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Determination of a generalized inverse of a Boolean relation matrix

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A criterion for the existence of a generalized inverse of a Boolean relation matrix is derived and a way of determining the generalized inverses is given.

1. Introduction and basic concepts

In his papers [3] and [4] Plemmons has motivated the research for generalized inverses of a Boolean relation matrix by applications in network and switching theory [5, 6], in nonnegative generalized inverses of matrices over the reals [4], and in the general theory of graphs [5]. In this paper we shall give a criterion for the existence of a generalized inverse, analogous to that of the existence of a solution for a Boolean matrix equation $AX=B$. Further, a way of determining all the generalized inverses of a Boolean relation matrix is given. The results here are based on the ideas of the paper [2].

By a Boolean relation matrix of order n is meant an $n \times n$ matrix of zeros and ones. The product, join and meet of such matrices are defined as in case of Boolean matrices of zeros and ones, see e.g. [1]. Any solution of the Boolean relation matrix equation

$$A=AXA \tag{1}$$

is called a generalized inverse of the given matrix A.

Any Boolean relation matrix $B=[b_{ij}]$ can be mapped onto a bipartite graph $G(B)=(V_B \cup V'_B, E_B)$, where the vertices of V_B correspond to the rows of B and those of V'_B to the columns of B. An undirected edge (x, y') belongs to $G(B)$ if and only if $b_{ij}=1$ in B, where i corresponds to x and j to y'. Conversely, any bipartite, undirected graph, for which the numbers of elements in V_B and V'_B equal, i.e. $|V_B|=|V'_B|$, can be translated into a Boolean relation matrix.

Consider the product $B_1 B_2$ of two Boolean realtion matrices B_1 and B_2. This product can be mapped onto a chain of bipartite graphs $G(B_1)$ and $G(B_2)$, denoted by $G(B_1)\,G(B_2)$, where the vertex sets V_1' and V_2 of $G(B_1)$ and $G(B_2)$, respectively, coincide. Let $B_1\,B_2=B_3=[b_{ij}^3]$. According to the definition of the Boolean matrix product, $b_{ij}^3=1$ if and only if there is a path of length two from a vertex $x \in V_1$ of $G(B_1)$ to a vertex $y' \in V_2'$ of $G(B_2)$, where i corresponds to x and j to y'. As an illustration, see the product $B_1\,B_2=B_3$ described in Figure 1, when

$$B_1=\begin{array}{cc} \begin{matrix} x' & y' & z' \end{matrix} & \\ \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} & \begin{matrix} x \\ y, \\ z \end{matrix} \end{array} \quad B_2=\begin{array}{cc} \begin{matrix} x' & y' & z' \end{matrix} & \\ \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} & \begin{matrix} x \\ y, \\ z \end{matrix} \end{array} \quad \text{and } B_3=\begin{array}{cc} \begin{matrix} x' & y' & z' \end{matrix} & \\ \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} & \begin{matrix} x \\ y. \\ z \end{matrix} \end{array}$$

According to the associativity of the Boolean matrix product any product of n Boolean relation matrices, where $n \geqslant 3$ and finite, can be represented as a chain of n bipartite graphs.

2. A criterion

Let A be a given Boolean relation matrix and consider the product AXA. For sake of clarity, we shall denote the first matrix of AXA by B_1, the second by B_2, and the third by B_3. Consider the chain graph $G(B_1)\,G(B_2)\,G(B_3)$ obtained from bipartite graphs $G(B_1)$, $G(B_2)$, and $G(B_3)$ by identifying the vertex sets V_1' and V_2, and the sets V_2' and V_3. The graphical description $G(B_1)\,G(B_2)\,G(B_3)$ of the matrix product AXA implies immediately that there is a generalized inverse for A, i.e. there is a solution for the equation (1), if and only if

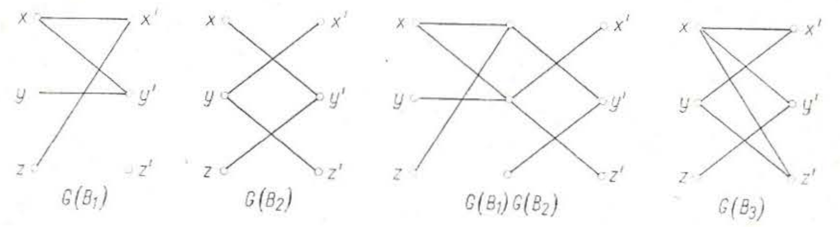

Fig. 1

(i) for any edge $(x, y') \in E_A$ there is at least one path of length three from $x \in V_1$ of $G(B_1)$ to $y' \in V_3'$ of $G(B_3)$ in the graph $G(B_1)\;G(B_2)\,G(B_3)$, and

(ii) for any edge $(z, w') \notin E_A$ there is no path of length three from $z \in V_1$ of $G(B_1)$ to $w' \in V_3'$ of $G(B_3)$ in the graph $G(B_1)\;G(B_2)\;G(B_3)$.

As the Boolean matrix product is distributive with respect to the join operation on matrices, there is a solution X_0 for 1 such that $Y \leqslant X_0$ for each solution Y for (1), if any solutions exist. Clearly $G(X_0)$ contains each edge not contradicting the condition (ii). In the following we shall determine a matrix M, or equivalently a bipartite graph $G(M)$ with $|V_M|=|V_M'|$, containing each edge not contradicting the condition (ii). According to the maximality of M. A has generalized inverse if and only if of a solution for a Boolean matrix equation $AX=B$, or $XA=B$, see e.g. [1].

Let $\Gamma_A\,x$ denote the set of vertices adjacent to x in the graph $G(A)$. In order that (ii) is valid, a vertex $x \in V_2$ $(=V_1')$ can be joined to a vertex $y' \in V_2'$ $(=V_3)$ only if for any $z \in \Gamma_{B_1}\,x$ the relation $\Gamma_{B_1}\,z \supseteq \Gamma_{B_3}\,y'$ holds. In other cases there would be a path of length three in $G(B_1)\,G(B_2)\,G(B_3)$ from a vertex $z_1 \in V_1$ to a vertex $w' \in \Gamma_{B_1}\,y'$, while $(z, w') \notin E_A$. Consider the translation of this condition, which determines the edges of the graph $G(M)$, into a serie of suitable Boolean matrix operations.

The notation D'' means the Boolean complement of the Boolean relation matrix D and D^T the transpose of D. Let us consider the matrix product $A\,(A^T)''=C=[c_{ij}]$. In the chain graph $G(A)\,G\big((A^T)''\big)$ the vertex sets V_A' of $G(A)$ and V_A' of $G((A^T)'')$ coincide. Assume that x corresponds to i and y to j. Then $c_{ij}=1$ if $\Gamma_A\,x \nsubseteq \Gamma_A\,y$, and $c_{ij}=0$, if $\Gamma_A\,x \subseteq \Gamma_A\,y$. Indeed, if $\Gamma_A\,x \subseteq \Gamma_A\,y$, then for any $z' \in \Gamma_A\,x$ the edge (z', y) does not belong to the graph $G\big((A^T)''\big)$ according to the completmentedness, and hence there are in $G(A)\;G\big((A^T)''\big)$ no path of length two from x to y, which implies $e_{ij}=0$. The proof for $c_{ij}=1$ is similar. Note that $B_1=B_3=A$, and thus we have found a matrix form to the condition $\Gamma_{B_1}\,z \supseteq \Gamma v_{B_3}\,y'$.

Consider now the matrix product $A^T\,C^T=A^T\,[A\,(A^T)'']^T=F=[f_{ij}]$. Let $f_{ij}=0=$ $=\bigcup\limits_{s=1}^{s=n} a_{is}^T\,c_{sj}^T$. Then for any $z \in V_A$, if $(x, z) \in E_A$, $(z, y) \notin R_{C^T}$, i.e. $\Gamma_A\,z \supseteq \Gamma_A\,y$, where i corresponds to x and j to y. If $f_{ij}=1$, then for some z, $(x, z) \in E_A$, also $(z, y) \in E_{C}\,r$, i.e. $\Gamma_A\,z \nsupseteq \Gamma_A\,y$. But then, according to the condition for the edge in $G(M)$, an edge $(x, y) \in E_M$ exactly then, when $f_{ij}=0$, and thus we have found the expression $\big(A^T\,[A\,(A^T)'']^T\big)''$ for M. The criterion written formerly in terms of M gives now the theorem

THEOREM 1. A Boolean relation matrix A has a generalized inverse if and only if $A=A\,\big(A^T[A(A^T)'']^T\big)''\,A$.

3. A determination method

In this section we shall follow the lines of the paper [2] without trying to find solution algorithms analogues to those proposed by Rudeanu in [7] and Ledley in [1] for the Boolean matrix equations $AX=B$ and $XA=B$. The way of this paper is appropriate for moderate values of n.

In the following we shall construct the graph $G(M'')$ by a graphical way; note that $B_1=B_3=A$ and $B_2=M''$. Consider a vertex $x \in V_1$. We can immediately determine the vertices in the sets $\{V_1'-\Gamma_{B_1}x\}$ and $\{w|\;\Gamma_{B_3}\,w \cap \{V_3'-\Gamma_{B_3}\,x\} \neq \emptyset\}$. Join any vertex $u \in \Gamma_{B_1}\,x$ to any vertex w and repeat this process for any x of V_1. The bipartite graph $G(B_2)$ of the chain graph $G(B_1)\,G(B_2)\,G(B_3)$ is $G(M'')$, since we have constructed all the edges contradicting the condition (ii) of the previous section and only those, as the construction immediately shows. From Theorem 1 it follows that one cannot from the graph $G(M'')$ conclude the existence of a generalized inverse for A.

The existence of a generalized inverse for A will be tested by verifying the validity of the condition (i) for any edge $(x, y') \in E_A$. This can be performed as follows: Join any vertex $u \in V_3$ $(=V_2')$, for which $y' \in \Gamma_{B_3} u$, to each vertex $z' \in \Gamma_{B_1} x \subset$ $\subset V_2$ $(=V_1')$ and remove all the edges contained in $G(M'')$. If the edge set E_B of the graph $G(B_2)$ in the graph $G(B_1)\, G(B_2)\, G(B_3)$ constructed by this manner is non-empty for aby edge $(x, y') \in E_A$, (i) and (ii) are valid (cf. the removal of the edges in $G(M'')$), and hence a generalized inverse exists. We shall formulate the observations above in a theorem giving a criterion for the generalized inverse for A.

Denote by $Z(x, y')$ the Boolean relation matrix of the edge $(x, y') \in E_A$ determined by the manner reported above.

THEOREM 2. Let A be a given Boolean relation matrix. A has a generalized inverse if and only if the matrix $Z(x, y')$ is non-zero for any edge $(x, y') \in E_A$. Furthermore, if there is a generalized inverse for A', then any Boolean relation matrix $Q \leqslant \bigcup\limits_{(x, y')} Z(x, y')$, where $(x, y'') \in E_A$, is a generalized inverse for A, if $Q \cap Z(x, y')$ is non-zero for any edge $(x, y') \in E_A$.

Proof. The validity of the first part of the theorem was shown previously. According to the construction rules of the graphs $G(Z(x, y'))$, $G(\bigcup\limits_{(x, y')} Z(x, y'))$ does not contain edges contradicting the condition (ii). Since $Q \leqslant \bigcup\limits_{(x, y')} Z(x, y')$, $G(Q)$ has this property as well. As for any $(x, y') \in E_A$ the meet $Q \cap Z(x, y')$ is a non-zero matrix, $G(Q)$ is a graph for which the condition (i) hold, and hence Q is a generalized inverse for A. This completes the proof.

Note that $\bigcup\limits_{(x, y')} Z(x, y') = M$, since only those edges contained in $G(M'')$ were removed by the construction of $G(Z(x, y'))$. Further, if there is a generalized inverse for A. Theorem 2 offers a way of enumerating all the generalized inverses for A.

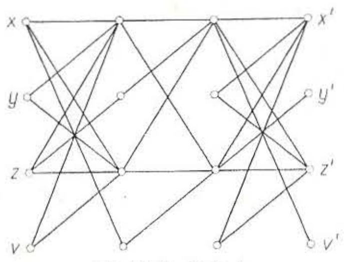

Fig. 2

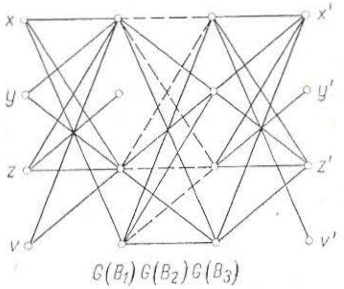

Fig. 3

Finally, consider an example. Let A be a given Boolean relation matrix,

$$A = \begin{array}{c} \begin{array}{cccc} x' & y' & z' & v' \end{array} \\ \left[\begin{array}{cccc} 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{array}\right] \end{array} \begin{array}{c} \\ x \\ y \\ z \\ v \end{array}. \quad \text{Then } M = (A^T [A (A^T)'']^T)'' = \begin{array}{c} \begin{array}{cccc} x' & y' & z' & v' \end{array} \\ \left[\begin{array}{cccc} 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 \end{array}\right] \end{array} \begin{array}{c} \\ x \\ y \\ z \\ v \end{array}$$

which is a generalized inverse for A, as one can readily verify. Figure 2 shows the construction of the graph and let us consider nearer the vertex x. Now $\Gamma_{B_1} x =$ $= \Gamma_{B_3} x = \{x', z', v'\}$, $\{V_1' - \Gamma_{B_1} x\} = \{y'\} = \{V_3' - \Gamma_{B_3} x\}$, and $\{w |\, \Gamma_{B_3} w \cap \{y'\} \neq \emptyset\} =$ $= \{z\}$, which can be easily seen from the figure. According to the construction rule of $G(M'')$, (x, z'), (z, z'), and (v, z') belong to the edge set $E_{M''}$.

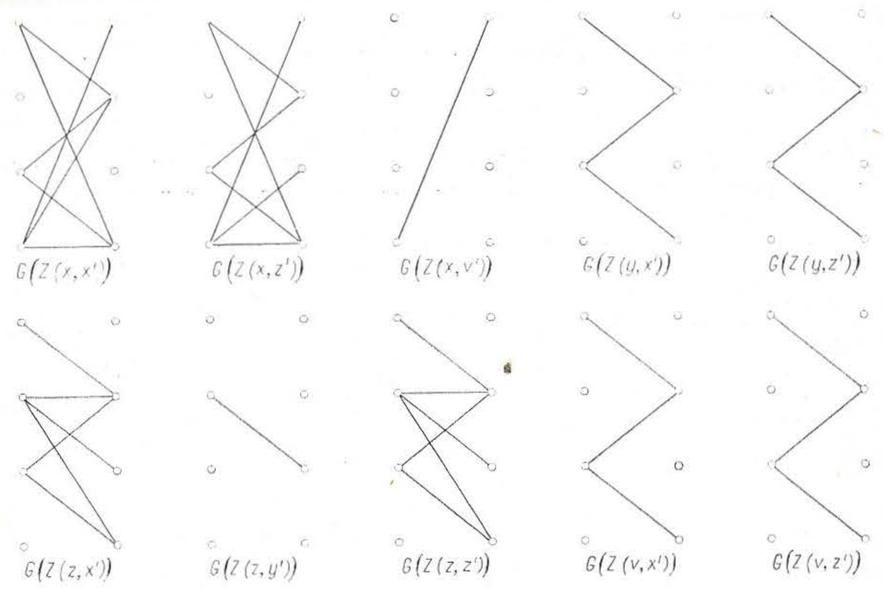

Fig. 4

In Figure 4 all the graphs $G(Z(x, y'))$ are given, and Figure 3 shows the construction of $G(Z(x, x'))$. Now $\Gamma_{B_1} x = \{x', z', v'\}$ and for each vertex t of the set $\{x, y, z, v\}$, $x' \in \Gamma_{B_3} t$. Hence, in $G(B_2)$ any $t \in \{x', y', z', v'\}$ is joined by an edge to any $q \in \{x, y, v\}$. The dotted edges correspond to the edges of $G(M'')$. All the generalized inverse can now be formed according to Theorem 2.

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Wyznaczanie uogólnionej macierzy odwrotnej względem (boolowskiej) macierzy relacji

W pracy podano kryterium istnienia uogólnionej macierzy odwrotnej względem (boolowskiej) macierzy relacji oraz przedstawiono metodę jej wyznaczania.

Определение обобщенной матрицы обратной по отношению к булевой матрице соотношений

В работе дан критерий существования обобщенной матрицы обратной по отношению к булевой матрице соотношений а также представлен метод ее определения.

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