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Functional aspects of dynamic programming. General results*

by

FRANCO GIANNESSI
University of Venice
Institute of Mathematics

The method of dynamic programming is analysed from the general viewpoint of a functional space. A sufficient condition is given in order that the method can be applied to extremum problems. Some applications to the calculus of variations will be given in a paper to appear.

Introduction

The method of dynamic programming [1] is becoming more and more important in the functional field, for instance for its applications to the calculus of variations [2, 3]. Moreover, in many problems, related to the use of the method of dynamic programming, the problem arises to know how far the method works. These reasons made me sure of the importance of studying the foundations of the method. About the case of real-valued functions of real variables many results already exist; see for instance [1] and [7]. In the present paper the method of dynamic programming is analysed from the general viewpoint of a functional space. After some notations (sec. 1), some introductory properties are given (sec. 2). Then, the multi-stage composition of a set is analysed from a functional viewpoint (sec. 3). At last, a sufficient condition in order that the method can be applied in a functional space is given (sec. 4). In a further paper some applications to the calculus of variations are given.

1. Some notation

Let (X, T), (Ω, T') be topological spaces and x, ω be elements of X, Ω respectively. Define a parametric partition of X as a set ofsubsets of X such that any two of them are disjoint; such that their union is X itself; and such that a one-one cor-

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respondence between them and the elements of Ω is given. Denote by $X(\omega)$, $x(\omega)$ the subset of X corresponding to the element $\omega \in \Omega$, and the generic element of $X(\omega)$, respectively. A parametric partition of X will be pointed out by writting:

$$X = \{x(\omega) \in X(\omega) : \omega \in \Omega\}. \tag{1.1}$$

The parametric partition (1.1) will be said to have the property C, if

(i) the subsets $X(\omega)$ are closed;

(ii) the function g: $\Omega \to \{X(\omega) : \omega \in \Omega\}$ is upper semicontinuous, in the sense that, given any open set $X^* \subset T$, the set $\Omega^* = \{\omega \in \Omega : X(\omega) \subset X\}$ is an open set of T'.

At last, let f: $X \to \bar{R}$ be a given function from $X \subset T$ into the subset $\bar{R}$ of the reals, and let $f_\omega : X(\omega) \to R : \omega)$ the restriction of f to $X(\omega)$. Define

$$e(\omega) = \operatorname*{Inf}_{x \in X(\omega)} f(x) = \operatorname{Inf} f_\omega(x(\omega));$$

$$E(\omega) = \operatorname*{Sup}_{x \in X(\omega)} f(x) = \operatorname{Sup} f_\omega(x(\omega));$$

so that we can state two introductory lemmas.

2. Some lemmas

We will now state in a functional space two known properties of real-valued functions of reals variables[1].

LEMMA 1. If X is compact; if the parametric partition (1.1) has the property C; and if f is lower (upper) semicontinuous, then $e(\omega)$ is lower ($E(\omega)$ is upper) semicontinuous.

P r o o f. We will demonstrate Lemma 1 with regard to the lower semicontinuity. To this aim it is enough to show that, for any real α, the set $\Omega(\omega) = \{\omega \in \Omega : e(\omega) > \alpha\}$ is open. If[2] $\sim \Omega(\alpha) \neq \emptyset$, then $\Omega(\alpha)$ is open by definition. Otherwise, consider $\omega_0 \in \sim \Omega(\alpha)$. As a closed subset of a compact set is compact too[3], by (i) of property C, $X(\omega)$ is compact. As the image of a lower semicontinuous function from a compact set into the reals has a finite infimum and contains it[4], a $x_0 \in X(\omega_0)$ exists, such that $f(x_0) = e(\omega_0) \leqslant \alpha$, this inequali-

[1] See [5] and [6] for Lemma 1, and [7] for Lemma 2.

[2] By the symbol $\sim S$ we denote the complement of the set S.

[3] See [4] p. 61, Theorem 17′ (ii).

[4] In fact, it is known (see 4 p. 62, Theorem 18) that, if the function $f : X \to Y$ is continuous, in the sense that the inverse image of any open set of Y (on depending of the particular topology of Y) is an open set of X (on depending of the particular topology of X), then Y is a compact set if X is. Thus, as the lower semicontinuity of $f : X \to R$ is a particular case of the continuity when the open sets are ordinary real intervals, the image of the restriction, f_ω, of f to the compact set $X(\omega)$ (such a restriction is lower semicontinuous because such if f) has a finite cover of open sets of the kind $(\alpha, +\infty)$, and thus it has a finite infimum $e(\omega)$. The same conclusion is obtained, if $f : X \to \bar{R}$ is supposed to be upper semicontinuous. Assume now that $e(\omega)$ does not belong to the image of f_ω. Then, the function $f_\omega - e(\omega)$ would never be zero on $X(\omega)$, and the function $1/(f_\omega - e(\omega))$ would be upper semicontinuous on $X(\omega)$, and here upper bounded, as $X(\omega)$ is compact. On the other hand, $1/(f_\omega - e(\omega))$ would be upper unbounded, as $e(\omega)$ is the infimum of f_ω. The proof ab absurdo is now completed.

ty being based on the fact that $\omega_0 \in \sim \Omega(\alpha)$. On the contrary, if $x_0 \in X(\omega_0)$ from the inequality $f(x_0) \leqslant \alpha$ it follows that $e(\omega_0) \leqslant \alpha$. Define $\mathscr{X}(\alpha) = \{x \in X : f(x) > \alpha\}$, so that the equalities $\Omega(\alpha) = \{\omega \in \Omega : X(\omega) \cap \mathscr{X}(\alpha) = \emptyset\} = \{\omega \in \Omega : X(\omega) \subset \mathscr{X}(\alpha)\}$ hold. Now remark that, because of the lower semicontinuity of f, $\mathscr{X}(\alpha)$ is an open set of X, as it is the inverse image of the open set $(\alpha, +\infty)$ under the function f. Then, by (ii) of property C, $\Omega(\alpha)$ is open on Ω. This completes the proof.

The part of Lemma 1 related to $E(\omega)$ may be shown either directly in a quite similar way, or applying the above conclusions to $-f$.

LEMMA 2. The inequalities

$$\operatorname*{Inf}_{x \in X} f(x) = \operatorname*{Inf}_{\omega \in \Omega} [\operatorname*{Inf}_{x \in X(\omega)} f_\omega(x)]; \tag{2.1}$$

$$\operatorname*{Sup}_{x \in X} f(x) = \operatorname*{Sup}_{\omega \in \Omega} [\operatorname*{Sup}_{x \in X(\omega)} f_\omega(x)]; \tag{2.2}$$

hold. Moreover, if X and Ω are compact, if the parametric partition has the property C, if the function f is lower, upper semicontinuous, then the equalities

$$\min_{x \in X} f(x) = \min_{\omega \in \Omega} [\min_{x \in X(\omega)} f_\omega(x)]; \tag{2.3}$$

$$\max_{x \in X} f(x) = \max_{\omega \in \Omega} [\max_{x \in X(\omega)} f_\omega(x)]; \tag{2.4}$$

hold, respectively.

P r o o f. Define $M = \operatorname*{Sup}_{x \in X} f(x)$, $\mathscr{M} = \operatorname*{Sup}_{\omega \in \Omega} E(\omega)$. To demonstrate (2.2) we have to show that the equality $M = \mathscr{M}$ holds. As $X(\omega) \subset X$, the inequality $M \geqslant E(\omega)$ holds, $\forall \omega \in \Omega$, and implicates $M \geqslant \mathscr{M}$. On the other hand, the inequelities $\mathscr{M} \geqslant E(\omega) \geqslant f_\omega$, $\forall \omega \in \Omega$ together with the equality $\bigcup_{\omega \in \Omega} X(\omega) = X$, implicate $\mathscr{M} \geqslant f(x)$, $\forall x \in X$, and thus $\mathscr{M} \geqslant M$. It follows $M = \mathscr{M}$. To show (2.4) remark now that, because of the compactness of X and of the upper semicontinuity of f, M belongs to the image of f. Moreover, the property C, together with upper semicontinuity of f, implicates, by Theorem 1, the upper semicontinuity of $E(\omega)$. This fact, together with the compactness of Ω, implicates that $\mathscr{M}$ belongs to the image of the function $E(\omega)$. In a quite similar way (2.1) and (2.3) may be shown. This completes the proof.

Remark that the assumptions of the two preceeding lemmas, are verified if X has only a finite number of elements; this fact often happens in the applications.

3. Some remarks

Consider the following P r o p e r t y C′:

(i) the subset $X(\omega)$ are closed;

(ii) if the set $X^* \subset X$ is closed on X, then the set $\{\omega \in \Omega : X(\omega) \bigcap X^* \neq \emptyset\}$ is closed on Ω.

Then, it is easy to show that property C $\Leftrightarrow$ property C′.

P r o o f. Consider the set $X^* \subset X$ and closed on X, and define $W = \{X(\omega):X(\omega) \quad X^* \neq \emptyset\}$; so that the equality $\sim W = \{X(\omega):X(\omega) \subset \sim X^*\}$ holds. Then, by (ii) of property C′, $\sim X^*$ is open on X and its inverse image under the function g is open on Ω, i.e. g is upper semicontinuous. Then, property C′ $\Rightarrow$ property C. Vice versa, consider the set $X^* \subset X$ and open on X, and define $W = \{X(\omega):X(\omega) \subset X^*\}$; so that the equality $\sim W = \{X(\omega):X(\omega) \bigcap \sim X^* = \emptyset\}$ holds. Then, by (ii) of property C, $\sim X^*$ is closed on X and its inverse image under g is closed on Ω. This completes the proof.

4. Multi-stage composition of a set

Before going on to analyse the method of dynamic programming on a general topological space, we have now to generalize to such a space the idea of multi-stage composition of a set of the Euclidean space.

Given a set $X \subset T$, consider the subsets $X_1, X_2, \ldots, X_n$, of X, such that

(i) $X_1 \subset X_2 \subset \ldots \subset X_N \subset T$;

(ii) for any $j = 1, \ldots, N$ a topological space (Ω_j, T_j') is given, and a parametric partition of X_j, i.e.

$$X_j = \{x_j(\omega_j) \in X_j(\omega_j):\omega_j \in \Omega_j\} \tag{4.1}$$

exists, such that, given any subset $X_{j+1}(\omega_{j+1})$ of X_{j+1}, a bijective and bicontinuous (homeomorphism)[5], i.e.

$$\varphi_{j+1}:X_j \to X_{j+1}(\omega_{j+1}),$$

and a closed subset $\bar{\Omega}_j(\omega_{j+1})$ of Ω_j exist, such that the set $\{\varphi_{+1}(x_j(\omega_j)) \in \varphi_{j+1}(X_j(\omega_j)): :\omega_j \in \Omega_j(\omega_{j+1})\}$ is a parametric partition of $X_{j+1}(\omega_{j+1})$;

(iii) $\{x_N(\omega_N) \in X_N(\omega_N):\omega_N \in \Omega_N\}$ is a parametric partition of X.

Then, we define a N-stage composition of X as the set of subsets

$$X_1(\omega_1), \ldots, X_N(\omega_N), \tag{4.2}$$

satisfying the equations

$$X_{j+1}(\omega_{j+1}) = \{\varphi_{j+1}(x_j(\omega_j)) \in \varphi_{j+1}(X_j(\omega_j)):\omega_j \in \bar{\Omega}_j(\omega_{j+1})\},\ j = 1, \ldots, N. \tag{4.3}$$

The sets (4.2) will be said first stage, ..., N-th stage of the N-stage composition, respectively. At last, the N-stage composition, (4.2)–(4.3) will be said to have the property C, if the parametric partitions (4.3) have it, whatever $\omega_{j+1} \in \Omega_{j+1})$ $j = 1, \ldots, N-1$, may be.

Now remark that, if X is compact and the N-stage composition (4.2)–(4.3) has the property C, then the subsets $\varphi_{j+1}(X_j(\omega_j))$ of $X_{j+1}(\omega_{j+1})$ are compact[6]. Thus, as

[5] $X_j(\omega_j)$ has the same meaning as $X(\omega)$, but it is referred to X_j instead of to X. ω_j, $x_j(\omega_j)$ are generic elements of Ω_j, $X_j(\omega_j)$ respectively. For sake of simplicity we write φ_{j+1} instead of $\varphi_{\omega j+1}$ as we should; in fact, the function φ_{j+1} depends on the fixed element ω_{j+1} and not only on the stage $j+1$.

[6] As they are closed subsets of the compact set X.

the inverse function

$$\varphi_{j+1}^{-1}:\varphi_{j+1}\left(X_j(\omega_j)\right) \to X_j(\omega_j) \tag{4.4}$$

is continuous, the set $X_j(\omega_j)$[7] is compact,

Remark also that, if X is compact and the N-stage composition (4.2)–(4.3) has the property C, then the parametric partition, (4.1) has the property C too. In fact, the image of the closed set $\varphi_{j+1}\left(X_j(\omega_j)\right)$ is closed, as it is an homeomorphism; thus $X_j(\omega_j)$ is closed. Moreover, the intersection of $X_j(\omega_j)$ and every set $X_j^* \subset X_j$ and closed on X_j is closed[8]; and, as (4.2)–(4.3) has the property C, the set of ω_j, corresponding (in the parametric partition) to the image of such an intersection under the function φ_{j+1}, is closed on Ω_j.

5. Some examples

Now we will give some examples of a N-stage composition of a set to explain the preceding considerations.

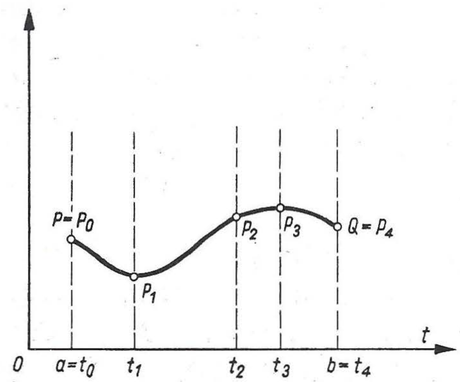

Fig. 1

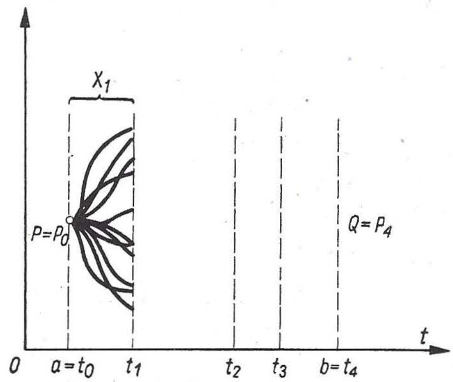

Fig. 2

(a) Define X as the set of continuous single-valued functions, defined on a (closed) interval $[a, b]$, and joining two points P and Q of abscisses a and b, respectively (Fig. 1). The natural metric $d(x', x'') = \max\limits_{[a,b]}|x'-x''|$, $\forall\, x', x'' \in X$, determines a set T of open sets, so that (X, T) is a topological space (as it is a metric space)[9].

Let t_1, t_2, t_3 be a given abscisses, with $a < t_1 < t_2 < t_3 < b$, and P_1, P_2, P_3 denote the points of a geometric element of X, having abscisses t_1, t_2, t_3, respecti-

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[7] $X_j(\omega_j)$ is the image of the function (4.4).

[8] In fact, the equality $X_j^* \cap X_j(\omega_j) = X_j^* \cup X_j(\omega_j)$ holds; moreover, as the union of two open sets of a topological space is open too, the set $X^* \cap X_j(\omega_j)$ is closed.

[9] (X, T) is well known and it is usually denoted by $C[a, b]$.

vely. For every $j = 1, 2, 3, 4$ define X_j as the set of the restrictions of the elements of X to $[a, t_j]$, so that we have[10]

$$X_1 \subset X_2 \subset X_3 \subset X_4 = X.$$

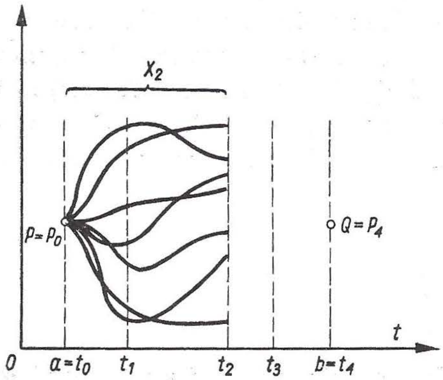

Fig. 3

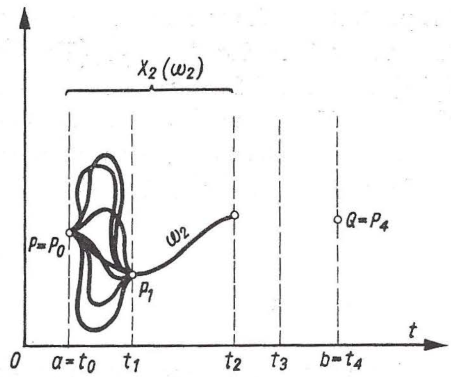

Fig. 4

Figures 2 and 3 show the sets X_1, X_2 respectively. Define Ω_j as the set of the restrictions, ω_j, of a generic element of X to $[t_{j-1}, t_j]$; the same kind of the above metric determines a set T_j' of open sets, so that (Ω_j, T_j') is a topological space. Now define $X_j(\omega_j)$ as the set of the elements of X_j, which coincide with ω_j on $[t_{j-1}, t_j]$. Then, $\{x_j(\omega_j) \in X_j(\omega_j) : \omega_j \in \Omega_j$ is evidently a parametric partition of X_j.

Figure 4 shows the set $X_2(\omega_2)$, ω_2 being the function pictured on t_1, t_2. Remark also that $X_1(\omega_1) = \omega_1$, as shown by Fig. 2.

Foe every $j = 1, 2, 3$, given an $\omega_{j+1} \in \Omega_{j+1}$, define $\overline{\Omega}_j(\omega_{j+1})$ as the set of the functions of Ω_j, which equal the function ω_{j+1} at $t = t_j$; so that $\overline{\Omega}_j(\omega_{j+1}) \subset \Omega_j$. Figure 5 shows the given element ω_{j+1} and $\overline{\Omega}_j(\omega_{j+1})$ at $j = 3$. It follows that $X_{j+1}(\omega_{j+1})$ may be regarded as the set of the functions of $X_j(\omega_j)$, $\omega_j \in \overline{\Omega}_j(\omega_{j+1})$, "extended" with the function ω_{j+1}. Figure 6 shows, at $j = 3$, the given element ω_{j+1}, the corresponding $\overline{\Omega}_j(\omega_{j+1})$; the set $X_j(\omega_j)$ with $\omega_j = \omega_j^i$, $i = 1, 2, 3$; and the set $X_4(\omega_4)$, regarded as before indicated.

Now remark that the function φ_{j+1} may now be regarded in the following way

$$\varphi_{j+1} \colon \bigcup_{\omega_j \in \overline{\Omega}_j(\omega_{j+1})} X_j(\omega_j) \to X_{j+1}(\omega_{j+1}) \tag{5.1}$$

and is evidently bijective, as the equality

$$x'_{j+1}(\omega_{j+1}) = x''_{j+1}(\omega_{j+1}),\ x'_{j+1}(\omega_{j+1}),\ x''_{j+1}(\omega_{j+1}) \in X_{j+1}(\omega_{j+1})$$

implicates the same equality on the interval $[a, t_j]$, i.e. the equality of the corresponding functions of the domain of (5.1). Also the bicontinuity of (5.1) is quite evident, if we observe that the same kind of topology has been adopted for the domain

[10] We regard the space $C[c, d]$, with $a < c < d < b$, as subspace of $C[a, b]$.

and the image of (5.1): the image and the inverse image, under (5.1), of an open set is open.

A 4-stage composition of the set X has been thus obtained by the sets $X_j(\omega_j)$, $j = 1, 2, 3, 4$.

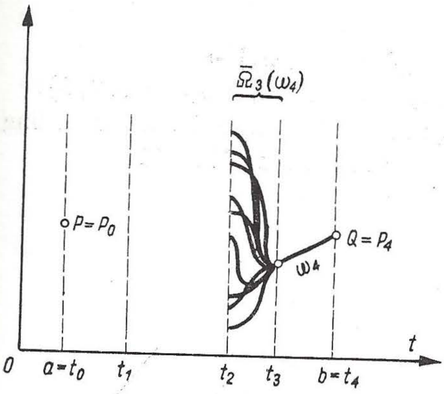

Fig. 5

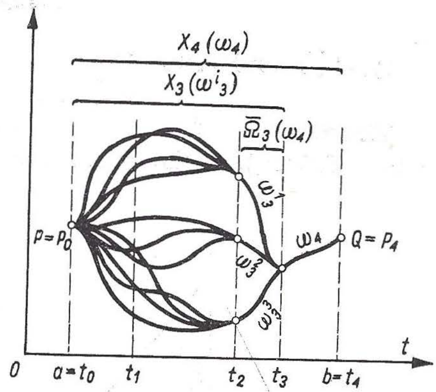

Fig. 6

(b) Define X as the set

$$X = \{(y_1, y_2, y_3): y_1 + y_2 + y_3 \leqslant a;\ y \geqslant 0,\ i = 1, 2, 3\},\ a > 0,$$

of the 3-dimensional Euclidean space[11]. For every $j = 1, 2, 3$ define

$$X_j = \{(y_1, \ldots, y_j): y_1 + \ldots + y_j \leqslant a;\ y_i \geqslant 0,\ i = 1, \ldots, j\},$$

so that we have

$$X_1 \subset X_2 \subset X_3 = X.$$

Figure 7 shows a geometric interpretation of X_1, X_2, X_3 as a segment, triangle, tetrahedrom, respectively. Define $\Omega_j = \{\omega : 0 \leqslant \omega_j \leqslant a\}$; so that ω_j is now a real number and $\Omega_1 = \Omega_2 = \Omega_3 = [0, a]$[12]. Define $X_j(\omega_j) = \{(y_1, \ldots, y_j): y_1 + \ldots + y_j = \omega_j;\ y_i \geqslant 0,\ i = 1, \ldots, j\}$. Now it is easy to remark that $\{x_j(\omega_j) \in X_j(\omega_j): \omega_j \in \Omega_j\}$ a parametric partition of X_j[13].

Figure 8 shows a geometric interpretation of $X_1(\omega_1)$, $X_2(\omega_2)$, $X_3(\omega_3)$, as a point, as a segment, as a triangle, respectively.

For every $j = 1, 2$, given an $\omega_{j+1} \in \Omega_{j+1}$, define $\bar{\Omega}_j(\omega_{j+1}) = \{\omega_j : 0 \leqslant \omega_j \leqslant \omega_{j+1}\}$ so that $\Omega_j(\omega_{j+1}) \subset \Omega_j$.

Figure 9 shows $\bar{\Omega}_j(\omega_{j+1})$ at $j = 2$. It follows that $X_{j+1}(\omega_{j+1})$ may be regarded as the set of $(j+1)$-dimensional vectors, whose first j coordinates $y_1, \ldots, y_j$ are given by the coordinates of the points of $X_j(\omega_j)$, $\omega_j \in \bar{\Omega}_j(\omega_{j+1})$, respectively, and whose $(j+1)$-th coordinate is $y_{j+1} = \omega_{j+1} - \omega_j$. Then, the function φ_{j+1} which may be regarded in the same way as (5.1), is now evidently bijective and bicontinuous.

[11] The meaning of (X, T) in now trivial.
[12] The meaning of (Ω_j, T) is now trivial.
[13] $x_j(\omega_j)$ is now the vector $(y_1, \ldots, y_j) \in X_j(\omega_j)$.

Figure 10 shows φ_{j+1} at $j = 2 : \varphi_3$ appears to be the following one-one correspondence between the triangle $0AB$, i.e. $\bigcup_{\omega_2 \in \Omega_2 \omega_3} X_2(\omega_2)$ and the triangle ABC, i.e. $X_3(\omega_3)$:

$$\varphi_{j+1} : \bigcup_{\omega_2 \in \bar{\Omega}_2 \omega_3} X_2(\omega_2) \to X_3(\omega_3)$$

or

$$\varphi_{j+1} : \{(y_1, y_2) : y_1 + y_2 = \omega_2, y_1 \geqslant 0; y_2 \geqslant 0, 0 \leqslant \omega_2 \leqslant \omega_3\} \to$$
$$\to \{(y_1, y_2, y_3) : y_1 + y_2 + y_3 = \omega_3, y_j \geqslant 0, j = 1, 2, 3\}.$$

The restriction of φ_3 to $X_2(\omega_2)$, i.e. the segment DE, given us the corresponding subset of $X_3(\omega_3)$, i.e. the segment FG, as shown by the arrows of Fig. 10.

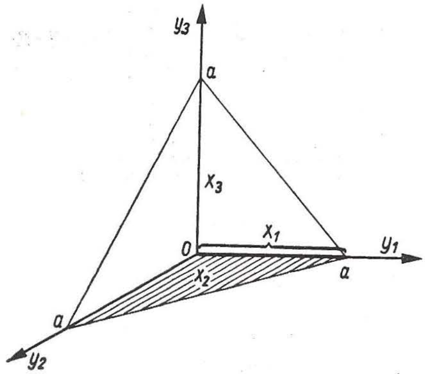

Fig. 7

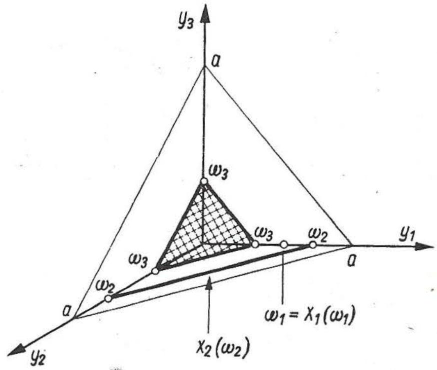

Fig. 8

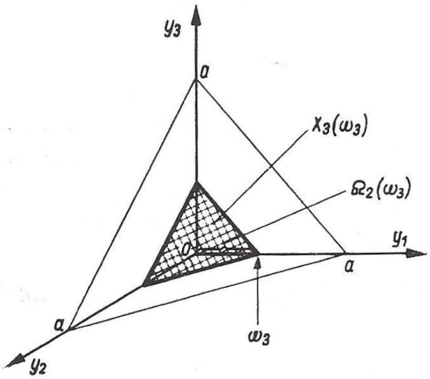

Fig. 9

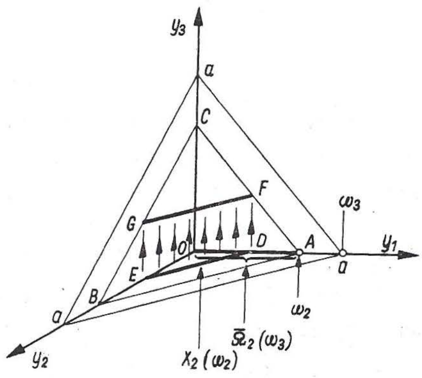

Fig. 10

The set $X_1(\omega_1)$, $X_2(\omega_2)$, $X_3(\omega_3)$ thus obtained give us a 3-stage composition of X.

6. Multi-stage dynamical problems

The preceding lemmas and definitions enable us to consider now the extremum problems. To this aim and without loss of generality consider the following problem

$$\mathscr{P} : \max_{x \in X} f(x),$$

where $f : X \to \bar{R}$ is the function of section 1. We assume the following

HYPOTHESIS I. The set X is compact and the function f is upper semicontinuous by which at least an optimal solution of $\mathscr{P}$ exists.

We will say that the problem $\mathscr{P}$ may be regarded as a N-stage dynamical problem when the following conditions are satisfied.

(i) At least a N-stage composition of X exists; such a composition is given by (4.2)–(4.3).

(ii) For every $j = 1, \ldots, N-1$ it is possible to define a function $f_j: X_j \to \overline{R}_j$, whose restriction to $X_j(\omega_j)$ is said $f\omega_j$, such that the restriction of $f_{\omega j+1}$ to subset $\varphi_{j+1}(X_j(\omega_j))$ of $X_{j+1}(\omega_{j+1})$, i.e. $f_{\omega j+1}(\varphi_{j+1}(x_j(\omega_j)))$, is a function of the kind

$$G_{j+1}(f_{\omega j}(x_j(\omega_j)), \omega_j, \omega_{j+1})$$

where $G_{j+1}(u, \omega_j, \omega_{j+1})$ is a real function from the Cartesian product $R_j \times \Omega_j \times \Omega_{j+1}$ into the reals increasing in relation to u.

When $\mathscr{P}$ may be regarded as a N-stage dynamical problem, the following problems are considered

$$\mathscr{P}_j(\omega_j): \sup_{x(\omega j) \in X_j(\omega j)} f_{\omega j}(x(\omega_j)), j = 1, \ldots, N, \tag{6.1}$$

whose feasible regions satisfy the equalities

$$X_{j+1}(\omega_{j+1}) = \{\varphi_{j+1}(x_j(\omega_j)) \in (\varphi_{j+1}(\omega_j)) : \omega_j \in \overline{\Omega}_j(\omega_{j+1})\}, j = 1, \ldots, N-1; \tag{6.2}$$

and whose extremizing functions satisfy the equalities

$$f\omega_{j+1}\left(\varphi_{j+1}(x_j(\omega)_j)\right) = G_{j+1}(f_{\omega j}(x_j(\omega_j), \omega_j, \omega_{j+1}), j = 1, \ldots, N-1. \tag{6.3}$$

The problems (6.1) are said first stage, ..., N-th stage of the representation of $\mathscr{P}$ as a N-stage dynamical problem.

Moreover, if the N-stage composition sub (i) has the property C, wesay that the representation (6.1)–(6.3) of $\mathscr{P}$ as N-stage dynamical problem has the property C.

New remark that the considerations we made at the end of section 4 could be here repeated. In particular, if X is compact and if the representation (6.1)–(6.3) of $\mathscr{P}$ has the property C, then the sets $X_j(\omega_j)$ are compact, as they are closed subsets of a compact set. In such a case the supremum of $\mathscr{P}_j$ belongs to the image of $f_{\omega j}$.

Now define

$$F_j(\omega_j) = \sup_{x_j(\omega j) \in X_j(\omega j)} f_{\omega j}(x_j(\omega_j)), j = 1, \ldots, N; \tag{6.4}$$

then, we can state the following theorem on which the method of dynamic programming is based.

7. The fundamental theorem of dynamic programming

(A) The set $X \subset T$ and the function $f: X \to \overline{R}$ are given. Assume that the problem

$$\mathscr{P}: \sup_{x \in X} f(x)$$

may be regarded as a N-stage dynamical problem, and that (6.1)–(6.3) be a representation of $\mathscr{P}$ as N-stage dynamical problem.

Then the following equalities hold

$$F_{j+1}(\omega_{j+1}) = \operatorname*{Sup}_{\omega_j \in \bar{\Omega}_j(\omega_{j+1})} G_{j+1}\big(F_j(\omega_j), \omega_j, \omega_{j+1}\big), \quad j = 1, \dots, N-1; \tag{7.1}$$

$$\operatorname*{Sup}_{\omega_j \in \Omega_j} F_j(\omega_j) = \operatorname*{Sup}_{x_j \in X_j} f_j(x_j), \; j = 1, \dots, N, \tag{7.2}$$

where f_j is the restriction of f to X_j.

(B) Moreover, if we assume that

(i) the sets X and X_j are compact;

(ii) the representation (6.1)–(6.3) of $\mathscr{P}$ as N-stage dynamical problem has the property C;

(iii) the functions f and f_j are upper semicontinuous on X and X_j, respectively; then the equalities

$$F_{j+1}(\omega_{j+1}) = \max_{\omega_j \in \bar{\Omega}_j(\omega_{j+1})} G_{j+1}\big(F_j(\omega_j), \omega_j, \omega_{j+1}\big), \; j = 1, \dots, N-1 \tag{7.3}$$

$$\max_{\omega_j \in \Omega_j} F_j(\omega_j) = \max_{x_j \in X_j} f_j(x_j), \; j = 1, \dots, N, \tag{7.4}$$

hold.

(C) Assume that (i)–(iii) sub (B) hold; denote by $x_j^*(\omega_j)$, ω_j^* optimal solutions of problems $\mathscr{P}_j(\omega_j)$, (7.3), respectively. Then

$$x_{j+1}^*(\omega_{j+1}) = \varphi_{j+1}\big(x_j^*(\omega_j^*)\big)$$

is an optimal solution of $\mathscr{P}_{j+1}(\omega_{j+1})$.

P r o o f. (A) As (6.2) is a parametric partition of $X_{j+1}(\omega_{j+1})$ by Lemma 2 the following equalities hold

$$F_{j+1}(\omega_{j+1}) = \operatorname*{Sup}_{x_{j+1}(\omega_{j+1}) \in X_{j+1}(\omega_{j+1})} f_{\omega_{j+1}}\big(x_{j+1}(\omega_{j+1})\big) =$$

$$= \operatorname*{Sup}_{\omega_j \in \bar{\Omega}(\omega_{j+1})} \; \operatorname*{Sup}_{\varphi_{j+1}(x_j(\omega_j)) \in \varphi_{j+1}(X_j(\omega_j))} f_{\omega_{j+1}}\big(\varphi_{j+1}\big(x_{j+1}(\omega_{j+1})\big)\big) j = 1, \dots, N-1. \tag{7.5}$$

Now remark that, given $(\omega_j, \omega_{j+1}) \in \Omega_j \times \Omega_{j+1}$, the only argument of the function (6.3) is the generic element, $x_j(\omega_j)$, of the inverse image of $\varphi_{j+1}\big(x_j(\omega_j)\big)$. Thus, the following equalities follow

$$\begin{aligned}
\operatorname*{Sup}_{\varphi_{j+1}(x_j(\omega_j)) \in \varphi_{j+1}(X_j(\omega_j))} & f_{\omega_{j+1}}\big(\varphi_{j+1}\big(x_j(\omega_j)\big)\big) = \\
&= \operatorname*{Sup}_{x_j(\omega_j) \in X_j(\omega_j)} G_{j+1}\big(f_{\omega_j} x_j(\omega_j)\big), \omega_j, \omega_{j+1}\big) = \\
&= G_{j+1}\big(\operatorname*{Sup}_{x_j(\omega_j) \in X_j(\omega_j)} f_{\omega_j}\big(x_j(\omega_j), \omega_j, \omega_{j+1}\big) = \\
&= G_{j+1}\big(F_j(\omega_j), \omega_j, \omega_{j+1}\big),
\end{aligned} \tag{7.6}$$

the second one of them holding, as $G_{j+1}(u, \omega_j, \omega_{j+1})$ has been assumed to be an increasing function on respect of u. (7.5) and (7.6) with (6.3) and Lemma 2, implicate (7.1).

(B) From (i), (ii) it follows that the sets $X_j(\omega_j)$ are compact $\forall \omega_j \in \Omega_j$; and that the parametric partition

$$X_j = \{x_j(\omega_j) \in X_j(\omega_j) : \omega_j \in \Omega_j\}$$

has the property C. Because of this fact and of the upper semicontinuity of f_j on X_j, and by Lemma 1 it follows that $F_j(\omega_j)$ is upper semicontinuous on Ω_j. Thus, becuase of the continuity of $G_{j+1}(u, \omega_j, \omega_{j+1})$ it follows that $G_{j+1}\big(F_j(\omega_j), \omega_j, \omega_{j+1}\big)$ is upper semicontinuous on the Cartesian product $\overline{\Omega}_j(\omega_{j+1}) \times \Omega_{j+1}$, which, under the present assumptions, is compact. All these facts, together with the compactness of X and X_j, turn (7.1), (7.2) into (7.3), (7.4) respectively.

(C) Remark that the following equalities evidently hold

$$F_j(\omega_j) = f_{\omega_j}\big(x_j^*(\omega_j)\big);\ F_{j+1}(\omega_{j+1}) = \max_{\omega_j \in \overline{\Omega}_j(\omega_{j+1})} G_{j+1}\Big(f_{\omega_j}\big(x_j^*(\omega_j)\big), \omega_j, \omega_{j+1}\Big)$$

$$= G_{j+1}\Big(f^*_{\omega_j}\big(x_j^*(\omega_j^*)\big), \omega_j^*, \omega_{j+1}\Big) = f_{j+1}\Big(\varphi_{j+1}\big(x_j^*(\omega_j^*)\big)\Big);$$

and they complete the proof of (C) and of the theorem.

As a complement of the preceding theorem, remark now that, instead of the symbols $X_1, \ldots, X_N$ of section 4, we could introduce a parameter t; N determination $t_1, \ldots, t_N$ of t; and the sets $X(t)$, $\Omega(t)$, such that

$$X(t_j) = X_j; \quad \Omega(t_j) = \Omega_j, \quad j = 1, \ldots, N.$$

With obvious changes in the remaining part of the symbols introduced since section 4, instead of (7.3) we would find

$$F\big(\omega(t_{j+1})\big) = \max_{\omega(t_j) \in \overline{\Omega}(t_j, \omega(t_{j+1}))} G_{j+1}\Big(F\big(\omega(t_j)\big), \omega(t_j), \omega(t_{j+1})\Big). \tag{7.7}$$

In this case it is possible to obtain a punctual relation for the function F, by evaluating (7.7) when t_j tends to t_{j+1}.

References

1. Bellman R., Dynamic programming. A Rand Corporation Research Study, Princeton Univ. Princeton, N.J. 1957.
2. Dreyfus S. E., Dynamic progamminrg and the calculus of variations. New York 1965.
3. Jacobson D. H., Mayne D. Q., Differential dynamic programming. New York 1970.
4. Maddox I. J., Elements of functional analysis. Cambridge 1970.
5. Scozra Dragoni G., Sui minimi e massimi parziali per le funzioni di più variabili. *Rend. Acc. Naz. Lincei* **6** (1927) 579–583.
6. Scorza Dragoni G., Un problema sui minimi e massimi parziali di una funzione. *Rend. Acc. Naz. Lincei* **11** (1930) 865–872.
7. Volpato M., Sui fandamenti analitici della programmazione dinamica. In: Studi e modelli di ricerca operativa. Torino 1971.

Aspekty funkcyjne programowania dynamicznego. Wyniki ogólne

Zanalizowano metodę programowania dynamicznego w przestrzeni funkcyjnej. Podano warunek konieczny stosowalności metody do zagadnień ekstremalnych. W następnej pracy zostaną podane pewne zastosowania metody programowania dynamicznego w rachunku wariacyjnym.

Функциональная сторона динамического программирования. Общие результаты

Анализируется метод динамического программирования в функциональном пространстве. Приводится необходимое условие применяемости метода для экстремальных задач. В следующей работе будут рассмотрены некоторые применения метода динамического программирования в варианционном исчислении.