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Optimum delivery problems*

by

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The paper presents a general scheme from production to sale of some goods with the aim of defining the delivery networks are mainly presented. A criterion for optimum choicce of a network is given.

For the daisy networks a particular heuristic algorithm is also shown.

1. Introduction

Our general problem is determining a system for deliv^ry some commodities (or goods) produced by a company.

For sake of simplicity, we suppose that the system can be synthetized as in Fig. 1. The problem consists in the optimization of the distribution network by which the goods are transferred from the central store to various peripherical warehouses.

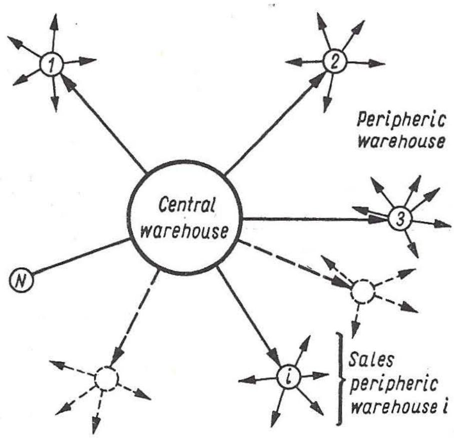

Fig. 1

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We could use also this terminology: there is a warehouse and many customers. In fact, usually the problem has at least two levels: on terminology is appropriated to the first level, the other for the second one.

Generally speaking, problems concerning transports can classified in the following way:

(i) project (or strategical) problems: in our case this means determination of the structure and the parameters concerning the transport fleet;

(ii) management (or tactical) problems: in our case this means the use of the fleet in an optimal way.

It is now to emphasize that the direct formulation of project problems does not appear convenient, since they imply too radical hypothesys about management on the contrary, if the management problems are resolveable with a simple algorithm, the project problems will be solved using such algorithm over and over.

For these reasons, we will work out in this seminar some problems concerning management only.

Moreover, when we wish to solve some problems about a complex sector of the organization of a company using mathematical methods, the procedure, by which we can obtain a real improvement, is to subdivide the system into subsystems, to find out their mathematical description and successively determine the optimal solution. But following this procedure, we can run into the danger of obtain a partial vision of the true problems. For this reason we must bear in mind the interconnection with other sectors of company.

With reference to the specific topics of this seminar, this means that transportation problems cannot be considered completely separated from those concerning production, marketing, or sale estimation, informative systems and so on.

For example, in relation to problems concerning production, this means that the system to be optimized is the one synthetized in Fig. 2, where the state variables are the quantity of goods existing at warehouse and the control variables are the

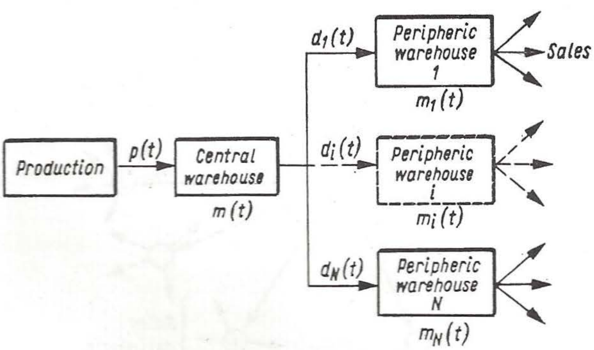

Fig. 2

production flow $p(t)$ and the transport flow $d(t)$, one far any warehouse. Moreover, as another example, we can say that the mention made in reference to sales estimations, implies interpretation in a statistic sense of many of the considerations which follow.

A comprehension of all these problems goes beyond the limits imposed at this seminar strictly dedicated to the problems concerning transport network. We must only note that projects problems become again useful if we have in mind to study the problem with all these interactions.

Meanwhile only the following problems will be examined:

(i) determination of a sub-optimal daisy network (for the terminology used here—see point 3);

(ii) criteria for determination the topology of the optimal or sub-optimal network.

2. Determination of a sub-optimal daisy network

2.1. The problem

As we said before, the problems is this: a warehouse, where a commodity is stored, must supply n customers at various location using vehicles of known capacity.

The objective is to determine the feasible routes of the vehicles that make the routing cost minimum or near minimum. The problem has been widely dealt with in the literature [1, 7, 9]: it is known that the optimal solution by Integer Programming or Dynamic Programming implies heavy computation times and large memory storage, due to the high number of variables to be considered, even for limited size problems. The computational effort become prohibitive in medium or large problems.

This accounts for the development of heuristic methods able to give near-optimal solution by reasonable effort.

For these reasons, some years ago we tried to find a heuristic method [2], which has been tested on various examples: the computational results show that improved solutions as well as competitive computational times are achieved with respect to heuristic methods proposed in literature [5, 6, 8, 10, 13, 14].

2.2. Mathematical formulation

The stated problem is dealt with as a weighted covering problem of the graph $G(N, X)$, where: $N = \{n_k, k = 0, 1, \ldots, n\}$ is a set of nodes which represent the location of the warehouse ($k = 0$) and of the customers ($k = 1, \ldots, n$); $X = x_{ij}$, $(i, j = 1, \ldots, n)$ is a set of edges which represent the route links available for the transport. With the aim to simplify notation only, we suppose the graph is complete. Every node n_k, $k = 1, \ldots, n$, is associated with a positive scalar q_k representing the requirement for commodity of the customer n_k.

Every edge x_{ij} is associated with a positive scalar c_{ij} representing the routing cost on the link $(i-j)$.

The problem is determine the x_{ij} for which we have:

$$\min_{\{x_{ij}\}} \left(z = \sum_{i=0}^{n} \sum_{j=0}^{n} c_{ij} x_{ij} \right)$$

under the constraints:

(1) $x_{ij} = 0$ or 1;
(2) every vehicle covers a tour passing through a subset of nods, including n_0;
(3) all vehicles have the same capacity Q;
(4) the total load allocated to a vehicle cannot exceed its capacity Q;
(5) the vehicle fleet consists of M trunks $\left(QM > \sum_{i=1}^{n} q_i\right)$;
(6) the requirements of all customers must be met;
(7) every customer can be supplied by a single vehicle ($Q > \max q_i$).

In order to describe the algorithm, the following notation are introduced:

$$N^k = \{N_r^k | r = 1, \ldots, R_k\},$$

where $N_r^k \subseteq N$ is the r-th subset of d_r nodes, including n_0 and n_k, which can be supplied by a single vehicle meeting the constraint 4.

A vehicle can supply the customers of the set N_r^k by $d_r!$ feasible routes; let be H_r^k the hamiltonian path defined on the set N_r^k and c_r^k the routing cost on the route H_r^k.

Let be

$H^k = \{H_r^k\}|r = 1, \ldots, R_k\}$,

$\overline{N_r^k} = N-(N_r^k - n_0)$,

X_r^k the subset obtained by deleting in X the elements x_{ij} associated with the links of H_r^k,

H_s^k the feasible route selected by the algorithm to deliver the customer n_k,

c_s^k the routing cost on the route H_s^k,

H_0^k the feasible route supplying the customer n_k in the optimal solution,

c_0^k the routing cost on the route H_0^k.

2.3. The algorithm

The basic idea of the algorithm is the decomposition of the problems is a sequence of sub-problems each aiming to select a feasible route to deliver a given customer. The sequence of sub-problems is stated by an ordering criterion; for example, we can use the ordered set S:

$$S = \{s_k | c_{0k} \geqslant c_{0e}, (1 \leqslant k < e \leqslant n).$$

Let be n_k the first element in S.

According to the principles of Dynamic Programming in turns out:

$$\min \sum\sum_{x_{ij} \in X} c_{ij}x_{ij} = c_0^k + \min \sum\sum_{x_{ij} \in X_0^k} c_{ij}x_{ij}. \tag{1}$$

Consider the function

$$P_N(\lambda) = \lambda \sum_{n_e \in N} c_{0e}, \qquad \lambda > 0.$$

Clearly for every choice N_r^k, $r = 1, \ldots, R_k$, there exists a value of the variable λ, $\lambda = \lambda_{N_r^k}$, such that

$$P_{\overline{N_r^k}}(\lambda^0_{N_r^k}) = \lambda^0_{N_r^k} \sum_{n_e \in \overline{N_r^k}} c_{0e} = \min \sum\sum_{x_{ij} \in X_r^k} c_{ij}x_{ij}.$$

By putting in function $P_{\overline{N}_r^k}(\lambda)$, eqn. (1) becomes

$$\min \sum_{x_{ij} \in X} \sum c_{ij} = \min_{\{N_r^k \in N^k, \lambda\}} \{c_r^k + P_{\overline{N}_r^k}(\lambda)\}.$$

Obviously for a given λ, $\lambda = \bar{\lambda}$, it tourns out

$$\min_{\{N_r^k \in N^k\}} \{c_r^k + P_{\overline{N}_r^k}(\bar{\lambda})\} \geqslant \min_{\{N_r^k \in N^k, \lambda\}} \{c_r^k + P_{\overline{N}_r^k}(\lambda)\}.$$

The proposed method is a heuristic one because the route delivering the customer n_k is selected by the choice criterion

$$\min_{\{N_r^k \in N^k\}} \{c_r + P_{\overline{N}_r^k}(\bar{\lambda})\}. \tag{2}$$

The criterion is based on the underlying assumption that function $P_{\overline{N}^k}(\lambda)$ represents an effective approximation of the minimum routing cost to deliver the set $\overline{N}_r^k$.

The algorithm is a iterative one; in a iteration the route delivering the customer n_k is selected by the choice criterion (2), $\bar{\lambda}$ given $(\lambda = \bar{\lambda})$.

We obtained all the iterations, changing the value of λ $(0 < \lambda < \infty)$.

Let be $G(\lambda)$ the function of the total cost $(G(\bar{\lambda}))$ is the total cost determined by the algorithm, when we use the value $(\lambda = \bar{\lambda})$; Figure 3 shows an example of this function. The function $G(\lambda)$ has two important properties.

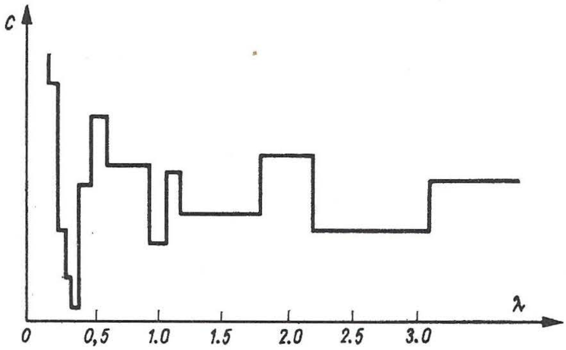

Fig. 3

P r o p e r t y 1. Given two values of λ, λ_1, and λ_2 $(\lambda_1 \neq \lambda_2)$ if

$$N_{s|\lambda=\lambda_1}^k = N_{s|\lambda=\lambda_2}^k \quad \forall n_k \in (N - n_0)$$

it turns out hat

$$N_{s|\lambda=\bar{\lambda}}^k = N_{s|\lambda=\lambda_1}^k = N_{s|\lambda=\lambda_2}^k \quad \forall n_k \in (N - n_0) \quad \forall \bar{\lambda}, \lambda_1 \leqslant \bar{\lambda} \leqslant \lambda_2.$$

P r o p e r t y 2. The exists a value of λ, $\lambda = \lambda_{\max}$, such that

$$N_{s|\lambda=\bar{\lambda}}^k = N_{\bar{s}\lambda=\lambda_{\max}}^k \quad \forall n_k \in (N - n_0) \quad \forall \bar{\lambda}, \bar{\lambda} \geqslant \lambda_{\max}$$

The proof of the properties is obvious.

These two properties reduce very much the number of iterations.

2.4. Observations and generalizations

This algorithm can be considered remarkable for many points of view.

(i) The solution is obtained very often good, mainly in comparison to the solutions found with the other known methods.

(ii) The time requirement is of the same order of magnitude of the other methods:

(iii) This algorithm can be applied also to other problems of large scale systems [11]. The succes dipends on a good choice of the estimation function (like function $P(\lambda)$ in this case).

(iv) We can consider this algorithm as one belonging to a new general class of algorithms [3].

We can use the choice criterion (2), changing at any step the values of λ. The total cost G is now a function $G\ (\lambda_1, \lambda_2, \ldots, \lambda_M)$:, the problem is to find

$$\min_{\{\lambda_i\}} G(\lambda_1, \lambda_2, \ldots, \lambda_M).$$

We can use either a branch and bound method, or some approximated methods, like those shown before.

In [11] and [12], for example, a method is shown, which gives better results in comparison to the algorithm shown before.

3. Criteria for the determination of the topology of the optimal or near-optimal network [4]

Before considering the general lines of the problem, it would be advisable to introduce a suitable terminology. We define:

Radial transport: a transport suppling only a customer;

radial network: a set of radial transports that entirely supply all customers;

petal transport: a transport suppling more than one customer;

daisy network: a set of petal transports that entirely supply all customers (see the problem shown before).

The network is named daisy, even in the case when the algorithm of the optimization determine radial etransport solution for some customers.

A double mixed network: is a set of transports which determine a radial network and a daisy one; in case of double mixed network a radial and a petal transport supply all customers.

It is easy to verify that some companies use daisy networks, some other radial ones; double mixed networks are generally not used. On the other hand, it will be easy to observe that these are preferable for particular conditions. To my knowledge, there are no criterion in literature about the choice of the topology; for these reason it could be of some interest to make some considerations about this problem.

It is remarkable to note that the use of double mixed network is strictly connected to some errors concerning sale estimations especially with regard to some type of commodity (for this reason the best results are obtained considering disaggregated [4] instead of aggregated commodities as in the following assumption).

Now we will determine the optimal solution only regarding costs, disregarding considerations about warehouses and transports capacity.

The are two kinds of costs:

(a) A cost G^t taking into account transports only.

(b) A cost G^* taking into account stock breacks, goods unselled and warehouse costs.

Let be

$$\mu = Q^r/Q,$$

where Q^r is commodity transported by the radial network and Q the total commodity.

It is easy to understand the shape of the functions $G^t(\mu)$ and $G^*(\mu)$, shown in Fig. 4, where $G^{t,r}$ and $G^{t,m}$ are the cost of the radial or daisy network respectively. For understanding the shape of the function $G^*(\mu)$, we must notice that daisy trans-

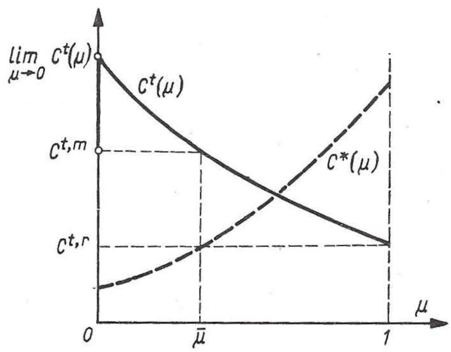

Fig. 4

port network allow us to follow in a better way the sale trend; $G^t(\mu)$ is a decreasing function, with a discontinuity in the point $\mu = 0$, because for any value of μ different from zero, we must use a double mixed network and that means many not fully utilized transports.

The optimization problem is now trivial and it is not important to discuss it here, but it is important to note that for some values of parameters, the double mixed network could be the best.

4. Conclusions

The aim of the paper is showing two different kinds of problems regarding delivery: one is determining appropriate algorithms (see point 2), once the question is put in a clear form; the second is just having a general view of the whole sector (see point 1) or trying to understand the reason of some management practice, with the aim of finding a correct riformulation of the problem (see point 3).

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Zagadnienia optymalnej dostawy

Przedstawiono ogólny schemat przepływu dóbr od produkcji do sprzedaży i zdefiniowano sieci dostawcze. Podano kryterium optymalnego wyboru sieci. Dla sieci podstawowych podano pewien algorytm heurystyczny.

Вопросы оптимальных поставок

Представлена общая схема прохождения материальных ценностей от производства к продаже и дано определение сети поставок. Приведен критерий оптимального выбора сети. Для основных сетей дан некоторый эвристический алгоритм.