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Adaptive stochastic approximation and sample mean algorithms for tracking

by

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The stochastic approximation and sample mean with constant memory algorithms are used for tracking a randomly time-varying process in random environment. A mean square error and convergence properties of both methods are investigated. When variances of the measuring noise and the process to be tracked are not known, the method of adaptation is presented. The results of computer simulation are given, as well.

1. Introduction

It is often necessary to obtain estimates of the instantaneous values of time-varying parameters from noisy measurements. This problem arises in tracking of moving targets, identification of nonstationary plants and many situations connected with measuring of time-varying quantities. Solving this problem can be realized with optimal schemes as Wiener or Kalman filters (see e.g. [1] and [2]). However, these schemes require accurate prior statistical information about process and disturbances. When it is impractical or impossible to arrive at this information, sub-optimal and adaptative techniques must be considered [3].

Another way of solving this problem is the use of heuristic algorithms, like this in [4].

This paper describes two adaptive algorithms for tracking; the first one based on the stochastic approximation method, the second one on the sample mean with a constant memory.

2. Stochastic approximation algorithm

Let the tracked process be described by equations (1) and (2):

$$x_{n+1}=x_n+\varepsilon_n, \tag{1}$$

$$y_n=x_n+z_n, \tag{2}$$

where: (1) x_n is the process submitted to increments ε_n, and y_n is its observation masked by noise z_n; (2) $\{\varepsilon_n\}=\varepsilon_n, \varepsilon_{n-1}, \dots, \varepsilon_0$ and $\{z_n\}=z_n, z_{n-1}, \dots, z_1$ are realizations of white, independent, stationary random sequences with zero mean and bounded variances.

The aim of a tracking algorithm is to give as good as possible estimate $\hat{x}_n$ of the instantaneous value of the process x, basing on observation sequence $\{y_n\}=$ $=y_n, y_{n-1}, \dots, y_1$.

The form of equations (1) and (2) generating the process and its observations, suggests the use of stochastic approximation method for tracking. Up to now it has been employed when $\varepsilon_n=0$, $n=1, 2, \dots$ [5], which corresponds to the estimation of a constant parameter, e.g. the mean of the random value.

Furthermore it was generalized on the case of process generated by the equation

$$x_{n+1}=a_n x_n$$

with known coefficients a_n and named dynamic stochastic approximation [6, 7].

Now, we will use the stochastic approximation algorithm for tracking the process (1).

The algorithm has the form

$$\hat{x}_{n+1}=\hat{x}_n+\lambda_n(y_{n+1}-\hat{x}_n) \tag{3}$$

where $\hat{x}_n$ — estimate of x_n (with $\hat{x}_0$ chosen arbitrarily), λ_n — weighting coefficient.

When the constant parameter is estimated, algorithm (3) converges to the true value in the mean square, under the following conditions [8]

$$\lim_{n\to\infty} \lambda_n=0, \qquad \sum_{n=0}^{\infty} \lambda_n=\infty. \tag{4}$$

These conditions does not include the condition

$$\sum_{n=0}^{\infty} \lambda_n^2<\infty$$

which is needed for convergence with probability 1.

Now it is reasonable to expect different properties of the algorithm (3) and different conditions on $\{\lambda_n\}$. Each term of the sequence $\{\lambda_n\}$ weighs the effect of a new observation on the instantaneous value of the estimate. Particularly: if $\lambda_n=1$, then all the previous but last observations are ignored and $\hat{x}_{n+1}=y_{n+1}$; if $\lambda_n=0$, then the last observation is ignored and $\hat{x}_{n+1}=\hat{x}_n$.

The tracking of the process is impossible without taking into account new observations, so it should be

$$\lim_{n\to\infty} \lambda_n>0$$

which is in contrary to (4).

Thus, let us examine the properties of algorithm (3). Define the tracking error after n-th step by

$$\Delta x_n \triangleq x_n-\hat{x}_n \tag{5}$$

and substituting the expressions (1) and (2) into (3) and (5) we obtain

$$(\Delta x_{n+1})^2=(\Delta x_n+\varepsilon_n)^2(1-\lambda_n)^2-2\lambda_n(1-\lambda_n)(\Delta x_n+\varepsilon_n)z_{n+1}+\lambda_n^2 z_{n+1}^2.$$

Making use of assumptions on ε and z and taking expectation of both sides of the above expression we have the recursive relation for the mean square error of the estimates

$$b_{n+1}=(b_n+\sigma_\varepsilon^2)(1-\lambda_n)^2+\lambda_n^2\sigma_z^2, \tag{6}$$

where

$$\sigma_\varepsilon^2 \triangleq E\varepsilon^2; \qquad \sigma_z^2 \triangleq Ez^2; \tag{7}$$

$$b_n \triangleq \underset{\{z_n\},\{\varepsilon_n\}/b_0}{E}(x_n-\hat{x}_n)^2; \quad b_0 \triangleq \underset{x_0}{E}(x_0-\hat{x}_0)^2. \tag{8}$$

Setting the first derivative of b_{n+1} with respect to λ_n equal to zero, and solving for λ_n we obtain the sequence $\{\lambda_n^o\}$ that minimizes (6). The result is

$$\lambda_n^o=\frac{b_n+\sigma_\varepsilon^2}{b_n+\sigma_\varepsilon^2+\sigma_z^2}. \tag{9}$$

Substituting the expression of (9) into (6) and setting

$$b_{n+1}=b_n=b_\infty^o$$

one obtains the minimized limiting mean square error

$$b_\infty^o=\frac{\sigma_\varepsilon^2}{2}\left(\sqrt{1+4\frac{\sigma_z^2}{\sigma_\varepsilon^2}}-1\right) \tag{10}$$

and the optimal limiting, constant weighting coefficient

$$\lambda_\infty^o=\frac{\sigma_\varepsilon^2}{2\sigma_z^2}\left(\sqrt{1+4\frac{\sigma_z^2}{\sigma_\varepsilon^2}}-1\right). \tag{11}$$

It should be noted that if $\sigma_\varepsilon^2\neq 0$, the optimal sequence $\{\lambda_n^o\}$ goes to nonzero limiting value as n increases without limit, which confirme our previous expectations. That is, algorithm (3) with sequence $\{\lambda_n\}$ given by (9) is an optimal (in the mean square sense) stochastic approximation algorithm for tracking the process (1).

It appears that this algorithm is analogical to the Kalman filter obtained for equations (1) and (2). The only difference is that the Kalman filter is really a one-step predictor while algorithm (3) realizes pure filtration. However, if we design the Kalman filter for pure filtration (see e.g. [1]) it becomes identical with the stochastic approximation algorithm (3).

The identity of the optimal stochastic approximation algorithm and the Bayesian estimator was shown in [9] in the case of estimation of a constant value ($\sigma_\varepsilon^2=0$).

It is seen from (9), that the optimal algorithm (3) is a nonstationary one. However, it tends to the stationary (time-invariant) algorithm, with constant weighting coefficient given by (11). This stimulates the following question then, how will the algorithm with a weighting coefficient constant from the beginning be working, and what will the optimum value of such a coefficient be?

Substituting $\lambda_n = \lambda = \text{const.}$ and $b_{n+1} = b_n = b_\infty$ in (6), and solving for λ we obtain

$$b_\infty = \frac{\sigma_\varepsilon^2 (1-\lambda)^2 + \lambda^2 \sigma_z^2}{2\lambda - \lambda^2} \qquad (12)$$

It can easily be shown, that expression (12) has a minimum given by (10) with λ given by (11). It means that the limiting mean square errors of the optimal stationary and the optimal nonstationary algorithms are equal.

3. Sample mean constant memory (SMCM) algorithm

Now, let us examine the properties of the simple algorithm

$$\hat{x}_n = \frac{1}{N} \sum_{i=n-N+1}^{n} y_i \qquad (13)$$

which computes the sample mean of N last observations. If the number of observations exceeds N then the mean square estimation error given by this algorithm is

$$b = \frac{(N-1)(2N-1)}{6N} \sigma_z^2 + \frac{1}{N} \sigma_\varepsilon^2. \qquad (14)$$

It is seen from the above expression that the m.s.e. consists of two components. The first one, dependent on process variation, and therefore proportional to σ_z^2. The latter one, dependent on the measurement noise, and proportional to σ_ε^2. The increase of N results in a decrease of the effect of the measurement noise, but at the same time decreases the ability of the SMCM algorithm to follow the changes of the process. Thus, the optimal value of N, that minimized (14), would be a compromise between the noise damping and the ability of tracking the process. It occurs for

$$N = \sqrt{3\frac{\sigma_\varepsilon^2}{\sigma_z^2} + \frac{1}{2}}. \qquad (15)$$

Expression (15) gives the solution for N in real numbers, however, while the algorithm (13) requires natural numbers for N. Thus, as an optimal value N^0 of N, this natural number neighbouring with solution (15) should be chosen, for which the error (14) is smaller.

In order to compare the quality of the algoritms (13) and (3), let us allow for a moment that N^0 could be a real number given by (15). Substituting (15) to (14) we have

$$b^o = \frac{\sigma_\varepsilon^2}{2}\left(\frac{4}{3}\sqrt{\frac{1}{2} + 3\frac{\sigma_z^2}{\sigma_\varepsilon^2}} - 1\right). \qquad (16)$$

It is easily seen, when one compares (16) and (10) that $b^o \geqslant b^o_\infty$ for $N \geqslant 1$.

Plots of the m.s.e. versus $\alpha \triangleq \sigma_z^2/\sigma_\varepsilon^2$, which depicts the functional relations of (10) — curve *1*, and (14) (with natural N^0) — curve *2*, are shown in Fig. 1. The curve *2* is a piece-wise linear, tangent to the unshown plot of the error (16) in points,

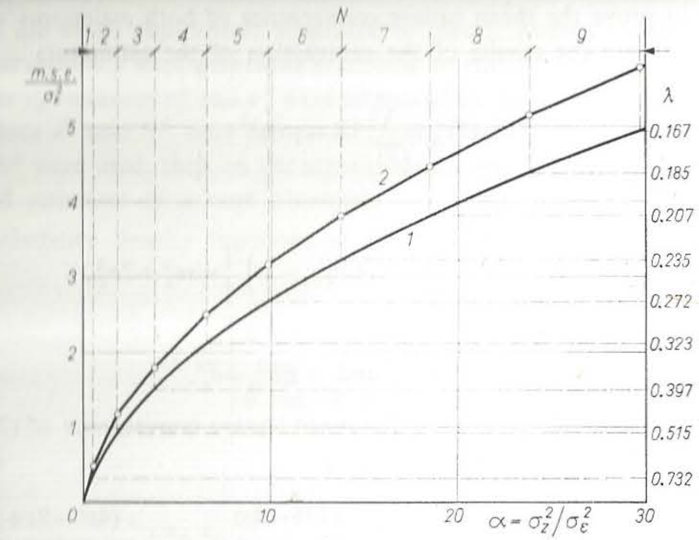

Fig. 1. Mean square error of the TISA (*1*) and SMCM (*2*) algorithms. The points marked by circles indicate where the optimal SMCM algorithm changes N to $N+1$

for which N given by (15) is a natural number. The points marked by circles correspond to the values of α for which setting of two succesive natural numbers for N in (14) gives the same value of error. These values of α are given by

$$\alpha_N = \frac{\sigma_z^2}{\sigma_\varepsilon^2} = \frac{2N^2 + 2N - 1}{6}; \quad N = 1, 2, 3, ..., \qquad (17)$$

and indicate when the optimal algorithm changes N to $N+1$.

4. Adaptation

The optimization of the time-invariant stochastic approximation (TISA) and SMCM algorithms requires knowledge of α. If α is not known there is a possibility of estimating it, on the base of observations y_n. A priori unknown value of α is then being replaced by actual estimates $\hat{\alpha}_n$ in the tracking algorithms.

Let us consider an estimation of α by independent estimation of σ_z^2 and σ_ε^2 with the following expressions

$$\hat{\sigma}_{zn}^2 = \frac{1}{n(k-l)} \sum_{i=1}^{n} [(y_{i+k} - y_i)^2 - (y_{i+l} - y_i)^2], \qquad (18)$$

$$\hat{\sigma}^2_{zn}=\frac{1}{2n(k-l)}\sum_{i=1}^{n} k(y_{i+l}-y_i)^2-l(y_{i+k}-y_i)^2, \tag{19}$$

where k, l — natural numbers chosen arbitrarily, and $k>l$.

In order to prove the mean square convergence of both estimators one should notice that they are the results of the subtraction of the estimators

$$\hat{e}_{sn}=\frac{1}{n}\sum_{i=1}^{n}(y_{i+s}-y_i)^2. \tag{20}$$

The mean value of the $\hat{e}_{sn}$ is

$$E\hat{e}_{sn}=E\left\{\frac{1}{n}\sum_{i=1}^{n}\left(\sum_{j=i}^{i+s-1}\varepsilon_j+z_{i+s}-z_i\right)^2\right\}=s\sigma_\varepsilon^2+2\sigma_z^2. \tag{21}$$

Hence

$$E\hat{\sigma}^2_{\varepsilon n}=\sigma_\varepsilon^2 \quad \text{and} \quad E\hat{\sigma}^2_{zn}=\sigma_z^2. \tag{22}$$

Now, it will be sufficient to show the mean square convergence of (20) to (21). After many transformations we have

$$E(\hat{e}_{sn}-E\hat{e}_{sn})^2=\frac{1}{n}\left[s^2 E\varepsilon^4+\frac{6+s}{3}Ez^4+\frac{s(36-2s)}{3}\sigma_z^2\,\sigma_\varepsilon^2+\frac{s(4s^2-9s+22)}{3}\sigma_\varepsilon^4\right]+$$
$$+\frac{1}{n^2}\left[\frac{s(s^2+1)}{3}E\varepsilon^4-2sEz^4-\frac{s(s^3-3s^2-s+3)}{3}\sigma_\varepsilon^4+2s\sigma_z^4\right]. \tag{23}$$

Thus, if the probability densities $p(\varepsilon)$ and $p(z)$ are symmetric ones, and all the moments arising in (23) are bounded, then

$$\lim_{n\to\infty} E(\hat{e}_{sn}-E\hat{e}_{sn})^2=0. \tag{24}$$

On the base of (22) and (24) one can conclude that the estimators (18) and (19) are convergent to σ_ε^2 and σ_z^2 accordingly, in the order of $1/n$.

The expressions for the variances of the estimators (18) and (19) are very similar to that of (23). They are functions of numbers k, l and 2-nd and 4-th moments of variables ε and z. However, the minimization of these variances with respect to k and l (or, in other words, the increase of the estimation efficiency) is not possible because of lack of any prior information about the moments mentioned above.

It should still be noted, that computing the estimates (18), (19) can be obtained via equivalent recursive formulas

$$\hat{\sigma}^2_{\varepsilon(n+1)}=\hat{\sigma}^2_{\varepsilon n}+\frac{1}{n}\left[\frac{(y_{n+k}-y_n)^2-(y_{n+l}-y_n)^2}{k-l}-\hat{\sigma}^2_{\varepsilon n}\right], \tag{25}$$

$$\hat{\sigma}^2_{z(n+1)}=\hat{\sigma}^2_{zn}+\frac{1}{n}\left[\frac{k(y_{n+l}-y_n)^2-l(y_{n+k}-y_n)^2}{2(k-l)}-\hat{\sigma}^2_{zn}\right]. \tag{26}$$

8. Numerical examples

We shall illustrate the work of the adaptive tracking algorithms by giving a numerical example. The example consists of a computer simulation as follows.

Making use of an algorithmic generator of pseudorandom numbers, process x and its observations y were generated according to equations (1) and (2). Basing on observations y_n variances σ_ε^2 and σ_z^2 were estimated by the algorithms (25), (26). Then, optimal values λ^0 and N^0 were computed from (11), (15) and (17). Current values of λ^0 and N^0 were used, then, in the algorithms (3) and (13), from which the TISA and SMCM estimates of x were obtained.

The probability density functions of the ε and z were assumed to have the form of

$$p(\varepsilon)=\begin{cases}\sqrt{6} \text{ for } -\frac{1}{2\sqrt{6}}<\varepsilon<\frac{1}{2\sqrt{6}},\\ 0 \text{ out of this interval}\end{cases}$$

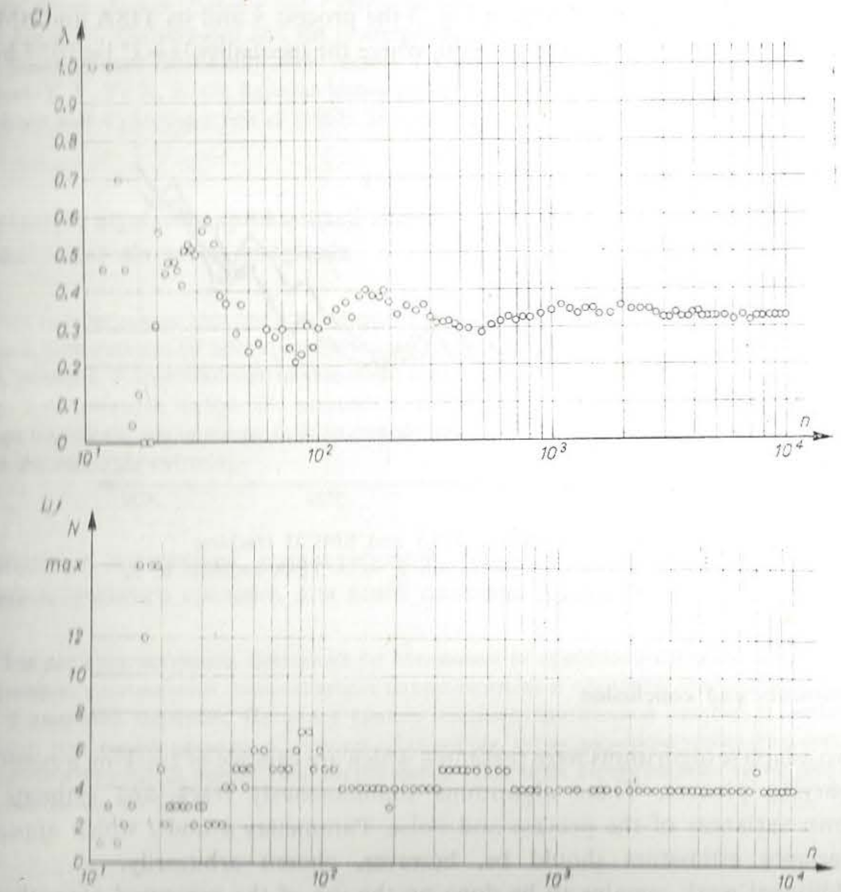

Fig. 2. Typical plots of a convergence of λ and N to its optimal values

$$p(z)=\begin{cases}1 & \text{for } -0.5<z<0.5\\ 0 & \text{out of this interval}\end{cases},$$

Since, for these densities we have $\alpha=\sigma_z^2/\sigma_\varepsilon^2=6$, then $\lambda^0=1/3$ and $N^0=4$. In the estimation algorithms (25), (26) $k=10$ and $l=5$ were taken.

Before obtaining the first values of the estimates $\hat{\sigma}_\varepsilon^2$ and $\hat{\sigma}_z^2$ (i.e. before the moment of $k+1=11$) in the tracking algorithms $\lambda=1$ and $N=1$ were taken.

Next, in order to ensure a correct work of these algorithms, it was assumed that

$$\text{if } \hat{\sigma}_z^2\leqslant 0 \text{ then } \lambda=1,\ N=1, \text{ and}$$

$$\text{if } \hat{\sigma}_z^2>0 \quad \text{and} \quad \hat{\sigma}_\varepsilon^2\leqslant 0 \quad \text{then} \quad \lambda=0,\ N=N_{\max},$$

where $N_{\max}$ denotes the capacity of the register the observations y_n were stored in.

As a result of simulation, the plots of λ and N versus the number of iterations are shown in Fig. 2. It is evident that they approach the optimal values given above.

For the purpose of illustrating how the both tracking algorithms work in the optimal regime ($\lambda\approx\lambda_\infty^o$ or $N=N^o$), in Fig. 3 the process x and its TISA and SMCM estimates are shown, for $n=3000$ to 3030, where the inequality $|\lambda-\lambda_\infty^o|<10^{-3}$ holds.

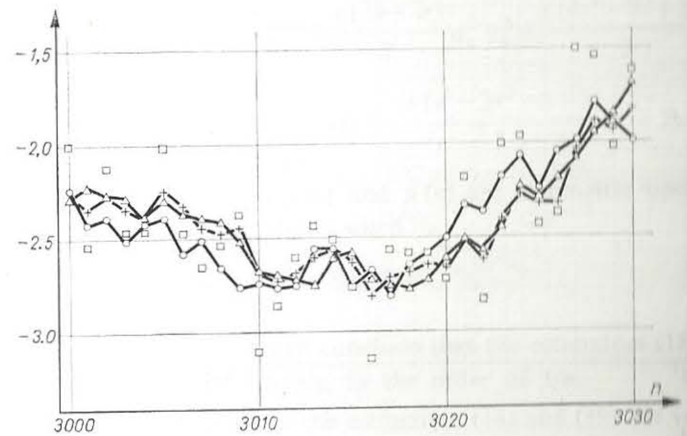

Fig. 3. Example of optimal TISA and SMCM tracking
○ — process x, □ — observation y, + — TISA estimate of x,
△ — SMCM estimate of x

4. Comments and conclusion

Two adaptive algorithms were presented which are capable of tracking a randomly time-varying process. These algorithms simultaneously track and estimate the unknown variances of the process and noise. Parameters k and l which appear in the variance estimators should be, however, chosen arbitrarily.

Additional work remains to be done on the use of the presented algorithms in the situation when the random sequencies $\{\varepsilon_n\}$ and $\{z_n\}$ are nonstationary ones.

Recent research indicates that if the variances of ε and z are changing randomly, it can simply be done by replacing $\gamma=1/n$ with $\gamma=$const. in variance estimators (25) and (26).

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Adaptacyjne algorytmy aproksymacji stochastycznej i średniej arytmetycznej dla celów nadążania

Przy rozwiązywaniu zagadnień związanych z nadążaniem za procesem przypadkowo zmiennym w czasie, wykorzystuje się aproksymację stochastyczną i algorytm średniej arytmetycznej ze skończoną pamięcią. Przeprowadzono analizę błędu średniokwadratowego oraz własności zbieżności dla każdej z omawianych metod. Dla przypadku, gdy wariancje szumu pomiarowego i procesu, za którym nadąża się, nie są znane, podano metodę adaptacji. Przedstawiono również wyniki modelowania na maszynie cyfrowej.

Адаптивные алгоритмы стохастической аппроксимации и арифметического среднего для цепей слежения

При решении вопросов, связанных со слежением за процессом случайно изменяющимся во времени, используется стохастическая аппроксимация и алгоритм арифметическото среднего с конечной памятью. Проведен анализ среднеквадратической ошибки и свойств сходимости для рассматриваемых методов. Для случая, когда дисперсии измерительного шума и слеженного процесса неизвестны, дается метод адаптации. Представлены также результаты моделирования на ЦВМ.