$

Now, because of equivalence established between (1.1), (1.2) and (3.3), (3.6) the following lemma can be proved easily.

LEMMA 4.1. The system (1.1) is controllable on $[0, k]$ iff

(A1) $\forall (y_0, f) \in Y_0\times\mathcal{F}\exists (y_1, v) \in Y_1\times\mathcal{V}$ such that by

$$y(0)=y_0+\hat{K}y_1$$

both (i) and (ii) are satisfied for the system (3.3)

(i) $y(t) \in \bar{Y}$ $\quad\forall t \in [0, 1]$

(ii) $y(1)=y_1$

Proof. Notice that $y(t) \in \bar{Y}$ on $[0, 1]$ is equivalent to $x(t)=0$ on $[k-1, k]$. Condition (ii) with $y(0)=y_0+\hat{K}y_1$ guarantees that (3.6) is satisfied. The equivalence between (1.1), (1.2) and (3.3), (3.6) completes the proof.

Now we are in a position to use the theorems given in section 2 for proving the main result of the paper, that is the controllability criterion.

THEOREM 4.1. The system (1.1) is controllable on $[0, k]$ iff

$$R_{\hat{B}} \subset R_{\hat{C}} + \mathrm{Mic}\,\bar{Y}, \tag{4.2}$$

$$Y_0 + \hat{K}\bar{\mathcal{A}}(\bar{B}) \subset (\hat{I} - \hat{K}\, e^{\bar{A}})\, \mathrm{Mic}\,\bar{Y} + \hat{K}\, \mathrm{Mcs}\,\bar{Y}, \tag{4.3}$$

where

$$\mathrm{Mic}\,\bar{Y} = \mathrm{Mic}\,(\hat{A}, \hat{C}, \bar{Y})$$

$$\mathrm{Mcs}\,\bar{Y} = \mathrm{Mcs}\,(\hat{A}, \hat{C}, \bar{Y})$$

$\bar{A} = \hat{A} + \hat{C}\hat{D}$ for some $\hat{D} \in R^{km \times kn}$ such that

$$\bar{A}\, \mathrm{Mic}\,\bar{Y} \subset \mathrm{Mic}\,\bar{Y}$$

$\bar{B}$ is an arbitrary $kn \times n$ matrix such that

$$R_{\bar{B}} \subset \mathrm{Mic}\,\bar{Y},\ R_{\hat{B}-\bar{B}} \subset R_{\hat{C}}.$$

Proof. At first notice that the existence of a matrix $\hat{D}$ follows from the fact that $\hat{A}\, \mathrm{Mic}\,\bar{Y} \subset R_{\hat{C}} + \mathrm{Mic}\,\bar{Y}$ (see [8], Lemma 3.2). This follows also trivially from Lemma 5.3 in this paper. The existence of $\bar{B}$ follows easily from (4.2).

Necessity. If (1.1) is controllable on $[0, k]$ then by Lemma 4.1 $\forall (0, f) \in Y_0 \times \mathcal{F}\ \exists y \in \hat{K} Y_1 : y$ is $\bar{Y}$-maintainable. Hence, making use of Corollary 2.2 yields (4.2). Now let $\bar{A}$, $\bar{B}$ be matrices having the properties required. For any $(y, f) \in \mathrm{Mic}\, Y \times \mathcal{F}$ choose the control

$$v_1(t) = \hat{C}^+ ((\bar{A} - \hat{A})\, y(t) + (\bar{B} - \hat{B}) f(t)). \tag{4.4}$$

Since $R_{\bar{A}-\hat{A}} = R_{\hat{C}\hat{D}} \subset R_{\hat{C}}$, $R_{\bar{B}-\hat{B}} \subset R_{\hat{C}}$ and $\hat{C}\hat{C}^+$ restricted to $R_{\hat{C}}$ is an identity operator, the equation (3.3) with $v = v_1$ may be written in the form

$$\dot{y}(t) = \bar{A}\, y(t) + \bar{B} f(t). \tag{4.5}$$

Provided that $y(0) = y \in \mathrm{Mic}\,\bar{Y}$ the solution to (4.5)

$$y(t) = e^{\bar{A}t}\, y + \int_0^t e^{\bar{A}(t-s)}\, \bar{B} f(s)\, ds$$

is in $\mathrm{Mic}\,\bar{Y}$ on $[0, 1]$, since $R_{\bar{B}} \subset \mathrm{Mic}\,\bar{Y}$ and $\mathrm{Mic}\,\bar{Y}$ is invariant under $\bar{A}$ and $e^{\bar{A}t}$. Thus it is obvious that $e^{\bar{A}}\, y + \int_0^1 e^{\bar{A}(1-s)}\, \bar{B} f(s)\, ds$ is the endpoint of a trajectory of (3.3) starting from $y \in \mathrm{Mic}\,\bar{Y}$ and completely belonging to $\mathrm{Mic}\,\bar{Y}$. Then by Theorem 2.3 the set $e^{\bar{A}}\, y + \int_0^1 e^{\bar{A}(1-s)}\, \bar{B} f(s)\, ds + \mathrm{Mcs}\,\bar{Y}$ consists of all vectors $\bar{Y}$-reachable from $y(0) = y$ at $t = 1$.

Let us note that

$$\{w \mid w = \int_0^1 e^{\bar{A}(1-s)}\, \bar{B} f(s)\, ds,\ f \in \mathcal{F}\} = \bar{\mathcal{A}}(\bar{B}),$$

the controllable subspace of the pair $(\bar{A}, \bar{B})$.

Controllability of (1.1) implies also by (4.2) and Corollary 2.3 that the vector $y(0) = y_0 + \hat{K} y_1$ in Lemma 4.1 must belong to $\mathrm{Mic}\, Y$.

Now, by Lemma 4.1 and considerations above the property (A1) implies the following

(A2) $\forall (y_0, w) \in Y_0 \times \bar{\mathcal{A}}(\bar{B})\quad \exists y_1 \in Y_1$:

(i) $y(0) = y_0 + \hat{K} y_1 \in \mathrm{Mic}\,\bar{Y}$,

(ii) $y(1) = y_1 \in e^{\bar{A}} (y_0 + \hat{K} y_1) + w + \mathrm{Mcs}\,\bar{Y}$.

The property (A2) is equivalent to

(A3) $\forall (y_0, w) \in Y_0 \times \bar{\mathcal{A}}(\bar{B})\quad \exists y \in \mathrm{Mic}\,\bar{Y}$:

$$y \in \hat{K} (e^{\bar{A}}\, y + w + \mathrm{Mcs}\,\bar{Y}) + y_0. \tag{4.6}$$

In fact, it is clear, by setting $y = y_0 + \hat{K} y_1$, that (A2) implies (A3). For the converse, let $P_0 \in R^{kn \times kn}$ be the orthogonal projector onto Y_0 and let $y_1 = \hat{K}^+ y$ where $\hat{K}^+$ is of the simple form

$$\hat{K}^+ = \begin{bmatrix} 0 & I & \cdots & 0 \\ 0 & 0 & \ddots & \\ \vdots & & \ddots & I \\ 0 & \cdots & & 0 \end{bmatrix} = \hat{K}^T. \tag{4.7}$$

One can easily verify that

$$P_0 + \hat{K}\hat{K}^+ = \hat{I}, \quad P_0 \hat{K} = 0, \quad P_0\, y_0 = y_0 \quad \text{for } y_0 \text{ in } Y_0.$$

Now, it follows from (A3)

$$P_0\, y \in 0 + y_0 = y_0.$$

This implies that $y_0 + \hat{K} y_1 = (P_0 + \hat{K}\hat{K}^+)\, y = y \in \mathrm{Mic}\,\bar{Y}$ which gives the property (A2) (i). Furthermore

$$y_1 = \hat{K}^+ y \in \hat{K}^+ \hat{K} (e^{\bar{A}}\, y + w + \mathrm{Mcs}\,\bar{Y}) + \hat{K}^+ y_0 = e^{\bar{A}} (y_0 + \hat{K} y_1) + w + \mathrm{Mcs}\,\bar{Y}$$

since $\hat{K}^+ Y_0 = 0$, $\hat{K}^+ \hat{K}$ is the identity on $\bar{Y}$, $\mathrm{Mcs}\,\bar{Y} \subset \bar{Y}$, and $e^{\bar{A}} y \in \mathrm{Mic}\,\bar{Y}$, $w \in \bar{\mathcal{A}}(\bar{B}) \subset \mathrm{Mic}\,\bar{Y} \subset \bar{Y}$, for by definitions of $\bar{A}$ and $\bar{B}$

$$R_{\bar{A}} \subset \mathrm{Mic}\,\bar{Y}, \quad R_{\bar{B}} \subset \mathrm{Mic}\,\bar{Y}.$$

Thus the equivalence (A2) $\Leftrightarrow$ (A3) is proved. Writing (4.6) in the form of

$$-y_0 - \hat{K} w \in (\hat{K} e^{\bar{A}} - \hat{I})\, y + \hat{K}\, \mathrm{Mcs}\,\bar{Y}$$

it is easily seen that

$$(A2) \Leftrightarrow (A3) \Leftrightarrow (4.3) \tag{4.8}$$

which completes the proof of necessity.

Sufficiency. Let (4.2) and (4.3) hold. First of all observe that by (4.8) it remains to prove that (4.2) and (A2) imply (A1). For any $(y_0, f) \in Y_0 \times \mathcal{F}$ choose the control

$$v = v_1 + v_2$$

where v_1 is defined by (4.4), to obtain the equation similar to (4.5)

$$\dot{y}(t)=\bar{A}\,y(t)+\bar{B}f(t)+\bar{C}\,v_2(t). \tag{4.9}$$

As was emphasized before $e^{\bar{A}}(y_0+\hat{K}y_1)+\int_0^1 e^{\bar{A}(1-s)}\,\bar{B}f(s)\,ds+Mcs\,\bar{Y}$ is the set of all points $\bar{Y}$-reachable at $t=1$ from $y(0)=y_0+\hat{K}y_1$ while the "disturbance" f influences the system (3.3). So the property (A2) means that for any $(y_0,f)\in Y_0\times\mathscr{F}$ there exists $y_1\in Y_1$ and $v_2\in\mathscr{V}$ such that the trajectory of the system (4.9) starting from $y_0+\hat{K}y_1$ attains y_1 at $t=1$ and $y(t)\in Mic\,\bar{Y}$ for all t considered. The control $v=v_1+v_2$ and the vector y_1 fulfill the requirements (A1) for the system (3.3). Thus the controllability conditions (4.2) (4.3) are proved.

As a special case of controllability property (3.1) one can consider the problem of controllability with respect to either x_0 or f. Namely, a vector x_0 is said to be controllable initial point if the initial pair $(x_0,0)\in R^n\times\mathscr{F}$ of (1.1) is controllable on some interval. Similarly a controllable initial function can be defined. The system (1.1) will be called *IP*-controllable (*IF*-controllable) if all initial points (initial functions) are controllable. From the proof of Theorem 4.1 one obtains immediately

COROLLARY 4.1. The system (1.1) is *IP*-controllable on $[0,k]$ if and only if

$$Y_0\subset(\hat{I}-\hat{K}\,e^{\bar{A}})\,Mic\,\bar{Y}+\hat{K}\,Mcs\,\bar{Y}. \tag{4.10}$$

COROLLARY 4.2. The system (1.1) is *IF*-controllable on $[0,k]$ if and only if

$$R_{\hat{B}}\subset R_{\hat{C}}+Mic\,\bar{Y},$$

$$\hat{K}\mathscr{I}(\bar{B})\subset(\hat{I}-\hat{K}\,e^{\bar{A}})\,Mic\,\bar{Y}+\hat{K}\,Mcs\,\bar{Y}. \tag{4.11}$$

For systems which are not *IP*-controllable one can get, in like manner, the description for the set of all controllable initial points.

COROLLARY 4.3. The set of all controllable initial points of the system (1.1) is of the form

$$P(Y_0\cap(\hat{I}-\hat{K}\,e^{\bar{A}})\,Mic\,\bar{Y}+\hat{K}\,Mcs\,\bar{Y})$$

where

$$R^{n\times kn}\ni P=[I,0,\dots,0] \tag{4.12}$$

A very simple necessary condition for controllability (*IP*-controllability) can be obtained from (4.10).

COROLLARY 4.4. If the system (1.1) is *IP*-controllable (controllable) on $[0,k]$ then

$$\dim P\,Mic\,\bar{Y}=n. \tag{4.13}$$

Proof. Applying the transformation (4.12) to both sides of (4.10) and observing that $P\hat{K}=0$, $PY_0=R^n$ one gets (4.13).

It is interesting to know, for systems which are not *IF*-controllable, a constructive description of a set of all controllable initial functions. This problem however remains unsolved and will not be treated in this paper.

5. Controllability on $[0,t_1]$, $k-1<t_1<k$

Let $t_1=k-1+\tau$, $\tau\in(0,1)$. The controllability of the system (1.1) on $[0,t_1]$ can be described by means of some dynamic properties of the system (3.3) quite similarly as in section 4 for the case $t_1=k$. Let us introduce then

LEMMA 5.1. The system (1.1) is controllable on $[0,t_1]$, $t_1=k-1+\tau$, $\tau\in(0,1)$, if and only if the system (3.3) has the property

(B1) $\forall(y_0,f)\in Y_0\times\mathscr{F}\ \exists(y_1,y_2,v)\in Y_1\times Y_1\times\mathscr{V}$ such that if $y(0)=y_0+\hat{K}y_1$ then
(i) $y(t)\in\bar{Y}$ on $[0,\tau]$,
(ii) $y(\tau)=y_2$,
(iii) $y(t)\in Y_1$ on $[\tau,1]$,
(iv) $y(1)=y_1$.

Proof. Controllability of the system (1.1) on $[0,t_1]$ implies that the condition $x(t)=0$ on $[t_1-1,k]$ may be fulfilled for any initial pair (x_0,f) by the choice of a proper control (e.g. after getting $x(t)=0$ on $[t_1-1,t_1]$ one may put $u(t)=0$ on $[t_1,k]$). The converse is obviously true. This can be expressed in terms of the equivalent system (3.3) with (3.6) as stated in the Lemma. In fact, (i) and (iii) are equivalent to $x(t)=0$ on $[t_1,k]$ which is easily seen from (3.2), (3.4), (4.1). Condition (iv) is required so that the equivalence between (3.3) (3.6) and (1.1) (1.2) holds. Condition (ii) can be omitted, but it is useful in the sequel.

The next two lemmas are needed in proving controllability criterion.

LEMMA 5.2. Let $\bar{Y}$, Y_1 be defined by (4.1) and let

$$Mic\,\bar{Y}=Mic(\hat{A},\hat{C},\bar{Y}),\quad Mic\,Y_1=Mic(\hat{A},\hat{C},Y_1),\ \text{then}$$

$$Mic\,\bar{Y}\supset Mic\,Y_1. \tag{5.1}$$

Proof. The proof follows trivially from the fact that $\bar{Y}\supset Y_1$ and the Definition 3.1.

LEMMA 5.3. There exists a $\hat{D}\in R^{km\times kn}$ such that for $\bar{A}=\hat{A}+\hat{C}\hat{D}$ the following inclusions hold

$$\bar{A}\,Mic\,Y_1\subset Mic\,Y_1, \tag{5.2}$$

$$\bar{A}\,Mic\,\bar{Y}\subset Mic\,\bar{Y}. \tag{5.3}$$

Proof. Since, by Lemma 5.2, $Mic\,\bar{Y}\supset Mic\,Y_1$ one can choose a basis $y_1,y_2,\dots,y_q$ for $Mic\,\bar{Y}$ such that $y_1,\dots,y_p$, $p\leqslant q$ is a basis for $Mic\,Y_1$. By definition $\hat{A}\,Mic\,\bar{Y}\subset Mic\,\bar{Y}+R_{\hat{C}}$ and $\hat{A}\,Mic\,Y_1\subset Mic\,Y_1+R_{\hat{C}}$. This implies $\hat{A}y_i=w_i+\hat{C}v_i$ for some $w_i\in Mic\,Y_1$ if $i\leqslant p$ and $w_i\in Mic\,\bar{Y}$, $i>p$, and for some $v_i\in R^{km}$, $i=1,\dots,q$. Now choose $\hat{D}$ such that $\hat{D}y_i=-v_i$. This is possible since $\dim\{y_1,\dots,y_q\}=q\geqslant\dim\{v_1,\dots,v_q\}$. Thus the proof is complete.

Now the controllability criterion can be proved in the form of

THEOREM 5.1. The system (1.1) is controllable on $[0, t_1]$, $k-1<t_1<k$ if and only if

$$R_{\hat{B}} \subset Mic\, Y_1 + R_{\hat{C}}, \tag{5.4}$$

$$\mathscr{A}(\bar{B}) \times (Y_0 + \hat{K}\mathscr{A}(\bar{B})) \subset E(Mic\, \bar{Y} \times Mic\, Y_1) + Mcs\, \bar{Y} \times \hat{K}\, Mcs\, Y_1, \tag{5.5}$$

where

$$E = \begin{bmatrix} \hat{I} & \hat{I} \\ \hat{I} & \hat{K} e^{\bar{A}} \end{bmatrix} \in R^{2kn \times 2kn},$$

$\bar{A} = \hat{A} + \hat{C}\hat{D}$ for some $\hat{D} \in R^{km \times kn}$ such that (5.2) and (5.3) are satisfied, $\bar{B}$ is an arbitrary $kn \times n$ matrix such that

$$R_{\bar{B}} \subset Mic\, Y_1 \quad \text{and} \quad R_{\bar{B}-\hat{B}} \subset R_{\hat{C}}.$$

Proof. By conditions (ii), (iii) of Lemma 5.1 controllability of (1.1) on $[0, t_1]$ implies that for any "disturbance" f there exists a Y_1-maintainable vector y_1. By Corollary 2.2 this implies (5.4). Now we have to prove only that if (5.4) holds the controllability of (1.1) on $[0, t_1]$ is equivalent to (5.5). Let $\bar{A}$, $\bar{B}$ be matrices as required in theorem. The existence of $\bar{A}$, $\bar{B}$ follows from Lemma 5.3 and from the inclusion (5.4) respectively. Note that, by Lemma 5.2, (5.4) implies (4.2). Applying Corollary 2.3, if (B1) holds then $y(0) = y_0 + \hat{K}y_1 \in Mic\, \bar{Y}$ and $y_2 \in Mic\, Y_1$. As in Theorem 4.1 one concludes that $e^{\bar{A}\tau} y(0) + w_1 + Mcs\, \bar{Y}$ is the set of all vectors $\bar{Y}$-reachable at $t=\tau$ from $y(0) \in Mic\, \bar{Y}$ where

$$w_1 = \int_0^{\tau} e^{\bar{A}(\tau-s)} \bar{B} f(s)\, ds. \tag{5.6}$$

Similarly $e^{\bar{A}(1-\tau)} y_2 + w_2 + Mcs\, Y_1$, where

$$w_2 = \int_{\tau}^{1} e^{\bar{A}(1-s)} \bar{B} f(s)\, ds, \tag{5.7}$$

is the set of all points Y_1-reachable at $t=1$ from $y(\tau) = y_2 \in Mic\, Y_1$. It is an elementary fact in controllability theory that the transformations $f \to w_1$ and $f \to w_2$ defined by (5.6), (5.7) are onto $\mathscr{A}(\bar{B})$. Since the integrals in (5.6), (5.7) are defined over disjoint intervals (except the point τ) it is clear that $f \to (w_1, w_2)$ is onto $\mathscr{A}(\bar{B}) \times \mathscr{A}(\bar{B})$. From above considerations it follows that the property (B1) implies

(B2) $\forall (y_0, w_1, w_2) \in Y_0 \times \mathscr{A}(\bar{B}) \times \mathscr{A}(\bar{B})\ \exists (y_1, y_2) \in Y_1 \times Mic\, Y_1$:

(i) $y_0 + \hat{K}y_1 \in Mic\, \bar{Y}$,

(ii) $y_2 \in e^{\bar{A}\tau}(y_0 + \hat{K}y_1) + w_1 + Mcs\, \bar{Y})$,

(iii) $y_1 \in e^{\bar{A}(1-\tau)} y_2 + w_2 + Mcs\, Y_1$.

It is easy to verify that, conversely, (B2) implies (B1). Finally the equivalence between (B2) and

(B3) $\forall (y_0, w_1', w_2) \in Y_0 \times \mathscr{A}(\bar{B}) \times \mathscr{A}(\bar{B})\ \exists (y, y_2') \in Mic\, \bar{Y} \times Mic\, Y_1$:

(i) $w_1' \in y + y_2' + Mcs\, \bar{Y}$,

(ii) $y_0 + \hat{K}w_2 \in y + \hat{K} e^{\bar{A}} \hat{K} e^{\hat{A}} y_2' + \hat{K}\, Mcs\, Y_1$,

can be proved.

For (B2)$\Rightarrow$(B3), define $y = y_0 + \hat{K}y_1$, $y_2' = -e^{-\bar{A}\tau} y_2$, $w_1' = -e^{-\bar{A}\tau} w_1$. Hence $y_2' \in Mic\, Y_1$, $w_1' \in \mathscr{A}(\bar{B})$ since from (5.2) and from the structure of controllable subspace it follows that the sets considered are invariant under $e^{-\bar{A}\tau}$. By substitution it is readily seen that B2(ii)$\Rightarrow$B3(i). The condition B2 (iii) implies that $y = y_0 + \hat{K}y_1 \in y_0 - \hat{K} e^{\bar{A}} y_2' + \hat{K}w_2 + \hat{K}\, Mcs\, Y_1$ and hence B3 (ii) follows.

For the converse, define $y_2 = -e^{\bar{A}\tau} y_2'$, $w_1 = -e^{\bar{A}\tau} w_1'$, and $y_1 = \hat{K}^{+} y$. Then we complete the proof in like manner as in case of (A3)$\Rightarrow$(A2) in the previous section.

The property (B3) is equivalent to (5.5) in an obvious way. Thus the proof of Theorem 5.1 is complete.

Considering special cases of *IP*- and *IF*-controllability one obtains the following criteria.

COROLLARY 5.1. The system (1.1) is *IP*-controllable on $[0, t_1]$, $k-1<t_1<k$ if and only if

$$Y_0 \subset (\hat{I} - \hat{K} e^{\bar{A}})\, Mic\, Y_1 + Mcs\, \bar{Y} + \hat{K}\, Mcs\, Y_1. \tag{5.8}$$

Proof. Setting $f=0$, it follows from the proof of Theorem 5.1 that *IP*-controllability is equivalent to (B3) with $w_1 = w_2 = 0$. Hence B3 (i) implies that $y = -y_2' + \bar{y}$ for some $\bar{y} \in Mcs\, \bar{Y}$. Substituting this to B3 (ii) yields

$$\forall y_0 \in Y_0\ \exists (y_2', \bar{y}) \in Mic\, Y_1 \times Mcs\, \bar{Y}: y_0 \in (\hat{K} e^{\bar{A}} - \hat{I})\, y_2' + \bar{y} + \hat{K}\, Mcs\, Y_1 \tag{5.9}$$

and this is equivalent to (5.8). Conversely, (5.9) implies that for each $y_0 \in Y_0$ there exists an $y = -y_2' + \bar{y} \in Mic\, Y_1 + Mcs\, \bar{Y} \subset Mic\, Y$ such that $y_0 \in y + \hat{K} e^{\bar{A}} y_2' + \hat{K}\, Mcs\, Y_1$ and this is the modification of (B3) for the case of *IP*-controllability.

COROLLARY 5.2. The system (1.1) is *IF*-controllable on $[0, t_1]$ iff (5.4) and (5.5) with $Y_0 = 0$ are satisfied.

Proof. Follows trivially from the Theorem 5.1.

One obtains easily the result analogous to Corollary 4.4.

COROLLARY 5.3. If the system (1.1) is *IP*-controllable (controllable) on $[0, t_1]$ then

$$\dim P(Mic\, Y_1 + Mcs\, \bar{Y}) = n. \tag{5.10}$$

Interesting that in case $t_1 = k-1+\tau$, $0<\tau<1$, the derived conditions (5.4), (5.5), (5.8), (5.10) does not depend on τ.

For a system with lag $h \neq 1$, $h>0$

$$\dot{x}(t) = Ax(t) + Bx(t-h) + Cu(t) \tag{5.11}$$

one can use the transformation $t = sh$, $x(sh) = z(s)$, $u(sh) = w(s)$ to obtain

$$\dot{z}(s) = hA\, z(s) + hB\, z(s-1) + hC\, w(s). \tag{5.12}$$

Thus the controllability of (5.11) on $[0, kh]$ is equivalent to controllability of (5.12) on $[0, k]$ and this is equivalent to (4.2), (4.3) with $e^{\bar{A}}$ replaced by $e^{h\bar{A}}$. This is evident since the other terms in the criterion does not depend on h, $h>0$, that is, $R_{\hat{B}} = R_{h\hat{B}}$, $Mic\,(\hat{A}, \hat{C}, \bar{Y}) = Mic\,(h\hat{A}, h\hat{C}, \bar{Y})$ etc. After similar substitution the Theorem 5.1 is valid for the system (5.11) in case $t_1 \neq kh$.

6. Examples

Example 1. Consider the system of the form (1.1) with matrices A, B, C as below

$$A=\begin{bmatrix}0&0&0\\1&0&-1\\0&0&1\end{bmatrix},\quad B=\begin{bmatrix}0&0&1\\0&1&0\\0&0&0\end{bmatrix},\quad C=\begin{bmatrix}0\\0\\1\end{bmatrix}.$$

Note that the system is relatively controllable on [0, 2] (see [2], [4], [5] for the criterion of relative controllability) since

$$\operatorname{rank}(C, AC, BC)=3\,.$$

It will be shown in what follows that the system considered is not controllable. Let $k\geq 2$ be an arbitrary integer and let us check the controllability on $[0, k]$. At first compute $Mic\ \bar{Y}$ using Lemma 2.2.

$$X_0=\bar{Y}^{\perp}=\left\{\begin{matrix}0&0&0\\ \cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot\\ 0&0&0\\ e_1&e_2&e_3\end{matrix}\right\},\quad R_{\hat{C}}^{\perp}=\left\{\begin{matrix}e_1&e_2&0&0&\dots&0&0\\ 0&0&e_1&e_2&&0&0\\ \cdot&\cdot&0&0&\dots&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&&0&0\\ 0&0&0&0&\dots&e_1&e_2\end{matrix}\right\}$$

where e_1, e_2, e_3 are the columns of the identity matrix $I\in R^{3\times 3}$ and, for conveniency, $\{A\}$ denotes the range R_A of a matrix A when the elements of A are written explicitly. Hence

$$X_0\cap R_{\hat{C}}^{\perp}=\left\{\begin{matrix}0&0\\0&0\\ \vdots&\vdots\\0&0\\e_1&e_2\end{matrix}\right\}$$

and

$$\hat{A}^T(X_0\cap R_{\hat{C}}^{\perp})=\left\{\begin{matrix}0&0&\\ \vdots&\vdots&\\0&0&\\e_3&e_2&\\0&e_1&-e_3\end{matrix}\right\}$$

since $B^T e_1=e_3$, $A^T e_1=0$, $B^T e_2=e_2$, $A^T e_2=e_1-e_3$. Now one obtains

$$X_1=X_0+\hat{A}^T(X_0\cap R_{\hat{C}}^{\perp})=\left\{\begin{matrix}0&0&0&0&0\\ \vdots&\vdots&\vdots&\vdots&\vdots\\0&0&\cdot&\cdot&\cdot\\e_3&e_2&0&0&0\\0&0&e_1&e_2&e_3\end{matrix}\right\}.$$

Similarly

$$X_2=X_0+\hat{A}^T(X_1\cap R_{\hat{C}}^{\perp})=\left\{\begin{matrix}0\\ \vdots\\0\\e_2\\e_1\\0\end{matrix}\right\}+X_1$$

and

$$X_k=\left\{\begin{matrix}e_1&e_2&0&\dots&0&0&0&0&0&0\\0&e_1&e_2&&&&&&&\\0&0&e_1&&&&&&&\\ \vdots&\vdots&\vdots&&e_2&0&0&&&\\0&0&0&&e_1&e_3&e_2&0&0&0\\0&0&0&\dots&0&0&0&e_1&e_2&e_3\end{matrix}\right\}.$$

Furthemore $X_k=X_{k+1}=\dots$ and thus $Mic\ \bar{Y}=X_k^{\perp}$.

Let us rewrite (4.2) in an equivalent form

$$R_{\hat{B}}^{\perp}\supset(Mic\ \bar{Y}+R_{\hat{C}})^{\perp}=(Mic\ \bar{Y})^{\perp}\cap R_{\hat{C}}^{\perp}$$

Here

$$R_{\hat{B}}^{\perp}=\left\{\begin{matrix}e_3&0&0&0\\0&I&0&0\\0&0&I&\cdot\\ \vdots&\vdots&\vdots&\vdots\\0&0&0&I\end{matrix}\right\}.$$

It is readily seen from above that (4.2) cannot be satisfied since a vector

$$\begin{bmatrix}e_1\\0\\ \vdots\\0\end{bmatrix}\in(Mic\ \bar{Y})^{\perp}\cap R_{\hat{C}}^{\perp}$$

is not in $R_{\hat{B}}^{\perp}$ and this implies, by Theorem 4.1, that the system considered is not controllable on $[0, k]$. Since k is an arbitrary integer so there is no a $t_1>0$ such that the system is controllable on $[0, t_1]$.

Example 2. Assume the matrices A, B, C of the system (1.1) are as follows

$$A=\begin{bmatrix}0&0\\0&1\end{bmatrix},\quad B=\begin{bmatrix}0&1\\-1&0\end{bmatrix},\quad C=\begin{bmatrix}1\\0\end{bmatrix}.$$

Consider the controllability on $[0, t_1]$, $2<t_1<3$. Here, by Lemma 2.2

$$Mic\ \bar{Y}=\begin{bmatrix}e_1&e_2&0\\0&0&e_2\\0&0&0\end{bmatrix},\quad Mic\ Y_1=\begin{bmatrix}e_2\\0\\0\end{bmatrix}.$$

This can be also verified directly using the definition of maximal controlled invariant.

Clearly $R_{\hat{B}} \subset Mic\ Y_1 + R_{\hat{C}}$ so that (5.4) is fulfilled. By the construction of the proof of Lemma 5.3

$$\hat{D}=\begin{bmatrix}0 & 0 & 0\\ D & 0 & 0\\ 0 & D & 0\end{bmatrix} \in R^{3\times 6}, \qquad D=[0, -1],$$

$$\bar{A}=\hat{A}+\hat{C}\hat{D}=\begin{bmatrix}A & 0 & 0\\ B_1 & A & 0\\ 0 & B_1 & A\end{bmatrix},$$

where

$$B_1=\begin{bmatrix}0 & 0\\ -1 & 0\end{bmatrix}.$$

One can choose $\bar{B}=\hat{B}$ or more simply

$$\bar{B}=\begin{bmatrix}-e_2 & 0\\ 0 & 0\\ 0 & 0\end{bmatrix} \in R^{6\times 2}.$$

It may be checked easily that $\bar{A}\bar{B}=\bar{B}$ which implies $\mathscr{A}(\bar{B})=\{\bar{B}\}$.

Now compute

$$\hat{K}\,e^{\bar{A}}=\begin{bmatrix}0 & 0 & 0\\ (e-1)\,A+I & 0 & 0\\ (e-1)\,B_1 & (e-1)\,A+I & 0\end{bmatrix} \in R^{6\times 6}.$$

taking into account the relations

$$B_1^2=0, \quad A^2=A, \quad AB_1+B_1A=B_1, \quad \bar{A}^2=\bar{A}.$$

Subsequently

$$Mcs\ Y_1=\mathscr{A}(Mic\ Y_1 \cap R_{\hat{C}})=0,$$

$$Mcs\ \bar{Y}=\mathscr{A}(Mic\ \bar{Y} \cap R_{\hat{C}})=\mathscr{A}\left(\begin{bmatrix}e_1\\ 0\\ 0\end{bmatrix}\right)=\left\{\begin{matrix}e_1 & 0\\ 0 & e_2\\ 0 & 0\end{matrix}\right\}.$$

Hence it may be verified that the subspace $E\,(Mic\ \bar{Y} \times Mic\ Y_1)+Mcs\ \bar{Y} \times \hat{K}\,Mcs\ Y_1$ includes the subspace $\mathscr{A}(\bar{B}) \times (Y_0+\hat{K}\,\mathscr{A}(\bar{B}))$. In fact, it follows from above that

$$\mathscr{A}(\bar{B}) \times (Y_0+\hat{K}\,\mathscr{A}(\bar{B}))=\left\{\begin{matrix}e_2 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & e_1 & e_2 & 0\\ 0 & 0 & 0 & e_2\\ 0 & 0 & 0 & 0\end{matrix}\right\} \subset R^{12},$$

$$\begin{bmatrix}I & I\\ I & \hat{K}\,e^{\bar{A}}\end{bmatrix}(Mic\ \bar{Y} \times Mic\ Y_1)=\left\{\begin{matrix}e_1 & e_2 & 0 & e_2\\ 0 & 0 & e_2 & 0\\ 0 & 0 & 0 & 0\\ e_1 & e_2 & 0 & 0\\ 0 & 0 & e_2 & e\cdot e_2\\ 0 & 0 & 0 & 0\end{matrix}\right\},$$

$$Mcs\ \bar{Y} \times Mcs\ Y_1=\left\{\begin{matrix}e_1 & 0\\ 0 & e_2\\ 0 & 0\\ 0 & 0\\ 0 & 0\\ 0 & 0\end{matrix}\right\},$$

where $0 \in R^2$ is the zero vector.

One can check easily that the vectors of $\mathscr{A}(\bar{B}) \times (Y_0+\hat{K}\,\mathscr{A}(\bar{B}))$ are linear combinations of the vectors from $E\,(Mic\ \bar{Y} \times Mic\ Y_1)+Mcs\ \bar{Y} \times \hat{K}\,Mcs\ Y_1$. Thus (5.5) is satisfied and the system considered is controllable on $[0, t_1]$, $t_1>2$. However it is not controllable on $[0, 2]$ since in this case

$$\dim P\ Mic\ \bar{Y}=\dim\left\{\begin{matrix}e_2\\ 0\end{matrix}\right\}=1<2$$

and the necessary condition (4.13) is not satisfied.

7. Conclusions

Explicit algebraic necessary and sufficient conditions of controllability to zero function for linear constant time-lag systems have been derived. The conditions are fully computable, that is, if one knows the matrices $A, B, \cdot C$ of the system (1.1) then all the terms appearing in the criteria can be easily computed and the conditions can be checked. This has been illustrated by two numerical examples. The criteria given in the paper have rather geometric than purely algebraic form. Nevertheless they are called to be algebraic since they can be easily expressed in an usual rank-matrix form according to the rule

$$R_A \subset R_B \Leftrightarrow \text{rank } B=\text{rank } [B, A].$$

The main results in the paper are also valid for systems with lag $h \neq 1$, $h>0$ provided that some small changes are made. The general scheme for obtaining controllability conditions based on equivalent system (3.3) with condition (3.6) can be applied for systems with many commensurable delays. Moreover, in like manner a wide class of optimal control problems for general nonlinear nonstationary time-lag system can be transformed into problems for some non-delayed system. This approach may have advantage, especially for some fixed time control problems. Although the equivalent non-delayed system has the dimension kn while the time-lag system is of the dimension n and $[0, k]$ is the interval considered, the continuity condition of the type (3.6) are equivalent to $(k-1)\,n$ scalar equations. Thus the true dimension of the problem is preserved. The disadvantage is that the formulation of a new non-delayed problem may be much more complex than of the problem for time-lag system.

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Algebraiczne kryteria sterowalności do funkcji zerowej dla liniowych stacjonarnych układów z opóźnieniem

Rozpatrzono liniowy układ dynamiczny z opóźnieniem opisywanym równaniem $\dot{x}(t)=Ax(t)+Bx(t-1)+Cu(t)$, w którym $x(t)\in R^n$, $u(t)\in R^m$; A, B, C — stałe macierze.

Otrzymano rozwiązanie ogólne problemu sterowalności do funkcji zerowej (zerowego stanu zupełnego) dla tego typu układów. Rozwiązanie jest podane w postaci sprawdzalnych numerycznie algebraicznych kryteriów wyrażonych za pomocą pewnych funkcji macierzy A, B, C. Kryteria te otrzymano stosując nową metodę umożliwiającą zastąpienie problemu sterowania w układzie z opóźnieniem równoważnym problemem w pewnym układzie bez opóźnienia spełniającym dodatkowe więzy równościowe narzucone na stan początkowy i końcowy.

Dla rozwiązania problemu w nowym sformułowaniu podano kilka lematów dotyczących utrzymywalności trajektorii oraz kompensacji zakłóceń w liniowych układach bez opóźnienia.

Алгебраические критерии управляемости к нулевой функции для линейных, стационарных систем с запаздыванием

В работе рассматривается линейная динамическая система с запаздыванием, представляемая уравнением $\dot{x}(t)=Ax(t)+Bx(t)+Cu(t)$, где $x(t)\in R^n$, $u(t)\in R^m$; A, B, C — постоянные матрицы. Получено общее решение проблемы управляемости к нулевой функции. Решение дается в форме явных, алгебраических условий построенных на основе некоторых функций из матриц A, B, C. Критерий управляемости получен при помощи нового метода, позволяющего заменить проблему управления в системе с запаздыванием равносильной проблемой для некоторой системы без запаздываний, в которой начальное и конечное состояние удовлетворяют дополнительное уравнение. Для решения эквивалентной проблемы управления предложено несколько теорем о способности удержания траектории и компенсации возмущений в линейных системах без запаздывания.