Control
and Cybernetics
VOL. 2 (1973) No. 3/4

The rate of profits in a model of economic growth*

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A model of full employment equilibrium growth is considered, in which the productions of the different sectors (goods) may grow at different rates, owing to changes in the consumption structure.

All productive sectors are considered as vertically integrated in the sense that all intermediate transactions are considered as internal aspects of a production process that leads from the original inputs (labour and capital goods) to the final products[1]. For simplicity, the model considers only two consumption sectors and only two capital goods sectors. It is assumed that capital goods are required only for the production of consumption goods and not for the production of themselves. Hence labour is the only input in the production of capital goods. It is also assumed that the capital goods, once produced, last for ever.

The techniques of production are defined by means of a set of labour input coefficients which, owing to the technological progress, change over time. It is assumed that it is possible to approximate the time paths of the labour coefficients by a step-wise function.

1. Most of the growth models to be found in the literature are concerned with an economic system in which either only one sector is considered or all different sectors grow at the same rate. This is a serious shortcoming. In fact, it is well known that when, owing to technical progress, income per head increases, the consumption of different goods does not increase at the same rate.

The model we shall consider in this paper allows for the possibility of different sectors growing at different rates. It derives logically from a model presented by

* Presented at Polish-Italian Meeting on "Modern applications of mathematical systems and control theory, in particular to economic and production systems", Cracow, Poland, 14–20 Sept. 1972.

Most of the material of this paper has been published, in a much more complete form, in T. Cozzi: Sviluppo e stabilità dell'economia (Torino 1969) Chpts. I and II, and constitued a part of the author's Ph. D. thesis at the University of Cambridge. However, the mathematical notation has been changed and some new results are here presented for the first time in sections 3–4.

[1] L. L. Pasinetti: A new theoretical approach to the problems of economic growth, in: Semaine d'étude sur le rôle de l'analyse économetrique dans la formulation de plans de developpement (Vatican City 1965). In fact, our model shares most of the assumptions made by Pasinetti and should be considered as an attempt at probing more deeply into some of the problems studied by him.

Pasinetti[2]. In fact, both models have the same structure and share most of the assumptions. In particular, they both assume that all productive sectors are *vertically integrated*, which means that only final goods are considered in the model. All *intermediate* goods are considered as internal moments in a production process that leads from the original factors to the final products. This procedure requires a peculiar definition of the input coefficiens defining the production processes. Suppose, for example, that in the final stages of production a certain commodity is produced by means of a certain amount of labour and certain amounts of intermediate goods. These amounts of intermediate goods have required, to be produced, certain amounts of labour and certain amounts of intermediate goods, and the latter in their turn were also produced by certain amounts of labour and goods, and so on. To avoid taking account of all these intermediate operations, it can be supposed that they have all taken place within a single sector which has ultimately produced the final commodity by the employement of all the amounts of labour that have gone into it either *directly* (in the final stage of production) or *indirectly* (in all the previous stages — these clearly infinite in number). The same argument can be repeated for the amounts of capital goods both directly and indirectly required for the production of the final good.

Thus the device of considering production sectors as vertically integrated enables us to represent the amounts of all the inputs that go directly or indirectly into a production process by the amounts of services provided by two types of factors of production, namely labour and capital.

As concerns labour, the model assumes that it is homogeneous as between different uses. The amount of labour utilized by the different sectors is measured in physical units, e.g. the number of workers or the number of working hours.

Capital — the total amount of capital goods utilized in the different sectors — will be measured in physical terms as well. As Pasinetti suggests, there is no need to adopt a unique unit of measurement for the capital goods utilized in different sectors. Instead it will be assumed that the capital utilized in each sector is measured in terms of a unit peculiar to itself, viz. in terms of sectoral *productive capacity*. For example, a unit of capacity for the i-th sector (production of corn) will consist of the whole amount of different capital goods needed to produce a unit of the i-th commodity (e.g. a cwt. of corn) in the unit of time. According to this system, naturally, there will be a unit of measurement of capital for each sector in which capital is utilized.

For simplicity, we shall assume that capital goods are required only for the production of consumption goods, whereas capital goods are not needed for the production of themselves. For this production only certain amounts of labour are needed (either directly or indirectly)[3].

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[2] L. L. Pasinetti, op. cit.

[3] The assumption that capital goods are produced solely by labour can be ruled out without changing the substance of the analysis. See Pasinetti op. cit. chpt. II sec. VII.

For the sake of simplicity we shall further assume that, once they are made, capital goods last for ever. Moreover, in order to keep the analysis as simple as possible, we shall assume that only two consumption goods are produced in the economy[4].

Since the amounts of capital goods utilized in each consumption sector are measured in different units, there will be as many processes (sectors) for the production of capital goods as there are sectors producing consumption goods. Hence we shall consider two capital goods sectors.

The processes of production are defined in the following way: to produce one unit of the i-th consumption good ($i = 1, 2$) it is necessary to employ directly or indirectly a_i units of labour and to have in stocks—as fixed capital—one unit of the capital goods produced by the i-th capital goods sector. To produce one unit o f the i-th capital good it is only necessary to employ directly or indirectly α_i units of labour.

The coefficients a_i and α_i may change in time owing to technical progress. But, contrary to what is usually done, we shall not assume that these coefficients decrease continuously over time. Instead we shall assume that they change at certain points of time: $t = 0, \tau_1, \tau_2, \ldots$, exogeneously given, and then stay constant for the whole stretch of time included between two of these successive points. In other words, we assume that a_i and α_i are step functions of time.

The first aim of the model is to determine the time-paths of the production of different goods on the hypothesis of sectoral equilibrium and full employment of the available labour force which grows at a constant exponential rate: $N(t) = N_0 \exp \varepsilon t$.

The demand for the consumption goods comes from the workers. Their demand is for themselves and their households. Let c_i be the per capita consumption of the i-th good. Then, in order to have equilibrium in consumption sectors, we should have:

$$x_i = c_i N_0 \exp \varepsilon t, \qquad i = 1, 2, \tag{1.1}$$

where x_i is the production of the i-th consumption good.

Note that, as we shall soon see, c_i is not constant but may change in time.

On the hypothesis already made, the equilibrium in capital goods sectors requires that

$$\xi_i = \dot{x}_i, \qquad i = 1, 2, \tag{1.2}$$

where ξ_i denotes the production of the capital goods to be used for the production of the i-th consumption good.

The full employment condition is expressed by

$$a_1 x_1 + a_2 x_2 + \alpha_1 \xi_1 + \alpha_2 \xi_2 = N_0 \exp \varepsilon t. \tag{1.3}$$

By substitution of (1.1) and (1.2) into (1.3) we get

$$a_1 c_1 + a_2 c_2 + \alpha_1 (\dot{c}_1 + \varepsilon c_1) + \alpha_2 (\dot{c}_2 + \varepsilon c_2) = 1. \tag{1.4}$$

Hence, given the time-paths of all input coefficients (a_i and α_i), there is one degree of freedom which can be removed by insering particular hypotheses on the behaviour of per capita consumption. We shall comment on two very simple cases.

[4] The extension to the case of n consumption goods may be made very easily.

1.1. Case I. *Proportional growth.* Assume that up to $t = 0$ the system was in full employment equilibrium with constant techniques ($a_i = \bar{a}_i$; $\alpha_i = \bar{\alpha}_i$), and constant per capita consumption coefficients. The system was then growing in size with an unchanging structure. The full employment condition (eq. (1.4)) was then expressed by:

$$\bar{a}_1\bar{c}_1 + \bar{a}_2\bar{c}_2 + \varepsilon\bar{\alpha}_1\bar{c}_1 + \varepsilon\bar{\alpha}_2\bar{c}_2 = 1. \tag{1.5}$$

Suppose now that at $t = 0$, owing to technical progress, we have a *reduction* in one, or some, of the input coefficients. Suppose also that, after the change at $t = 0$, these coefficients will undergo-no other change[5]. Hence

$$a_i(t) = a_i \leqslant \bar{a}_i; \quad \alpha_i(t) = \alpha_i \leqslant \bar{\alpha}_i; \quad i = 1, 2; \quad t \geqslant 0. \tag{1.6}$$

We can immediately take notice that, in order to keep satisfied the full employment condition, one or both of the consumption coefficients have to increase.

In case I we shall be concerned with what we call *proportional* growth, i.e. with a situation in which both consumption coefficients increase at the same rate

$$\frac{c_1(t)}{c_2(t)} = \frac{\bar{c}_1}{\bar{c}_2} = \beta_0, \tag{1.7}$$

where β_0 is a constant.

On the hypotheses stated it is easy to integrate (1.4) to get

$$c_2(t) = [\bar{c}_2 - (1/\delta_0)\exp\left(-\frac{\delta_0}{\gamma_0}t\right) + (1/\delta_0), \tag{1.8}$$

where

$$\delta_0 = a_1\beta_0 + a_2 + \varepsilon\alpha_1\beta_0 + \varepsilon\alpha_2, \tag{1.9}$$

$$\gamma_0 = \alpha_1\beta_0 + \alpha_2. \tag{1.10}$$

In words, the consumption per head of the second good increases exponentially from its initial value $\bar{c}_2$ to the value $1/\delta_0$.

1.2. Case II. *Changing consumption structure.* Consider now a situation in which equation (1.7) of the preceding paragraph is substituted by:

$$\dot{c}_1(t)/\dot{c}_2(t) = \beta_1 \neq \beta_0; \quad t \geqslant 0; \tag{1.11}$$

all other hypotheses being kept unchanged. We are in a world in which, from $t = 0$ on, the consumption per head of one good increases faster than that of the other.

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[5] It is easy to show that, if there are subsequent changes in the input coefficients at times $t = \tau_1, \tau_2, \ldots > 0$, the full employment equilibrium path of the system is of the same general type of that found out in the text. In fact, for any t in the interval $\tau_j \leqslant t \leqslant \tau_{j+1}$, we shall

$$a_i(t) = [a_i(\tau_j) - \bar{a}_i(\tau_i)]\exp[-\bar{\gamma}(\tau_j)(t-\tau_j)] + \bar{a}_i(\tau_j),$$

where $a_i(\tau_j)$ is the value the consumption coefficient has assumed for $t = \tau_j$, i.e. at the initial moment of the time interval we are considering; $\bar{a}_i(\tau_j)$ is the value the consumption coefficient would tend to assume in the long run if, after $t = \tau_j$ there were no further change in input coefficients. The speed with which $a_i(\tau_j)$ tends to $\bar{a}_i(\tau_j)$ is measured by $\bar{\gamma}(\tau_j)$, a parameter depending on the conditions prevailing in the economy at $t = \tau$. See T. Cozzi: Sviluppo e stabilità dell'economia (Torino 1969) Chpt. I.

Without loss of generality, lest us assume that $\beta_1 > \beta_0$ so that it is the consumption of the first good which increases faster.

Equation (1.11) can be integrated to give

$$c_1(t) = \beta_1 c_2(t) + (\beta_0 - \beta_1)\bar{c}_2. \tag{1.12}$$

Let us take notice that the linear relation between c_1 and c_2 may interpreted as an approximation of a more complicated relation. In fact it is always possible to take piece-wise approximations of any relation $c_1(t) = \varphi[c_2(t)]$ in the intervals $(0, \tau_1)$, (τ_1, τ_2), ... We assume that only one approximation is sufficiently good for the interval $t \geqslant 0$, but it is possible, show that no great complication is involved in taking as many piece-wise linear approximations as one likes[6].

On the hypothesis expressed by (1.2) it is easy to integrate (1.4) to get

$$c_2(t) = \left[\bar{c}_2 - \frac{\eta}{\delta_1}\right] \exp\left(-\frac{\delta_2}{\gamma_2} t\right) + \frac{\eta}{\delta_1}, \tag{1.13}$$

where

$$\delta_1 = a_1\beta_1 + a_2 + \varepsilon\alpha_1\beta_1 + \varepsilon\alpha_2, \tag{1.14}$$

$$\gamma_1 = \alpha_1\beta_1 + \alpha_2, \tag{1.15}$$

$$\eta = 1 - a_1(\beta_0 - \beta_1)\bar{c}_2 - \varepsilon\alpha_1(\beta_0 - \beta_1)\bar{c}_2. \tag{1.16}$$

In this case too the level of c_2 increases exponentially from $\bar{c}_2$ to its limit value η/δ_1. The time-path of c_1 can be easily found on the basis of (1.13) and (1.12).

Many more time-paths of the consumption per head of the two goods could be analysed by considering different hypotheses on the changes in the consumption structure. But for our present purposes it is sufficient to probe into these two cases only.

2.1. Let us make now some hypotheses that will enable us to determine the behaviour of the rate of profit. Precisely, let us assume that there is perfect competition so that a unique wage rate: w and *a* unique profit rate: r will rule all over the rule all over the economy. The latter will be equal to the market rate of interest if all investments are considered by businessman as riskless.

Furthermore, in long run equilibrium, perfect competition will enforce equality between costs and prices. If we assume that production takes place instantaneously, no cost needs to be charged for interest either on the value of intermediate goods or on wages paid out to obtain the final output. We shall further assume that the values of capital gains or losses brought about by changes in the prices of goods held in stocks by the firms, are not computed among production costs and therefore are not carried over into selling prices[7].

[6] See Cozzi op. cit. Chpt. I.

[7] This hypothesis has been christened by Solow as the "received doctrine". The doctrine was accepted by N. Georgescu-Roegen: Relaxation phenomena in linear dynamic models (in: Activity analysis of production and allocation. Ed. T. Koopmans. New York 1951), by L. L. Pasinetti op. cit., and many others. According to R. M. Solow: Competitive valuation in a dynamic input-output system (*Econometrica* 1959), the received doctrine can be accepted only if it is assumed that entrepreneurs expect proces to remain constant even if they actally change through time. If it is assumed, on the contrary, that entrepreneurs foresee perfectly the changes in proces, capital gains and losses have to be computed as production costs.

On these hypotheses, if we denote by p_i the price of the i-th consumption good and by π_i the price of the corresponding capital good, the price system is described by:

$$p_i = a_i w + r\pi_i, \quad i = 1, 2, \tag{2.1}$$

$$\pi_i = \alpha_i w, \quad i = 1, 2. \tag{2.2}$$

Equations (2.2) state that the prices of capital goods are exclusively determined by the level of labour costs per unit of output. It must be observed, however, that this is so only because we have assumed that capital goods are produced by utilizing labour alone and not capital goods as well. In fact, the prices of consumption goods which require capital goods for their production are to be found, as seen from (2.1), from the sum of two components which represent respectively labour costs and remuneration for the capital invested per unit of output. In fact, owing the our hypotheses, the value of the capital invested per unit of output of the i-th consumption good is given by π_i, and it is on this value that the rate of profit will be paid.

2.2. For the moment, let us make what is called the "extreme classical saving assumption", i.e. let us assume that:

(a) Taken as whole, workers save nothing. They receive their wages and spend the whole lot on the purchase of consumption goods.

(b) Profits are earned by capitalists where pure spirits in the sense that they consume absolutely nothing and save the whole amount of the profits they earn.

Furthermore, let us assume that equilibrium conditions are always maintained. Or, better, let us assume that there is a central authority that has the task of operating in such a way as to keep the economy in full employment equilibrium and that it does its job well. This means that, on the one hand, the central authority has to ensure that the overall level of savings is equal to the overall level of investments required for full employment in the economy. On the other hand, it has to ensure that overall investments are distributed among the different sectors in such a way as to keep the conditions of sectoral equilibrium for the central authority, besides making full use of all the instruments of monetary policy (including all the instruments of selective intervention at sectoral levels), to be in a position to intervene directly on the volume of total investment and on its sectoral distribution. For example, should investments by private entrepreneurs prove insufficient to keep the economy in full employment, the central authority might itself have to undertake additional investments in the appropriate sectors.

On the hypotheses stated, if we denote by x_i and ξ_i the full employment equilibrium levels of production, the corresponding value of investments is given by:

$$I = \pi_1\xi_1 + \pi_2\xi_2. \tag{2.3}$$

The level of savings, on the hypotheses (a) and (b), is given by the amount of profits earned by capitalists. Recalling that the value of the capital invested in each i-th consumption goods sector is given by $\pi_i x_i$ and that the rate of profit is r, the level of savings is defined by

$$S = r(\pi_1 x_1 + \pi_2 x_2). \tag{2.4}$$

By equating S to I, taking account of (2.1) and (2.2), we can solve for r to get:

$$r = \frac{\alpha_1 \xi_1 + \alpha_2 \xi_2}{\alpha_1 x_1 + \alpha_2 x_2}, \tag{2.5}$$

and, by remembering that, in equilibrium, $\xi_i = \dot{x}_i$, we can see that

$$r = \frac{\alpha_1 x_1}{\alpha_1 x_1 + \alpha_2 x_2} \frac{\dot{x}_1}{x_1} + \frac{\alpha_2 x_2}{\alpha_1 x_1 + \alpha_2 x_2} \frac{\dot{x}_2}{x_2} \tag{2.6}$$

which shows that the rate of profit is given by a weighted average of the sectoral rates of growth.

2.3. Let us now compare the behaviour of the rate of profit in the presence of technical progress on both the hypotheses outlined before.

2.3.1. On the hypothesis of proportional growth, it is easy to see from (2.6) that the rate of profit, let it be called $r^*(t)$, is given by

$$r^*(t) = \dot{x}_1/x_1 = \dot{x}_2/x_2, \tag{2.7}$$

i.e. it equals the rate of growth of the system. This was all to be expected since it was assumed that all wages are consumed, all profits invested and that sectoral outputs increase at one and the same rate. These are actually the same hypotheses that led von Neumann to demonstrate the equality between rate of profit and rate of growth[8].

We would like to point out however that, unlike what happens in von Neumann's model, in ours the growth rate of the system, and hence the rate of profit, does not always remain constant through time. In fact, constancy is only to be found it the system happens to grow proportionally with unchanging technical conditions. Should technical conditions change, both the rate of growth and the rate of profit will change over time. As will shortly be shown, these rates *normally* tend to decrease towards the value they ought to have in conditions of full employment proportional growth in the absence of technical progress. This value is given by ε, i.e. by the growth rate of the labour force. But it happens that, when technical progress reduces one (or more) of the labour coefficients, both the profit and the growth rates are suddenly driven upwards and, from the new level, they start to decline again.

To make these statements clear, let us refer back to what was said in section 1.1 above. There an economic system was considered which, until time $t = 0$, had been growing in equilibrium with constant technical conditions and constant per capita consumption, at a rate equal to ε. Hence also the rate of profit was equal to ε. At $t = 0$, owing to the reduction in technical coefficients, the per capita consumption begins to grow proportionately. As a result, there is a jump upwards in the rate of profit. In fact, by recalling that $x_i = c_i N_0 \exp \varepsilon t$, and making use of (1.8) we can see that

$$r^*(t) = \varepsilon + \frac{\dot{c}_1}{c_1} = \varepsilon + \frac{\dot{c}_2}{c_2} = \varepsilon + \frac{-\dfrac{\delta_0}{\gamma_0}\left[\bar{c}_2 - \dfrac{1}{\delta_0}\right]\exp\left(-\dfrac{\delta_0}{\gamma_0}t\right)}{\left[\bar{c}_2 - \dfrac{1}{\delta_0}\right]\exp\left(-\dfrac{\delta_0}{\gamma_0}t\right) + \dfrac{1}{\delta_0}}, \tag{2.8}$$

which shows that, at $t = 0$, $r^*(t)$ jumps from ε to a higher value: $r^*(0)$.

[8] J. von Neumann: A model of general equilibrium. *Review of Economic Studies.*

But, as it is immediately clear, in the absence of subsequence reductions in technical coefficients, the rate of growth of per capita consumption tends to zero as time increases. Hence we can conclude that $r^*(t)$ follows in time a downward path from $r^*(0)$ to its limit values ε.

It should also be clear that $r^*(t)$ would behave as described even if, owing to technical progress, we had new reductions in the labour coefficients after $t = 0$, i.e. at times $t = \tau_1, \tau_2, \ldots > 0$. The only difference would be that, at these dates, $r^*(t)$ instead of showing an increase from ε upward, would jump up from the value it had already reached. A typical time-path for $r^*(t)$ is shown by the continuous line in Fig. 1.

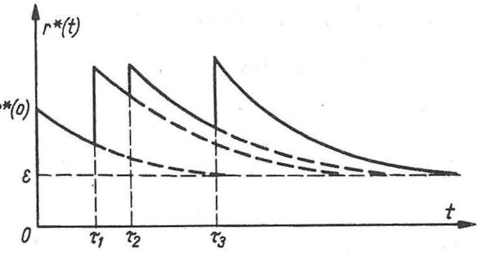

Fig. 1

To hint at the reason for this movement of the rate of profit we can refer to economic considerations. Take the case of a reduction of the labour coefficient in the sectors producing consumption goods. Then a number of workers who should have been engaged in producing consumption goods will be in excess requirements. If unemployment is to be avoided, they must be re-employed in the sectors producing capital goods. The result will be a higher output of the latter goods than there would have been in the absence of technical progress. The additional output of capital goods must be offset by an adequate additional amount of savings. The increase in the rate of profit will determine the creation of these extra savings. But latter, the sectors producing consumption goods will have to increase their employment at higher rates than those in the sectors producing capital goods, for otherwise the additional output of capital goods could not be absorbed. Therefore the output of investement goods will slow down. To maintain equilibrium, the rate of saving will have to decrease, which will be obtained by a reduction in the profit rate.

With merely a few slight changes, the same type of argument can be used to explain the time-path of the rate of profit, in the case of reductions of the labour coefficients in the sectors producing investment goods.

2.3.2. Let us consider now the economic system referred to in case II above. It is easy to show, by utilizing (2.6), (1.13), (1.12) and (1.1) and making some algebraical manipulations, that the time-path of the rate of profit $\hat{r}(t)$ is given by

$$\hat{r}(t) = \varepsilon + \frac{-\delta_1 [\bar{c}_2 - (\eta/\delta_1)] \exp\left(-\dfrac{\delta_1}{\gamma_1} t\right)}{\gamma_1 [\bar{c}_2 - (\eta/\delta_1)] \exp\left(-\dfrac{\delta_1}{\gamma_1} t\right) + \gamma_1(\eta/\delta_1) + \alpha_1(\beta_0 - \beta_1)\bar{c}_2} \tag{2.9}$$

Now, let us compare the time-path of $\hat{r}(t)$ with that of $r^*(t)$. It is easy to show that at time $t = 0$, we have $r^*(0) = \hat{r}(0)$[9], so that it can be stated the rates of profit in the two systems are equal when a change in the structure of consumption begins to take place.

It may further be noted that in both systems the rate of profit in the long run tends to fall to its minimum level ε. In fact, we see by inspection that $\lim_{t\to\infty} r^*(t) = \lim_{t\to\infty} \hat{r}(t) = \varepsilon$. It can therefore be asserted that if production techniques do not undergo further changes after that occurring at $t = 0$, the rate of profit in the long run is not affected by changes in consumption structures.

It can finally be noted if $\delta_0/\gamma_0 = \delta_1/\gamma_1$, it always emerges that $r^*(t) = \hat{r}(t)$ for all $t \geqslant 0$[10].

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[9] Note that, for $t = 0$ (eq. (2.8)) becomes

$$r^*(0) = \varepsilon + \frac{1-\delta_0\tau_2}{\gamma_0\tau_2},$$

while (2.9) becomes

$$\hat{r}(0) = \varepsilon + \frac{\eta-\delta_1\bar{c}_2}{\gamma_1\tau_2+\alpha_1(\beta_0-\beta_1)\tau_2}.$$

Substituting the values of δ_1, γ_1, η given by (1.14)–(1.16), into the latter expression, making a few simplifications and taking account of (1.9) and (1.10), we can verify that $r^*(0) = \hat{r}(0)$.

More generally it can be demonstrated that, if full employment is maintained, the rate of profit will not be affected by a chance in the structure of production (consumption) when the change begins to take place. Let us call this date $t = \tau$. Now consider (2.6) above, i.e. the expression giving the time-path of the rate of profit. Differentiate at $t = \tau$ this expression with respect to one of the rates of growth $(\dot{x}_1/x_1)$ to obtain

$$\frac{\partial r}{\partial[\dot{x}_1|x_1]} = \frac{\alpha_1 x_1}{\alpha_1 x_1+\alpha_2 x_2} + \frac{\alpha_2 x_2}{\alpha_1 x_1+\alpha_2 x_2}\,\frac{\partial[\dot{x}_2|x_2]}{\partial[\dot{x}_1|x_1]}, \tag{1}$$

where all the variables are evaluated at $t = \tau$.

Take now the full employment condition (1.3) which can be written as

$$x[a_1+\alpha_1(\dot{x}_1|x_1)]+x_2[a_2+\alpha_2(\dot{x}_2|x_2)]-N_0\exp(\varepsilon t) = 0. \tag{2}$$

By differentiating this expression at $t = \tau$ with respect to $\dot{x}_1/x_1$, we obtain

$$\frac{\partial[\dot{x}_2|x_2]}{\partial[\dot{x}_1|x_1]} = -\frac{\alpha_1 x_1}{\alpha_2 x_2}. \tag{3}$$

Hence substituting (3) into (2) we obtain

$$\partial r/\partial[\dot{x}_i|x_i] = 0, \tag{4}$$

which proves what was stated above.

[10] We can in fact (eq. (2.8)) write $r^*(t)$ as

$$r^*(t) = \varepsilon + \frac{1}{-\dfrac{\gamma_0}{\delta_0} - \left\{\dfrac{\gamma_0|\delta_0}{\delta_0[\bar{c}_2-(1|\delta_0)]}\right\}\exp\dfrac{\delta_0}{\gamma_0}t} \tag{1}$$

and (eq. (2.9)) we write $\hat{r}(t)$ as

$$\hat{r}(t) = \varepsilon\, \frac{1}{-\dfrac{\gamma_1}{\delta_1} - \left\{\dfrac{(\gamma_1|\delta_1)\eta+\alpha_1(\beta_0-\beta_1)\bar{c}_2}{\delta_1[\bar{c}_2-(\eta|\delta_1)]}\right\}\exp\dfrac{\delta_1}{\gamma_1}t}. \tag{2}$$

Now, as we know that $r^*(t) = \hat{r}(t)$ at $t = 0$, if we have $\delta_0/\gamma_0 = \delta_1/\gamma_1$, we must necessarily have equality between the two expressions in curly brackets in (1) and (2). It therefore follows that in this sense, the right-hand sides of (1) and (2) are equal for every value of t.

But, with the exception of these cases, the time-paths of the rates of profit in the two systems will be different. In particular it is easy to show that for every $t > 0$ except in the limit case when $t \to \infty$, we shall have $r^*(t) < \hat{r}(t)$ if and only if $(\delta_0/\gamma_0) > (\delta_1/\gamma_1)$ and viceversa[11].

To appreciate from an economic point of view the results so for obtained, let us take notice that from (1.9), (1.10) and (1.14), (1.15), we can write

$$\frac{\delta_0}{\gamma_0} = \frac{a_1\beta_0 + a_2}{\alpha_1\beta_0 + \alpha_2} + \varepsilon, \tag{2.10}$$

$$\frac{\delta_1}{\gamma_1} = \frac{a_1\beta_1 + a_2}{\alpha_1\beta_1 + \alpha_2} + \varepsilon. \tag{2.11}$$

[11] Denote respectively by h_0 and h_1 the expressions in curly brackets in (1) and (2) of the preceding footnote. We know that, for $t = 0$, $r^*(0) = \hat{r}(0)$. This implies that h_0 and h_1 are negative and have to satisfy to the following expression

$$[(-\gamma_0/\delta_0) - h_0] = [(-\gamma_1/\delta_0) - h_1] > 0. \tag{3}$$

This expression can be written as

$$\frac{\gamma_0}{\delta_0}\left[-1 - \frac{\delta_0}{\gamma_0} h_0\right] = \frac{\gamma_1}{\delta_1}\left[-1 - \frac{\delta_1}{\gamma_1} h_1\right]. \tag{4}$$

Suppose now that we have

$$(\delta_0/\gamma_0) > (\delta_1/\gamma_1). \tag{5}$$

From (4) we can then write

$$-1 - \frac{\delta_0}{\gamma_0} > -1 - \frac{\delta_1}{\gamma_1} h_1 \tag{6}$$

that is

$$-\frac{\delta_0}{\gamma_0} h_0 > -\frac{\delta_1}{\gamma_1} h_1, \tag{7}$$

where both members of the inequality are positive.

Now denote the denominators of (1) and (2) of the preceding footnote respectively by $\varphi^*(t)$ and $\bar{\varphi}(t)$ and differentiate these functions with respect to t to obtain

$$\frac{d}{dt}\varphi^*(t) = -\frac{\delta_0}{\gamma_0} h_0 \exp\left(\frac{\delta_0}{\gamma_0} t\right), \tag{8}$$

$$\frac{d}{dt}\hat{\varphi}(t) = -\frac{\delta_1}{\gamma_1} h_1 \exp\left(\frac{\delta_1}{\gamma_1} t\right). \tag{9}$$

Hence, for every $t > 0$ we shall have $\varphi^*(t) > \hat{\varphi}(t)$ if

$$-\frac{\delta_0}{\gamma_0} h_0 \exp\left(\frac{\delta_0}{\gamma_0} t\right) > -\frac{\delta_1}{\gamma_1} h_1 \exp\left(\frac{\delta_1}{\gamma_1} t\right). \tag{10}$$

But this is certainly verified if (5) and hence (7) are valid. If the opposite inequality would also be verified in (10). It can therefore be stated that for every $t > 0$

$$\varphi^*(t) \gtrless \hat{\varphi}(t) \text{ if and only if } (\delta_0/\gamma_0) \gtrless (\delta_1/\gamma_1). \tag{11}$$

Now remembering that $\varphi^*(t)$ and $\hat{\varphi}(t)$ are the denominators of the fractions appearing in (1) and (2) of the footnote 9, we may conclude that for every $t > 0$, except the limit case $t \to \infty$ we have

(a) $r^*(t) < \hat{r}(t)$ if and only if $(\delta_0/\gamma_0) > (\delta_1/\gamma_1)$,

(b) $r^*(t) > \hat{r}(t)$ if and only if $(\delta_0/\gamma_0) < (\delta_1/\gamma_1)$,

as we have asserted in the text.

Let us now denote by $F(\beta)$ the following expression

$$F(\beta) = \frac{a_1\beta + a_2}{\alpha_1\beta + \alpha_2} \tag{2.12}$$

and take notice that $F(\beta_0)$ represents the fraction of the right hand side of (2.10) and $F(\beta_1)$ that to be found in the right hand side of (2.11). Nothe further that $(\delta_0/\gamma_0) \gtreqless (\delta_1/\gamma_1)$ according as $F(\beta_0) \gtreqless F(\beta_1)$. Now differentiate (2.12) with respect to β to obtain after simplifications

$$\frac{d}{d\beta} F(\beta) = \frac{a_1\alpha_2 - a_2\alpha_1}{(\alpha_1\beta + \alpha_2)^2} \,. \tag{2.13}$$

The sign of (2.13) will be greather than, equal to, or less than zero according as $[a_1\alpha_2 - a_2\alpha_1]$ is greater than, equal to, or less than zero. This, in turn, means that if $\beta_1 > \beta_0$ that is if the growth of *per capita* consumption of the first commodity in the system with non-proportional growth is higher that that of the second, we shall have

$$F(\beta_1) \gtreqless F(\beta_0) \text{ and then } \frac{\delta_0}{\gamma_0} \lesseqgtr \frac{\delta_1}{\gamma_1} \text{ of there that } \frac{\alpha_2}{a_2} \gtreqless \frac{\alpha_1}{a_1}.$$

The opposite is obviously the case if $\beta_1 < \beta_0$. We are now in a position to translate the results obtained in economic terms. Let us note first of all that the ration α_1/a_1 and α_2/a_2 respectively measure the capital intensity (or organic composition of capital) in the first and the second sector. In effect, they are equal to the ratios between the value of the capital invested per unit of output in each sector of consumption goods and the value of the wages paid out directly or indirectly per unit of output in the sector.

It can therefore first of all be seen that a change in consumption structure has no effect on the rate of profit if the two sector of consumption goods have the same capital intensity. But if the first sector has a lower capital intensity than the second, a change in consumption structure increasing the consumption of the first commodity more than that of the second (as occurs if $\beta_1 > \beta_0$), has the effect of lowering the rate of profit with respect to the case of proportional growth. Conversely, if the first consumption good has a capital intensity higher than that of the second one, a change in consumption structure in favour of the first commodity has the effect of maintaining the rate of profit (which in the long run fall towards ε) at a higher level than it would have been in the case of proportional growth.

These conclusions are illustrated in the Figs. 2, 3, 4. These diagrams compare the paths of the rate of profit in the economic system in which there is proportional growth and in the one in which the consumption of the first good is growing more than that in the second (the case in which $\beta_1 > \beta_0$). Figure 2 illustrates the case in which the first consumption good has a capital intensity lower than that of the second which implies that the rate of profit is higher in the economy with proportional growth than in the other. Figure 3 represents the case in which it is the first sector that has the higher capital intensity. Figure 4 shows the case in which the two sectors have the same capital intensity so that the time-path of the rate of profit is not affected by changes in the consumption structure.

We have therefore arrived at the conclusion that for an economic system growing in full employment equilibrium, the time-path of the rate of profit depends on the *structure* of per capita demand for consumption goods, unless it so happens that these goods have the same capital intensity. The more the structure of per capita demand changes in favour of goods of a higher capital intensity, the higher will be the rate of profit—excepting for the point at which the consumption structure begins to change and the limit case of full employment equilibrium growth at the rate ε, in the absence of technical progress.

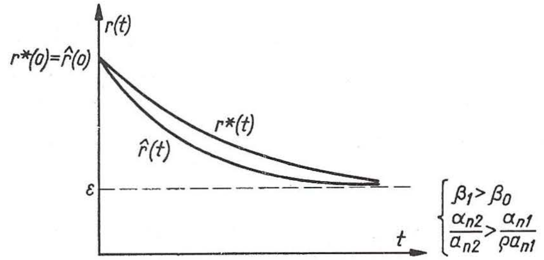

Fig. 2

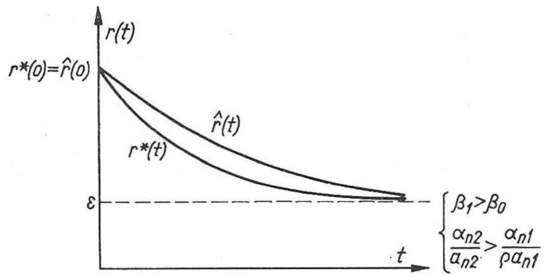

Fig. 3

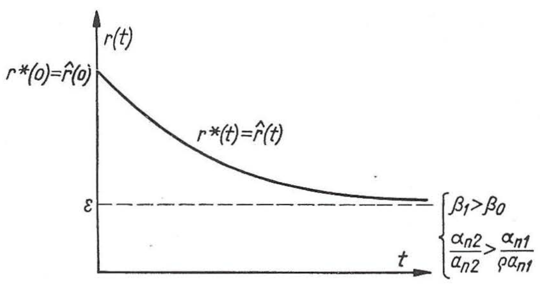

Fig. 4

The conclusions we have so far obtained point out once more the limitations implied in the use of aggregative models which cannot take account of the effects of the changes in consumption structure.

Moreover, our conclusions contrast with those one can derive from the neo-marginalistic theory of growth. According to this theory, if there exists the possibility of chosing among different production techniques, there will be an inverse relation between the rate of profit and the capital intensities of the chosen techniques. In our analysis there is no choice of techniques since in each sector there is only one technique. Hence, rigorously speaking, it should not be possible to compare our results with those of the marginalistic theory. However, even for the case in which only one technique exists in each sector, J. Meade maintained that the inverse relation between the rate profit and capital intensity holds good in the aggregate[12]. Meade's point is the following: if the rate of profit goes up, the prices of the goods produced by techniques of a higher capital intensity increase relatively to those of the goods produced by techniques of a lower capital intensity. Hence, consumers will demand less of the first goods and more of the second ones. If it is possible to define in some way the capital intensity for the economy as a whole (Meade defines it in terms of the capital-labour ration), it is clear that, when the rate of profit goes up, the capital intensity goes down not because it brings about a change in productive techniques but because it brings about a change in the structure of consumers' demand.

We think that Meade's analysis is essentially *static*: given the demand curve, the quantity demanded increases when the price decreases and viceversa. In this case we are considering movements along a given demand curve. Our analysis, on the contrary, is essentially *dynamic*. As time goes on, the demand curves shift either because some changes in consumers' tastes may take place or because the consumers' income has increased. A change in the structure of demand in favour of the goods produced by techniques of a higher capital intensity is associated with an increase in the rate of profit not with a reduction of it. This conclusion—by denying the existence, for the economy as a whole, of an inverse relation between rate of profit and capital intensity—comes to strenghten the criticisms raised in the recent literature, against the marginalistic theory of production and distribution.

3. Let us now give up the assumption of perfect competition while retaining all other assumptions. Hence the rate of profit may be different among sectors. Let

$$r_i(t) = \varepsilon + \varrho_i(t) \tag{3.1}$$

be the i-th sector's rate of profit, $\varrho_i(t)$ being the excess of $r_i(t)$ over its minimum level ε.

The equilibrium (saving equals investment) condition now reads:

$$\alpha_1 \dot{x}_1 + \alpha_2 \dot{x}_2 = [\varepsilon + \varrho_1(t)]\alpha_1 x_1 + [\varepsilon + \varrho_2(t)]\alpha_2 x_2. \tag{3.2}$$

This expression, by using (1.1) and performing some manipulations, can be written as

$$\left[H_1 \frac{\dot{c}_1}{c_1} + H_2 \frac{\dot{c}_2}{c_2} + \varepsilon\right] = [H_1 \varrho_1(t) + H_2 \varrho_2(t) + \varepsilon], \tag{3.3}$$

where

$$H_i = \frac{\alpha_i c_i}{\alpha_1 c_1 + \alpha_2 c_2}, \qquad i = 1, 2.$$

[12] J. E. Meade: A neo-classical theory of economic growth. London 1962.

The economic interpretation of (3.3) is straightforward. Its left-hand side gives the average rate of growth of the system and it has to be equal to the average rate of profit. Hence, we can see that, even when the profit rates are different among sectors, the average rate of profit is determined by the average rate of growth of the system[13].

Now, as we already know from the analysis developed in section 2 above, the average rate of growth, then equal to the unique rate of profit, is influenced, among other things, by the changes that may occur in the consumption structure. In particular, it was shown that a shift of consumption is favour of a good of a higher capital intensity has the effect of increasing both the average rate of growth and the rate of profit above the level they would otherwise have had—exception being made for the time at which the change in consumption structure takes place and for the limit case of full employment growth under unchanged technical conditions. The same conclusion holds good now owing to the equality of the average profit and growth rates.

By the same analysis developed above, it can further be shown that the average rate of profit is pusked upwards by the occurence of technical progress. But the increase in the average rate of profit does not necessarily mean that both sectoral rates are going to increase. The analysis so far developed has nothing to do with the problem of the determination of the rates of profit in each particular sector. In fact, by leaving this question completely open, our analysis is consistent with many theories of the determination of sectoral profit rates. For instance, it would be possible to fit into the model the hypothesis that the rate of profit in one of the two sectors is positively correlated with its rate of growth, or the hypothesis that an innovation in one sector increases the rate of profit in the sector concerned and not in the other[14], and many more alternative hypotheses.

What our analysis points to is only that an aggregative consistency (equilibrium) condition is always to be satisfied. This condition states that, in equilibrium and on our other assumptions, the average rate of profit for an economic system is determined by its average rate of growth so that, given the latter not all sectoral profit rates can be independently given (or can be independently explained). In our case, if one of these rates is given, the other is completely determined. In particular, given the average rate of growth of the system, a higher degree of monopoly in one sector—which means a higher profit rate in the sector—implies, ceteris paribus, a lower rate of profit in the other sector. In conclusion, it is not possible to have two in-

[13] It should be pointed out that the amount of profits is the source of savings. Hence, in equilibrium, there should be just that amount of profits necessary to finance growth. The other main function of profits in capitalist economies, i.e. that of measuring the degree of success obtained by entrepreneurs and of being the basis on which they take their decisions, is not considered explicity in the model. Its consideration is instead left to some asides (see, for instance, the discussion of the results of sect. 3 below).

[14] In this case the model will have some resemblance with the theory put forth by J. A. Schumpeter: The theory of economic development (Cambridge, Mass. 1939—ist German edition 1911). The difference is that for Schunpeter the increase in the rate of profit goes to the benefit of the innovating entrepreneurs and not to that of all entrepreneurs operating in the sector concerned.

dependent theories for the determination of the two full employment equilibrium sectoral rates of profit. The two theories have to be consistent, i.e. they have to satisfy the equilibrium condition (3.3).

It should be quite clear that, in reality, both sectors may take measures to increase their rate of profit. But—ceteris paribus, e.g. in the absence of the effects of changes brought about by technical progress—either one sector succeds in doing so and the other has to suffer a reduction in its rate below the level it would otherwise have attained, or the equilibrium of the system cannot be maintained. In the latter case, the time-paths of the real variables and, in particular, that of the sectoral rates of profit, will come to depend on the type of reactions of the economic operators to the conditions of disequilibrium and on the lags connected with these reactions. Lack of time prevents us to probe deeper into this matter, but we think safe to point out that, according to the different hypotheses made on the mechanism of reactions, the system may follow very different time-paths, some of them stable, some other unstable[15].

4. Lastly, let us comment briefly on the consequences of relaxing the "extreme classical saving hypothesis" so for made. Let us follow Kaldor[16] in assuming that there is saving both out of profits and out of wages though in different proportions. Denote by s_p and s_w the propensity to save out of profits and out of wages and assume that $1 \geqslant s_p > s_w > 0$. Then, on the hypotheses of the section 3 above, the equilibrium (saving = investment) condition now reads

$$\alpha_1 \dot{x}_1 + \alpha_2 \dot{x}_2 = s_c \sum_{i=1}^{2} [\varepsilon + \varrho_i(t)]\, \alpha_i x_i + s_w w[\alpha_1 x_1 + a_2 x_2 + \alpha_1 \dot{x}_1 + \alpha_2 \dot{x}_2]. \tag{4.1}$$

This expression, by using (1.1) and performing same simple but long manipulations, can be transformed into:

$$\left[H_1 \frac{\dot{c}_1}{c_1} + H_2 \frac{\dot{c}_2}{c_2} + \varepsilon\right] = (s_c - s_w)[H_1 \varrho_1(t) + H_2 \varrho_2(t) + \varepsilon] + \frac{s_u}{k}, \tag{4.2}$$

where k, the over-all capital-output ratio, has the following definition:

$$k = \frac{\pi_1 x_1 + \pi_2 x_2}{p_1 x_1 + p_2 x_2 + \pi_1 \dot{x}_1 + \pi_2 \dot{x}_2} \tag{4.3}$$

and where the expression (2.1) above defining the prices of the consumption goods has been modified into:

$$p_i = a_i w + [\varepsilon + \varrho_i(t)]\pi_i \tag{2.1b}$$

in order to take account of the difference among the sectoral rates of profit.

It is easy to recognise into (4.2) the results on the relation between rate of profit and rate of growth obtained by Kaldor with reference to an aggregative model.

[15] For an example of an analysis of the stability of the model outlined in this paper—see Cozzi op. cit. Chpt. III.

[16] N. Kaldor: Alternative theories of distribution. *Review of Economic Studies* 1955–1956 No. 2.

Hence, all Kaldor's considerations apply to our model. In particular, it is of interest to us the conclusion that, given the rate of growth, the (average) rate of profit is the higher the lower are the propensities to save both out of profits and out of wages. Now, the advertising and other sales promotion activities made by the firms tend to reduce the propensities to save all over the economy. Consequently we should have an increase in the equilibrium rate of profit. In this way, an increase in the degree of monopoly in one sector, brought about by an increase in the sales promotion activities, may not only push up the profit rate in the sector by reducing that in other sectors (see section 3 above), but may also cause an increase in the profit rate all over the economy. This effect is to be added to the other effects on the rates of profit that an increase in the degree of monopoly may bring about by changing the speed of technical progress or by influencing the sectoral and over-all rates of growth.

Lastly, it may be useful to stress once more that, in order to reach more definite conclusions on the actual behaviour of the rates of profit in real economics, the analysis of equilibrium here developed has to be supplemented by an appropriate analysis of its stability.

Stopa zysku w modelu wzrostu gospodarczego

Rozważono model zrównoważonego wzrostu przy założeniu całkowitego zatrudnienia, przy czym produkcja poszczególnych sektorów (dóbr) może wzrastać z różną szybkością wynikającą ze zmian w strukturze konsumpcji.

Założono, że wszystkie sektory produkcji są zintegrowane pionowo w tym sensie, że wszystkie tranzakcje pośrednie są traktowane jako wewnętrzne aspekty procesu produkcji, który od wejść pierwotnych (praca i środki produkcji) prowadzi do wyrobów końcowych[1]. Dla uproszczenia model zawiera tylko dwa sektory wytwarzające dobra konsumpcyjne i dwa sektory wytwarzające środki produkcji. Założono oprócz tego, że środki produkcji potrzebne są tylko do produkcji dóbr konsumpcyjnych, wobec czego jedynym wejściem w sektorze wytwarzającym środki produkcji jest praca. Zakłada się także, że raz wyprodukowane środki produkcji trwają do końca rozważanego okresu.

Techniki produkcji zdefiniowano za pomocą zbiorów współczynników nakładów pracy, które w wyniku postępu technicznego zmieniają się w czasie. Założono, że zmiany w czasie współczynników nakładów pracy można aproksymować funkcją schodkową.

Норма прибыш в модели хозяйственного роста

Рассмотрена модель уравновешенного роста при предположении полного использования трудовых ресурсов, причём производство отдельных секторов (ценностей) может возрастать с разной скоростью, вытекающей из изменений в структуре потребления.

[1]L. L. Pasinetti: A new theoretical approach to the problems of economic growth. W: Semaine d'étude sur le rôle de l'analyse économetrique dans la formulation de plans de developpement (Vatican City 1965). Przedstawiony w niniejszej pracy model zawiera większość założeń przyjętych przez Pasinettiego i należy go traktować jako próbę głębszego wniknięcia w pewne problemy studiowane przez tego autora.

Предполагается, что все секторы производства вертикально интегрированы в том смысле, что все промежуточные сделки считаются внутренней стороной процесса производства, который от первичных входов (труд и средства производства) ведет к конечным изделиям[1]

Для большей простоты модель содержит только два сектора, выпускающих потребительские товары и два сектора, изготовляющих средства производства. Кроме этого предполагается, что средства производства нужны только для производства потребительских товаров, а поэтому единственным входом сектора, изготовляющего средства производства, является труд. Предполагается также, что раз изготовленные средства производства не изнашиваются до конца рассматриваемого периода.

Техника производства формулируется с помощью множеств коэффициентов трудовых затрат, которые в результате технического прогресса изменяются во времени. Предполагается, что изменения во времени коэффициентов трудовых затрат можно аппроксимировать с помощью ступенчатой функции.

[1] L. L. Pasinetti: A new theoretical approach to the problems of economic growth. — Semaine d'étude sur le role de l'analyse économetrique dans la formulation de plans de developpement (Vaticon City 1965). Представленная в данной работе модель содержит большинство предположений принятых у Пасинетти и следует ее воспринимать как попытку более глубокого проникновения в некоторые проблемы исследуемые этим автором.