Control
and Cybernetics
VOL. 2 (1973) No. 3/4

Controlled and conditioned invariance in the synthesis of unknown-input observers and inverse systems*

by

GIOVANNI MARRO
University of Bologna
Institute of Automatics

In this paper the concept of controlled invariance and its dual, that of conditioned invariance, are employed in order to derive a general and systematic procedure for the synthesis of devices which reproduce the state or the unknown input of linear time-invariant dynamic systems. The main featires which distinguish this approach from previous ones are the simple and unitary geometric treatment and the possibility of taking into account stability constraints on the devices to be synthesized.

Introduction

Controlled and conditioned invariance have been introduced in 1969 by G. Basile, R. Laschi and the author [1, 2] and have been applied for an unified state space approach to several structural problems of linear system theory, such as observability with lack of input knowledge [3, 4], decoupling and constrained reproducibility of output trajectories [5, 6], parametric intensitivity [7]. In 1970 independently Wonham and Morse developed an algorithm similar to that of controlled invariance and applied it to the synthesis of algebraic and dynamic decoupling controllers [8, 9].

In the present paper all the most important properties of controlled invariance are reviewed and for the first time their dual properties, concerning conditioned invariance, are derived and discussed. It is shown that, as controlled invariance is a very efficient tool for the synthesis of special purpose controllers, conditioned invariance can be similarly used for the synthesis of unknown-input observers and inverse systems.

The inversion of dynamical systems, that is the derivation of the inputs when only the outputs are accessible, has been treated by Brockett [10], who first gave

---

* Presented at the Polish-Italian Meeting on "Modern applications of mathematical systems and control theory, in particular to economic and production systems", Cracow, Poland, Sept. 14–20, 1972. This research has been supported by National Research Council, Rome, Italy.

a recirsive algorithm for the synthesis of inverses of single input-single output dynamical systems and an invertibility condition expressed in terms of the system matrices. Sain and Massey [11] derived a slight different invertibility condition, while Dorato [12] and Silverman [13] presented recursive algorithms for checking the invertibility of a system which can be viewed as extensions to multivariable systems of the Brockett's one.

The invertibility criteria developed by Sain and Massey and Silverman have been further investigated and discussed by Pal Singh [14], Panda [15] and their authors [16].

The geometric approach to the synthesis of inverse systems herein developed, based on the concepts of controlled and conditioned invariance, is completely different from those presented in the above mentioned literature. Its most important feature is the possibility of synthesizing the observer or the inverse system whose state has the maximal dimension and which satisfies the requirement of being asymptotically stable.

The paper is organized as follows. In Section 2 few general definitions and notations concerning stability properties of invariant subspaces state space are presented, in Section 3 the most important properties of controlled and conditioned invariants are reviewed and discussed, while in Section 4 some results on stability problems connected with the structural changes which can be obtained by means of state to input feedback and output to state feedbackare pointed out. In Section 5 unknown-input state observers for purely dynamical systems are presented as the dual of decoupling controllers and finally in Section 6 the derived results are extended to the case of non purely dynamical linear plants and to the synthesis of inverse systems.

The following notations are used through the paper. Vectors are denoted by lowercase boldface letters (**a**, **b**), linear transformations or matrices by capital boldface letters (**A**, **B**). By $\mathbf{x} \in R^n$ is meant that **x** is an n-vector. Subspaces or, more generally, sets of vectors, are denoted by capital script letters ($\mathscr{A}$, $\mathscr{B}$). As particular cases, $\mathscr{R}(\mathbf{A})$, $\mathscr{N}(\mathbf{A})$ denote the range and the null-space of the linear transformation **A**. The dimension of a subspace X is denoted by dim ($\mathscr{X}$). $\mathbf{A}\mathscr{X}$ represents the image of the set $\mathscr{X}$ under the linear transformation **A**, $\mathbf{A}^{-1*}\mathscr{Y}$ the inverse image of $\mathscr{Y}$ under the linear transformation **A**, i.e. the locus of vectors which are mapped into $\mathscr{Y}$ by **A**. $\mathscr{X}^1$ is the orthogonal complement of X, $\mathbf{A}^T$ the transpose of **A**.

2. Stability properties of invariant subspaces

Consider the purely dynamical plant described by the equations

$$\dot{\mathbf{x}} = \mathbf{A}x+\mathbf{B}\mathbf{u}, \tag{1a}$$

$$\mathbf{y} = \mathbf{C}\mathbf{x}, \tag{1b}$$

where the vectors $\mathbf{x} \in R^n$, $\mathbf{u} \in R^m$, $\mathbf{y} \in R^s$ represent the *state*, the *input* and the *output* respectively.

The plant (1) can be represented by the block diagram shown in Fig. 1, in which the algebraic operators **B** and **C** which relate the input to the *forcing actions* $\mathbf{f} \in R^n$ and the state to the output are distinguished from the strictly dynamic part of the system.

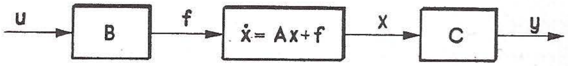

Fig. 1. The plant considered

It is well-known that the input-output structure of the system depends on the relative positions of some characteristic subspaces of the state space, as the subspaces which are invariants under the linear transformation **A**, the subspace of the forcing actions $\mathscr{R}(\mathbf{B})$ and that of the inaccessible states $\mathscr{N}(C)$.

In the following few definitions concerning stability properties of **A**-invariant subspaces, of primary importance for the approaches to synthesis problems which will be derived in next sections, are briefly reported.

Given any **A**-invariant $\mathscr{X} \in R^n$, i.e. any subspace $\mathscr{X}$ such that $\mathbf{A}\mathscr{X} \subseteq \mathscr{X}$, let $k = \dim(\mathscr{X})$ and express the state coordinates with respect to a basis whose first k element belong to $\mathscr{X}$, so that the matrix **A** assumes the from

$$\begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ 0 & \mathbf{A}_{22} \end{bmatrix}. \tag{2}$$

The eigenvalues of **A** are partitioned into two sets: *k eigenvalues associated to $\mathscr{X}$ or internal with respect to $\mathscr{X}$*, those of $\mathbf{A}_{11}$, which characterize the free trajectories of (1) ranging over $\mathscr{X}$, and *$n-k$ eigenvalues external with respect to $\mathscr{X}$*, those of $\mathbf{A}_{22}$, which characterize the free evolution of the projection of the state along $\mathscr{X}$ on any complement of $\mathscr{X}$, as, for instance, the orthogonal projection of the state on $\mathscr{X}^1$.

P r o p e r t y 1. The set of the eigenvalues internal with respect to the sum $\mathscr{X}_1 + \mathscr{X}_2$ of two **A**-invariants (which also is a **A**-invariant) is the union of the sets of the eigenvalues internal with respect to $\mathscr{X}_1$ and $\mathscr{X}_2$ respectively.

P r o p e r t y 2. The set of the eigenvalues external with respect to the intersection $\mathscr{X}_1 \cap \mathscr{X}_2$ of two **A**-invariants (which also an **A**-invariant) is the union of the sets of the eigenvalues external with respect to $\mathscr{X}_1$ and $\mathscr{X}_2$ respectively.

DEFINITION 1. An **A**-invariant $\mathscr{X} \subseteq R^n$ is said *internally stable* if the real parts of all eigenvalues internal with respect to $\mathscr{X}$ are negative. If not, it is said *internally unstable*.

DEFINITION 2. An **A**-invariant $\mathscr{X} \subseteq R^n$ is said *externally stable* if the real parts of all eigenvalues external with respect to $\mathscr{X}$ are negative. If not, it is said *externally unstable*.

Clearly the sum of two **A**-invariants internally stable is an **A**-invariant internally stable, the intersection of two **A**-invariants externally stable is an **A**-invariant externally stable.

DEFINITION 3. The *subspace of the stable modes of* **A**, which will be denoted by $\mathcal{S}_-(\mathbf{A})$, is the sum of all **A**-invariants internally stable.

DEFINITION 4. The *subspace of the unstable modes of* **A**, which will be denoted by $\mathcal{S}_+(\mathbf{A})$, is the intersection of all **A**-invariants externally stable.

The computation of $\mathcal{S}_-(\mathbf{A})$ and $\mathcal{S}_+(\mathbf{A})$ is easily performed by putting the matrix **A** in the Jordan canonical form. In fact, if $\mathbf{A} = \mathbf{T}^{-1}\mathbf{\Lambda}\mathbf{T}$, where

$$\mathbf{\Lambda} = \begin{bmatrix} \mathbf{\Lambda}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{\Lambda}_2 \end{bmatrix}$$

is the Jordan canonical form with rows and columns ordered in such a way that the k eigenvalues with negative real parts are the first k elements of the main diagonal, the subspaces of stable und unstable modes are given by

$$\mathcal{S}_-(\mathbf{A}) = \mathcal{R}(\mathbf{T}^{-1}\mathbf{R}_1), \qquad \mathcal{S}_+(\mathbf{A}) = \mathcal{R}(\mathbf{T}^{-1}\mathbf{R}_2),$$

with

$$\mathbf{R}_1 = \begin{bmatrix} \mathbf{I}_k \\ \mathbf{0} \end{bmatrix}, \qquad \mathbf{R}_2 = \begin{bmatrix} \mathbf{0} \\ \mathbf{I}_{n-k} \end{bmatrix},$$

where $\mathbf{I}_k$, $\mathbf{I}_{n-k}$ are identity matrices having the subscripted dimensions.

The following Properties 3 and 4 are easily derivable consequences of the previous definitions.

Property 3. An **A**-invariant $\mathcal{X} \subseteq R^n$ is internally stable if and only if $\mathcal{X} \subseteq \mathcal{S}_-(\mathbf{A})$.

Property 4. An **A**-invariant $\mathcal{X} \subseteq R^n$ is externally stable if and only if $\mathcal{X} \supseteq \mathcal{S}_+(\mathbf{A})$.

Now refer again to the block diagram shown in Fig. 1. Obviously this decomposition of the plant into three blocks is fictitious, because the forcing actions f are not directly accessible for intervention and the state x is not directly accessible for

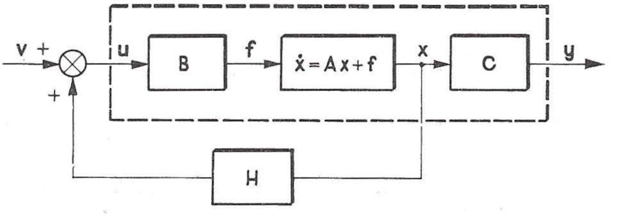

Fig. 2. State to input feedback

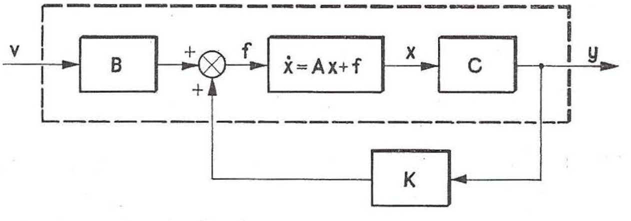

Fig. 3. Output to forcing action feedback

measurement. Many recent results in linear system theory are based on the so-called "state feedback" (or more precisely, "state to input feedback") and are derived under the hypothesis that the state is directly accessible, so that the plant can be controlled as shown in Fig. 2, that is by feeding back to the input a linear function of the state. The dual of the state feedback is the "output to forcing action feedback" shown in Fig. 3. Of course, both the feedback connections represented in Fig. 2 and in Fig. 3 are not realizable in the practice, but represent very useful schemes for studying controllers and observers. It is well-known that the feedback from the state to the input can be obtained by means of an observer of the luenberger's type [17–19].

The feedback system shown in Fig. 2 is described by the equations

$$\dot{\mathbf{x}} = (\mathbf{A}+\mathbf{BH})\mathbf{x}+\mathbf{B}v \quad (3)$$

$$\mathbf{y} = \mathbf{Cx} \quad (3b)$$

and that shown in Fig. 3 by the equations

$$\dot{\mathbf{x}} = (\mathbf{A}+\mathbf{KC})\mathbf{x}+\mathbf{B}v \quad (4a)$$

$$\mathbf{y} = \mathbf{Cx}. \quad (4b)$$

The dynamic behaviour and the structure of the controlled plant can be changed by means of the feedback connection because in general the matrices $\mathbf{A}+\mathbf{BH}$ and $\mathbf{A}+\mathbf{KC}$ have eigenvalues and invariant subspaces different from those of $\mathbf{A}$. As far as the assignment of the eigenvalues is concerned, the following Theorems 1 and 2, dual each other, are easily derived as a slight extension of a well-known result of Langenhop [20] and Wonham [21].

THEOREM 1. The subspace

$$\mathrm{mi}(\mathbf{A}, \mathscr{R}(\mathbf{B})+\mathscr{S}_-(\mathbf{A})\,^1 \quad (5)$$

can be made a $(\mathbf{A}+\mathbf{BH})$-invariant internally stable by a proper choice of the matrix $\mathbf{H}$. Conversely any subspace which, by a proper choice of $\mathbf{H}$, can be made a $(\mathbf{A}+\mathbf{BH})$-invariant internally stable is contained in it.

Proof. The subspace $\mathrm{mi}(\mathbf{A}, \mathscr{R}(\mathbf{B}))$ is known to be the locus of the reachable states of the system (1). Express the state coordinates with respect to a basis chosen in the following way: a first set of vectors spanning the subspace $\mathrm{mi}(\mathbf{A}, \mathscr{R}(\mathbf{B})) \cap \mathscr{S}_-(\mathbf{A})$, a second and a third set completing the span of $\mathrm{mi}(\mathbf{A}, \mathscr{R}(\mathbf{B}))$ and $\mathscr{S}_-(\mathbf{A})$ respectively and a fourth set arbitrary to complete the state space, so that the matrices $\mathbf{A}$ and $\mathbf{B}$ assume the form

$$\begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} & \mathbf{A}_{13} & \mathbf{A}_{14} \\ \mathbf{0} & \mathbf{A}_{22} & \mathbf{0} & \mathbf{A}_{24} \\ \mathbf{0} & \mathbf{0} & \mathbf{A}_{33} & \mathbf{A}_{34} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{A}_{44} \end{bmatrix} \begin{bmatrix} \mathbf{B}_1 \\ \mathbf{B}_2 \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} \quad (6)$$

[1] The notation $\mathrm{mi}(\mathbf{A}, \mathscr{X})$ refers to the minimum $\mathbf{A}$-invariant containing the subspace $\mathscr{X}$, which can be computed by the sequence of subspaces $\mathscr{Z}_0 = \mathscr{X}$, $\mathscr{Z}_i = \mathscr{X}+\mathbf{A}\mathscr{Z}_{i-1}$, $i = 1, 2, \dots$, which converges at most in $(n-1)$ steps.

where, by definition, the eigenvalues if $\mathbf{A}_{33}$ have negative real parts and those of $\mathbf{A}_{44}$ have non-negative real parts. Denoting by

$$[\mathbf{H}_1\mathbf{H}_2\mathbf{H}_3\mathbf{H}_4] \tag{7}$$

the feedback matrix similarly partitioned, the $\mathbf{A}+\mathbf{BH}$ matrix of the system with state to input feedback assumes the form

$$\left[\begin{array}{cc|cc} \mathbf{A}_{11}+\mathbf{B}_1\mathbf{H}_1 & \mathbf{A}_{12}+\mathbf{B}_1\mathbf{H}_2 & \mathbf{A}_{13}+\mathbf{B}_1\mathbf{H}_3 & \mathbf{A}_{14}+\mathbf{B}_1\mathbf{H}_4 \\ \mathbf{B}_2\mathbf{H}_1 & \mathbf{A}_{22}+\mathbf{B}_2\mathbf{H}_2 & \mathbf{B}_2\mathbf{H}_3 & \mathbf{A}_{24}+\mathbf{B}_2\mathbf{H}_4 \\ \hline \mathbf{0} & \mathbf{0} & \mathbf{A}_{33} & \mathbf{A}_{34} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{A}_{44} \end{array}\right]. \tag{8}$$

It has been shown [21] that the eigenvalues of the matrix shown wit dotted lines in (8) can be arbitrarily assigned by means of a proper choice of the matrices $\mathbf{H}_1$ and $\mathbf{H}_2$ so that the subspace (5), which is a $(\mathbf{A}+\mathbf{BH})$-invariant, can be made internally stable. On the other hand, the maximum (with respect to the choice of $\mathbf{H}$) subspace of the stable modes of $\mathbf{A}+\mathbf{BH}$ is clearly the subspace (5). Hence because of Property 3, any internally stable $(\mathbf{A}+\mathbf{BH})$-invariant must be contained in it. Q.E.D.

THEOREM 2. The subspace

$$\mathrm{MI}(\mathbf{A}, \mathcal{N}(\mathbf{C}) \cap \mathcal{S}_+(\mathbf{A}) \tag{9}$$

can be made a $(\mathbf{A}+\mathbf{KC})$-invariant externally stable by a proper choice of the matrix $\mathbf{K}$. Conversely, any subspace which, by a proper choice of $\mathbf{K}$, can be made a $(\mathbf{A}+\mathbf{KC})$-invariant externally stable contains it.

P r o o f. Theorem 2 is the dual of Theorem 1. Consider the subspace

$$\mathrm{mi}\left(\mathbf{A}^{\mathsf{T}}, \mathcal{R}(\mathbf{C}^{\mathsf{T}})\right)+\mathcal{S}_-(\mathbf{A}), [2] \tag{10}$$

which is clearly the orthogonal complement of (9). In force of Theorem 1 it can be made a $(\mathbf{A}^{\mathsf{T}}+\mathbf{C}^{\mathsf{T}}\mathbf{K}^{\mathsf{T}})$-invariant internally stable by means of a proper choice of $\mathbf{K}$. But the orthogonal complement of a $(\mathbf{A}^{\mathsf{T}}+\mathbf{C}^{\mathsf{T}}\mathbf{K}^{\mathsf{T}})$-invariant internally stable is a $(\mathbf{A}+\mathbf{KC})$-invariant externally stable. Furthermore, the orthogonal complement of any $(\mathbf{A}+\mathbf{KC})$-invariant externally stable, which is a $(\mathbf{A}^{\mathsf{T}}+\mathbf{C}^{\mathsf{T}}\mathbf{K}^{\mathsf{T}})$-invariant internally stable, therefore contained in (10) in force of Theorem 1, must clearly contain the subspace (9). Q.E.D.

3. Definitions and properties of controlled and conditioned invariance

Note that in the statements of Theorems 1 and 2 of the previous section the subspaces $\mathrm{mi}\left(\mathbf{A}, \mathcal{R}(\mathbf{B})\right)$ and $\mathrm{MI}\left(\mathbf{A}, \mathcal{N}(\mathbf{C})\right)$ appear. They are known to be the *subspace of controllability* (i.e. the locus of the states reachable from the origin) and

[2] The notation $\mathrm{MI}(\mathbf{A}, \mathcal{X})$ refers to the maximum A-invariant contained in the subspace $\mathcal{X}$, which can be computed by the sequence of subspaces
$\mathcal{Z}_0 = \mathcal{X}$, $\mathcal{Z}_i = \mathcal{X} \cap \mathbf{A}^{-1*} \mathcal{Z}_{i-1}$, $i = 1, 2, \ldots$, which converges at most in $(n-1)$ steps.

the *subspace of unobservability* (i.e. the locus of the initial states not detectable from any input-output record of finite lengtht) respectively. Many authors have pointed out the duality of the concepts of controllability and observability, a duality which appears clearly also from the proofs of Theorems 1 and 2. This duality can further extend by considering and solving the following problems, which are cornerstones in the geometrical development of the synthesis of controllers and observers:

(i) determine under what conditions there exists at least one matrix **H** such that a given subspace $\mathscr{J} \subseteq R^n$ is a $(\mathbf{A}+\mathbf{BH})$-invariant;

(ii) determine under what conditions there exists at least one matrix **K** such that a given subspaces $\mathscr{J} \subseteq R^n$ is a $(\mathbf{A}+\mathbf{KC})$-invariant.

These problems are related to the structural changes obtained by the feedback connections shown in Fig. 2 and in Fig. 3.

For the sake of notational compactness, according [1, 2] it is convenient to extend the concept of invariance of a subspace under a linear transformation by means of the following definitions.

DEFINITION 5. Any subspace $\mathscr{J} \subseteq R^n$ such that $\mathbf{A}\mathscr{J} \subseteq \mathscr{J}+\mathscr{R}(\mathbf{B})$ is called a $(\mathbf{A}, \mathscr{R}(\mathbf{B}))$-*controlled invariant.*

The sum of two $(\mathbf{A}, \mathscr{R}(\mathbf{B}))$-controlled invariants being a $(\mathbf{A}, \mathscr{R}(\mathbf{B}))$-controlled invariant, the maximum $(\mathbf{A}, \mathscr{R}(\mathbf{B}))$-controlled invariant contained in a given subspace $\mathscr{X} \subseteq R^n$ is univocally defined. It will be referred to by the notation MCI $(\mathbf{A}, \mathscr{R}(\mathbf{B}), \mathscr{X})$[3].

DEFINITION 3. Any subspace $\mathscr{J} \subseteq R^n$ such that $\mathbf{A}(\mathscr{J} \cap \mathscr{N}(\mathbf{C})) \subseteq \mathscr{J}$ is called a $(\mathbf{A}, \mathscr{N}(\mathbf{C}))$-*conditioned invariant.*

The intersection of two $(\mathbf{A}, \mathscr{N}(\mathbf{C}))$-conditioned invariants being a $(\mathbf{A}, \mathscr{N}(\mathbf{C}))$-conditioned invariant, the minimum $(\mathbf{A}, \mathscr{N}(\mathbf{C}))$-conditioned invariant containing a given subspace $\mathscr{X} \subseteq R^n$ is univocally defined. It will be referred to by the notation mci$(\mathbf{A}, \mathscr{N}(\mathbf{C}), \mathscr{X})$[4].

The solutions of the problems (i) and (ii) previously stated are provided by the following Theorems 3 and 4.

THEOREM 3. A subspace $\mathscr{J} \subseteq R^n$ is a $(\mathbf{A}, \mathscr{R}(\mathbf{B}))$-controlled invariant if and only if there exists at least one matrix **H** such that $(\mathbf{A}+\mathbf{BH})\mathscr{J} \subseteq \mathscr{J}$.

Proof. If $\mathscr{J}$ is a $(\mathbf{A}, \mathscr{R}(\mathbf{B}))$-controlled invariant, i.e. such that

$$\mathbf{A}\mathscr{J} \subseteq \mathscr{J}+\mathscr{R}(\mathbf{B}), \tag{11}$$

and $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n \subseteq R^n$ $(h = \dim(\mathscr{J}))$ are linearly independent vectors belonging

[3] The subspace MCI $(\mathbf{A}, \mathscr{R}(\mathbf{B}), \mathscr{X})$ can be computed as the limit of the sequence $\mathscr{Z}_0 = \mathscr{X}$, $\mathscr{Z}_i = \mathscr{X} \cap \mathbf{A}^{-1*}(\mathscr{Z}_{i-1}+\mathscr{R}(\mathbf{B}))$, $i = 1, 2, \ldots$, which converges at most in $(n-1)$ steps. This algorithm has been stated for the first time in [22].

[4] The subspace mci $(\mathbf{A}, \mathscr{N}(\mathbf{C}), \mathscr{X})$ can be computed as the limit of the sequence $\mathscr{Z}_0 = \mathscr{X}$, $\mathscr{Z}_i = \mathbf{A}(\mathscr{Z}_{i-1} \cap \mathscr{N}(\mathbf{C}))$, $i = 1, 2, \ldots$, which converges at most in $(n-1)$ steps. This algorithm has been stated and proved for the first time in [1].

to $\mathscr{J}$, which can be regarded as columns of a $n \times h$ matrix $\mathbf{X}$, there exists vectors $\mathbf{x}'_i \in R^n$ and $\mathbf{u}_i \in R^m$ such that

$$\mathbf{A}\mathbf{x}_i - \mathbf{x}'_i + \mathbf{B}\mathbf{u}_i, \; i = 1, 2, \ldots, n, \tag{12}$$

or, in matrix form,

$$\mathbf{AX} = \mathbf{X}' + \mathbf{BU}. \tag{13}$$

Assume

$$\mathbf{H} = -\mathbf{U}(\mathbf{X}^\top \mathbf{X}^{-1})\mathbf{X}^\top. \tag{14}$$

A general vector $\mathbf{x} \in \mathscr{J}$ can be expressed as $\mathbf{X}\mathbf{h}$, where $\mathbf{h} \in R^n$. It is easily verified that $(\mathbf{A}+\mathbf{BH})\mathbf{X}\mathbf{h} = \mathbf{X}'\mathbf{h} \in \mathscr{J}$.

On the other hand, if relationship (11) does not hold, there exists at least one vector $\mathbf{x}_0 \in \mathscr{J}$ such that $\mathbf{A}\mathbf{x}_0$ cannot be expressed as the sum of two vectors $\mathbf{x}'_0 \in \mathscr{J}$ and $\mathbf{B}\mathbf{u}_0 \in \mathscr{R}(\mathbf{B})$, hence no $\mathbf{X}$ exists such that $(\mathbf{A}+\mathbf{BH})\mathbf{x}_0 \in \mathscr{J}$. Q.E.D.

THEOREM 4. A subspace $\mathscr{J} \subseteq R^n$ is a $(\mathbf{A}, \mathscr{N}(\mathbf{C}))$-conditioned invariant if and only if there exists at least one matrix $\mathbf{K}$ such that $(\mathbf{A}+\mathbf{KC})\mathscr{J} \subseteq \mathscr{J}$.

Proof. Theorem 4 is derived by duality from Theorem 3. In fact, the relationship

$$\mathbf{A}(\mathscr{J} \cap \mathscr{N}(\mathbf{C})) \subseteq \mathscr{J} \tag{15}$$

is equivalent to

$$\mathbf{A}^\top \mathscr{J}^\perp \subseteq \mathscr{J}^\perp + \mathscr{R}(\mathbf{C}^\top). \tag{16}$$

But (16), because of Theorem 3, is a necessary and sufficient condition for the existence of a matrix $\mathbf{K}$ such that $(\mathbf{A}^\top + \mathbf{C}^\top \mathbf{K}^\top)\mathscr{J}^\perp \subseteq \mathscr{J}^\perp$ or, equivalently, $(\mathbf{A}+\mathbf{KC})\mathscr{J} \subseteq \mathscr{J}$. Q.E.D.

It is remarkable, that, in order to transform a $(\mathbf{A}, \mathscr{R}(\mathbf{B}))$-controlled invariant into a $(\mathbf{A}+\mathbf{BH})$-invariant a complete knowledge of the state is not required. In fact in order to implement the proper state feedback only the konowledge of a projection of the state on the controlled invariant is sufficient. Likewise, in order to transform a $(\mathbf{A}, \mathscr{N}(\mathbf{C}))$-conditioned invariant into a $(\mathbf{A}+\mathbf{KC})$-invariant it is sufficient to intervent only on forcing actions belonging to a complement of it. That is stated in more precise terms in the following corollaries of Theorem 3 and 4.

COROLLARY 1. Given a $(\mathbf{A}, \mathscr{R}(\mathbf{B}))$-controlled invariant $\mathscr{J} \subseteq R^n$ and a linear map $\mathbf{P}$ from R^n such that $\mathscr{N}(\mathbf{P}) \cap \mathscr{J} = 0$, there exists at least one matrix $\mathbf{H}$ of proper dimensions such that $(\mathbf{A}+\mathbf{BHP})\mathscr{J} \subseteq \mathscr{J}$.

COROLLARY 2. Given a $(\mathbf{A}, \mathscr{N}(\mathbf{C}))$-conditioned invariant $\mathscr{J} \subseteq R^n$ and a linear map Q to R^n such that $\mathscr{R}(Q) + \mathscr{J} = R^n$, there exists at least one matrix $\mathbf{K}$ of proper dimensions such that $(\mathbf{A}+Q\mathbf{KC})\mathscr{J} \subseteq \mathscr{J}$.

4. Reachability and stability properties related to feedback connections

Referring again to the system shown in Fig. 2, let $\mathscr{J}$ be a $(\mathbf{A}, \mathscr{R}(\mathbf{B}))$-controlled invariant, so that the relationship $(\mathbf{A}+\mathbf{BH})\mathscr{J} \subseteq \mathscr{J}$ holds for a proper choice of $\mathbf{H}$.

In the forcing actions are restricted to belong to $\mathscr{J}$ by means of an algebraic block connected to the system as shown in Fig. 4 and characterized by a matrix which satisfies the condition

$$\mathscr{R}(\mathbf{BF}) = \mathscr{R}(\mathbf{B}) \cap \mathscr{J}, \tag{17}$$

all state trajectories which leave the origin range necessarily on $\mathscr{J}$.

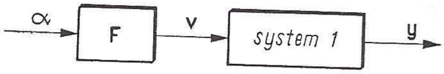

Fig. 4. Restricting the forcing action

The following Theorem 5 provides the maximum set of states reachable from the origin by means of trajectories restricted to range over $\mathscr{J}$. Because of linearity such a set is clearly a subspace and will be called the *maximum reachable subspace constrained by* $\mathscr{J}$ and denoted by MRS $(\mathbf{A}, \mathscr{R}(\mathbf{B}), \mathscr{J})$. Theorem 5 has been already stated in [1] and [8], but the proof given here is more intuitive and direct.

THEOREM 5. Given a $(\mathbf{A}, \mathscr{R}(\mathbf{B}))$-controlled invariant $\mathscr{J}$, the maximum reachable subspace constrained by $\mathscr{J}$ is defined by the relationship

$$\mathrm{MRS}(\mathbf{A}, \mathscr{R}(\mathbf{B}), \mathscr{J}) = \mathrm{mi}(\mathbf{A}+\mathbf{BH}, \mathscr{R}(\mathbf{B}) \cap \mathscr{J}), \tag{18}$$

where $\mathbf{H}$ is any matrix such that $(\mathbf{A}+\mathbf{BH})\mathscr{J} \subseteq \mathscr{J}$.

Proof. Consider another $\mathbf{H}$, say $\mathbf{H}'$, such that $(\mathbf{A}+\mathbf{BH}')\mathscr{J} \subseteq \mathscr{J}$. For every $\mathbf{x}_0 \in \mathscr{J}$, clearly $(\mathbf{A}+\mathbf{BH})\mathbf{x}_0 - (\mathbf{A}+\mathbf{BH}')\mathbf{x}_0 \in \mathscr{J}$, so that $\mathbf{B}(\mathbf{H}-\mathbf{H}')\mathscr{J} \subseteq \mathscr{J}$ and the sequence $\mathscr{Z}'_i \subseteq \mathscr{J}$ which defines $\mathrm{mi}(\mathbf{A}+\mathbf{BH}', \mathscr{R}(\mathbf{B}) \cap \mathscr{J})$, i.e.

$$\mathscr{Z}'_0 = \mathscr{R}(\mathbf{B}) \cap \mathscr{J} \tag{19}$$

$$\mathscr{Z}'_i = \mathscr{R}(\mathbf{B}) \cap \mathscr{J} + (\mathbf{A}+\mathbf{BH}+\mathbf{B}(\mathbf{H}'-\mathbf{H}))\mathscr{Z}'_{i-1}, \quad i = 1, 2, \ldots \tag{20}$$

is equal to the analogous sequence $\mathscr{Z}_i \subseteq \mathscr{J}$ which defines $\mathrm{mi}(\mathbf{A}+\mathbf{BH}, \mathscr{R}(\mathbf{B}) \cap \mathscr{J})$. In fact, by induction assume $\mathscr{Z}'_{i-1} = \mathscr{Z}_{i-1}$, so that

$$\begin{aligned}\mathscr{Z}'_i = \mathscr{R}(\mathbf{B}) \cap \mathscr{J} + (\mathbf{A}+\mathbf{BH}+\mathbf{B}(\mathbf{H}'-\mathbf{H}))\mathscr{Z}_{i-1} \subseteq \\ \subseteq \mathscr{R}(\mathbf{B}) \cap \mathscr{J} + (\mathbf{A}+\mathbf{BH})\mathscr{Z}_{i-1} + \mathbf{B}(\mathbf{H}'-\mathbf{H})\mathscr{Z}_{i-1} = \\ = \mathscr{R}(\mathbf{B}) \cap \mathscr{J} + (\mathbf{A}+\mathbf{BH})\mathscr{Z}_{i-1} = \mathscr{Z}_i.\end{aligned}$$

In perfectly analogous way it is possible to prove that $\mathscr{Z}_i \subseteq \mathscr{Z}'_i$, hence $\mathscr{Z}'_i = \mathscr{Z}_i$.

Q.E.D.

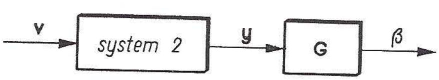

Fig. 5. Restricting the accessible states

Now the above result is dualized. Referring to the system shown in Fig. 3, let $\mathscr{J}$ be a $(\mathbf{A}, \mathscr{N}(\mathbf{C}))$-conditioned invariant, so that the relationship $(\mathbf{A}+\mathbf{KC})\mathscr{J} \subseteq \mathscr{J}$ holds for a proper choice of $\mathbf{K}$. Suppose that the set of unaccessible states is not longer $\mathscr{N}(\mathbf{C})$ but is augmented because of the lack of knowledge of the state components along $\mathscr{J}$. For instance this further uncertainty can be introduced by connecting to the output an algebraic linear block as shown in Fig. 5, characterized by a matrix G which satisfies the condition

$$\mathscr{N}(\mathbf{GC}) = \mathscr{N}(\mathbf{C}) + \mathscr{J}. \tag{21}$$

The following Theorem 6 provides the minimum set of states unobservable under these conditions. It is a subspace and will be called the *minimum unobservable subspace constrained by* $\mathscr{J}$ and denote by $\mathrm{mus}(\mathbf{A}, \mathscr{N}(\mathbf{C})), \mathscr{J}$.

THEOREM 6. Given $(\mathbf{A}, \mathscr{N}(\mathbf{C}))$-conditioned invariant $\mathscr{J}$, the minimum unobservable subspace constrained by $\mathscr{J}$ is defined by the relationship

$$\mathrm{mus}(\mathbf{A}, \mathscr{N}(\mathbf{C}), \mathscr{J}) = \mathrm{MI}(\mathbf{A}+\mathbf{KC}, \mathscr{N}(\mathbf{C}) + \mathscr{J}) \tag{22}$$

where $\mathbf{K}$ is any matrix such that $(\mathbf{A}+\mathbf{KC})\mathscr{J} \subseteq \mathscr{J}$.

Proof. Theorem 6 is the dual of Theorem 5 and can be proved with a procedure similar to that developed to prove Theorem 4 by considering the well-known duality theorem of Kalman [23]. Q.E.D.

The following Corollaries 3 and 4 are easily derivable consequences of Theorem 1, 2, 5 and 6.

COROLLARY 3. Given a $(\mathbf{A}, \mathscr{R}(\mathbf{B}))$-controlled invariant $\mathscr{J}$, the maximum subspace $\mathscr{J}' \subseteq \mathscr{J}$ which can be made a $(\mathbf{A}+\mathbf{BH})$-invariant internally stable by a proper choice of the matrix $\mathbf{H}$ is

$$\mathscr{J}' = \mathrm{MRS}(\mathbf{A}, \mathscr{R}(\mathbf{B}), \mathscr{J}) + (\mathscr{S}_{-}(\mathbf{A}+\mathbf{BH}) \cap \mathscr{I}\mathscr{J}) \tag{23}$$

where $\mathbf{H}$ is any matrix such that $(\mathbf{A}+\mathbf{BH})\mathscr{J} \subseteq \mathscr{J}$.

COROLARLY 4. Given a $(\mathbf{A}, \mathscr{N}(\mathbf{C}))$-conditioned invariant $\mathscr{J}$, the minimum subspace $\mathscr{J}' \supseteq \mathscr{J}$ which can be made a $(\mathbf{A}+\mathbf{KC})$-invariant externally stable by a proper choice of the matrix $\mathbf{K}$ is

$$\mathscr{J}'' = \mathrm{mus}(\mathbf{A}, \mathscr{N}(\mathbf{C}), \mathscr{J}) \cap (\mathscr{S}_{+}(\mathbf{A}+\mathbf{KC}) + \mathscr{J}), \tag{24}$$

where $\mathbf{K}$ is any matrix such that $(\mathbf{A}+\mathbf{KC})\mathscr{J} \subseteq \mathscr{J}$.

5. Partial state observers as dual of noninteracting controllers

In the present section the most important structural properties of non-interacting controlers and partial state observers will be derived by using the results previously stated and proved.

A quite interesting feature of the approach developed is to be constructive in the sense that, besides the proofs of theorem and assertions, it draws procedures for the design of actual control devices.

Assume that a model which reproduces the dynamic behaviour of the plant is connected as shown in Fig. 6. Of course if the initial states of the plant and of the model are congruent, at every instant of time the variables $\mathbf{x}'$ and $\mathbf{y}'$ represent estimates of the state $\mathbf{x}$ and the output $\mathbf{y}$ of the plant.

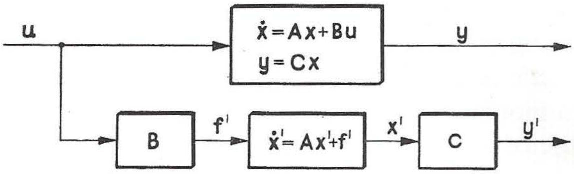

Fig. 6. The plant and the model

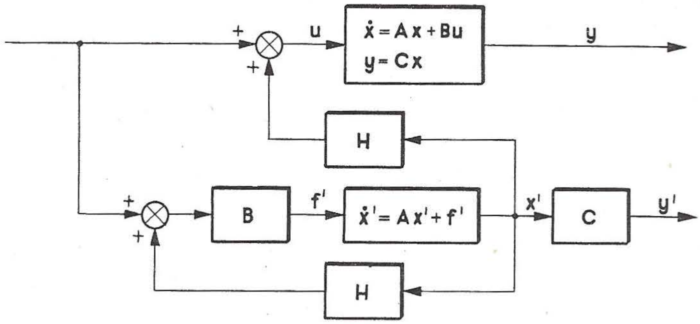

Fig. 7. A general controller or "dual observer"

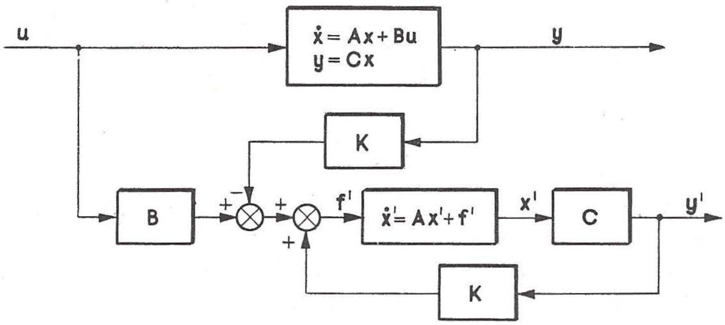

Fig. 8. A general observer

It is clear that this situation will remain also in the presence of the connections shown in Fig. 7, where the matrix $\mathbf{H}$ is completely arbitrary, because equal signals are summed to the inputs of the plant and of the model, or in the presence of the connections shown in Fig. 8, where the matrix $\mathbf{K}$ is also arbitrary, because the signals which are summed to the forcing actions of the model cancel each other. Note that, on the other hand, in force of Theorems 1 and 2 the stability of the model can always be assured if the plant is completely controllable in the case of Fig. 7 and if the plant is completely observable in the case of Fig. 8. This result is well-known and legitimately is considered a basis of system theory.

But a remarkable difference exists between the connections represented in Fig. 7 and in Fig. 8 since, while in the first case the plant follows the model, which can be considered a controller (or a "dual observer") as it has been called in [19], the second case clearly concerns a real observer, namely a device whose dynamic behaviour reproduces that of the plant and whose connection to the plant does not involve any perturbation.

From the general properties of controlled and conditioned invariants stated in the previous sections it follows that by a proper choice of the matrices **H** and **K** it is possible to specialize the general controller and the general observer in order to obtain a restricted control, as necessary for decoupling or non-ineraction, or a restricted observation, as necessary when, because of lack of input knowledge, a complete model is not realizable. These features of the devices shown in Fig. 7 and 8 are pointed out in the following Properties 5 and 6.

PROPERTY 5. Given a $(\mathbf{A}, \mathscr{R}(\mathbf{B}))$-controlled invariant $\mathscr{J}$, it is possible to synthesize a controller with state dimension equal to dim $(\mathscr{J})$ which drives the plant along all and only the state trajectories ranging over $\mathscr{J}$.

Proof. In force of Theorem 3 at least one matrix **H** exists such that $(\mathbf{A}+\mathbf{BH})\mathscr{J} \subseteq \mathscr{J}$. Hence under the state to input feedback shown in Fig. 7 $\mathscr{J}$ becomes an invariant in the state space of the controller. Furthermore Corollary 1 states that state coordinates of the controller restricted to $\mathscr{J}$ (for instance a projection of the state on $\mathscr{J}$) are sufficient for such a feedback, so that if the control action and the initial state are restricted to cause state trajectories ranging over $\mathscr{J}$ it is not necessary to reproduce in the controller the remaining of the state space. Q.E.D.

PROPERTY 6. Given a $(\mathbf{A}, \mathscr{N}(\mathbf{C}))$-conditioned invariant $\mathscr{J}$, it is possible to synthesize an observer with state dimension equal to n-dim$(\mathscr{J})$ such that the projection of the state on any complement of $\mathscr{J}$ along $\mathscr{J}$ is reproduced by the state of the observer.

Proof. In force of Theorem 4 at least one matrix **K** exists such that $(\mathbf{A}+\mathbf{KC})\mathscr{J} \subseteq \mathscr{J}$. Hence under the output to forcing action feedback shown in Fig. 8 $\mathscr{J}$ becomes an invariant in the state space of the observer. Furthermore Corollary 2 states that it is sufficient for this purpose that feedback acts only on the state coordinates of the observer restricted to a complement of $\mathscr{J}$ (for instance a projection of the state along $\mathscr{J}$ on any complement of $\mathscr{J}$), so that if only the projection of the state on a complement of $\mathscr{J}$ is sought it is not necessary to reproduce in the observer the remaining of the state space. Q.E.D.

In the practical implementations stability is the most important requirement for the partial controller considered in Property 5 or the partial observer considered in Property 6. If stability is required for these devices, in force of Corollaries 3 and 4 of previous section the controlled invariant $\mathscr{J}$ of Property 5 must be restricted to the controlled invariant $\mathscr{J}'$ defined by relationship (23) and the conditioned invariant $\mathscr{J}$ of Property 6 must be extended to the conditioned invariant $\mathscr{J}'$ defined by relationship (24).

Now the above presented properties of controlled and conditioned invariants are employed for the solution of the problem of synthesizing a stable unknown-input observer of the maximal dimension.

PROBLEM 1. Let **u** be accessible except for a given subspace $\mathscr{U} \subseteq R^m$. Synthesize a (stable) dynamical observer which provides the maximum information about the state of the plant.

A solution of Problem 1 is provided by the following procedure, which, even if outlined from a geometrical viewpoint, reduces to easily mechanizable matrix computations.

a. Determine the subspace

$$\mathscr{X}' = \mathrm{mci}(\mathbf{A}, \mathscr{N}(\mathbf{C}), \mathbf{B}\mathscr{U}), \tag{25}$$

i.e. the minimum conditioned invariant which contains the subspace of the unknown forcing actions. In force of Property 6 an observer with state dimension n-dim($\mathscr{X}'$) can be synthesized as shown in Fig. 9, choosing the matrix **K** in such a way that

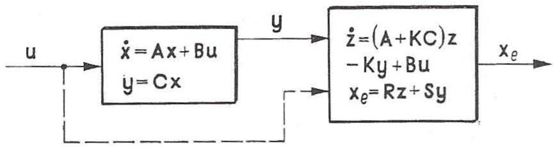

Fig. 9. An unknown-input dynamic observer

$(\mathbf{A}+\mathbf{KC})\mathscr{X}' \subseteq \mathscr{X}'$[5]. This is clearly the maximum observer whose dynamic behaviour is not affected by the unknown input. In fact the corresponding forcing action ranges on the $(\mathbf{A}+\mathbf{CK})$-invariant $\mathscr{X}'$, which is not reproduced in the observer.

b. If stability is required, in force of Corollary 4 the following $(\mathbf{A}, \mathscr{N}(\mathbf{C}))$-conditioned invariant must be considered instead of $\mathscr{X}'$:

$$\mathscr{X}'_s = \mathrm{MI}(\mathbf{A}+\mathbf{KC}, \mathscr{N}(\mathbf{C})+\mathscr{X}')) \cap (\mathscr{S}_+(\mathbf{A}+\mathbf{KC})+\mathscr{X}'). \tag{26}$$

Being $\mathscr{X}' \subseteq \mathscr{X}'_s$, the achievement of stability in general involves a reduction of the state dimension of the observer with respect to the case in which stability is not required.

The state z of the observer provides an estimate of the state in the plant except for the subspace $\mathscr{X}'$ (or $\mathscr{X}'_s$ if stability is required) and the output directly provides the state except for $\mathscr{N}(\mathbf{C})$, so that an estimate **x** whose uncertainty is restricted to $\mathscr{N}(\mathbf{C}) \cap \mathscr{X}'$ (or $\mathscr{N}(\mathbf{C}) \cap \mathscr{X}'_s$) can be obtained as a linear function $\mathbf{Rz}+\mathbf{Sy}$. To sum up, the following theorem has been proved.

THEOREM 7. If the input **u** of the plant is accessible except for a subspace $\mathscr{U} \subseteq R^m$, the minimum unobservability subspace which affects the estimate of the state provided by a dynamic device of the type shown in Fig. 9 is $\mathscr{N}(\mathbf{C}) \cap \mathscr{X}'$ (where $\mathscr{X}'$ is defined by (25)) if stability is not required for the observer, $\mathscr{N}(\mathbf{C}) \cap \mathscr{X}'_s$ where $\mathscr{X}'_s$ is defined by (26) if stability is required.

Hence using a dynamic observer of the type shown in Fig. 9, it is possible to derive the projection of the state on subspaces not larger than complements of

[5] Of course, the state dimension of the observer being reduced, a change of state coordinates is necessary (and some of the new coordinates are not reproduced), but for the sake of simplicity if does not appear in the equations reported in Fig. 10.

$\mathcal{N}(\mathbf{C}) \cap \mathcal{X}'$ or $\mathcal{N}(\mathbf{C}) \cap \mathcal{X}'_s$. If a greater information is sought, it is necessary to resort to a different class of observers, that is to devices employing dofferentiators. In fact by means of differentiators the direct information about the state provided by the output can be enlarged, obtaining in practice something analogous to a reduction of the subspace of the unaccessible states $\mathcal{N}(\mathbf{C})$. Therefore a general observer including differentiators consists of two parts in cascade; a device composed of differentiators and algebraic operators and a dynamic observer of the type shown in Fig. 9.

The minimum unobservability subspace with differentiators and algebraic operators is provided by the following Theorem 8, stated for the first time in [3]. Also, a possible constraint on the maximum allowed number of differentiators stages in cascade is taken into account in the statement.

THEOREM 8. If the input of the plant is accessible except for a subspace $\mathcal{U} \subseteq R^m$, the minimum unobservability subspace $\mathcal{X}''_k$ which affects the estimate of the state provided by a non-dynamic device including k stages of differentiators in cascade is given by the following sequence

$$\mathcal{X}''_0 = \mathcal{N}(\mathbf{C}),$$
$$\mathcal{X}''_i = \mathcal{N}(\mathbf{C}) \cap \mathbf{A}^{-1*}(\mathcal{X}''_{i-1} + \mathbf{B}\mathcal{U}), \quad i = 1, 2, \ldots, k. \tag{27}$$

If the number of cascaded differentiators stages is not bounded, the unobservability subspace clearly becomes

$$\mathcal{X}'' = \mathrm{MCI}(\mathbf{A}, \mathbf{B}\mathcal{U}, \mathcal{N}(\mathbf{C})). \tag{28}$$

Proof. Write equation (1b) as

$$\mathbf{q}_0 = \mathbf{M}_0 \mathbf{x}, \tag{29}$$

where $\mathbf{q}_0 = \mathbf{y}$, $\mathbf{M}_0 = \mathbf{C}$. From equation (29) the state can be recognized except for the subspace

$$\mathcal{X}''_0 = \mathcal{N}(\mathbf{M}_0) = \mathcal{N}(\mathbf{C}). \tag{30}$$

By differentiating (29) and employing (1a), the differential equation

$$\mathbf{q}_0 = \mathbf{M}_0 \mathbf{A}\mathbf{x} + \mathbf{M}_0 \mathbf{B}\mathbf{u} \tag{31}$$

is obtained. Let $\mathbf{P}_0$ be a projecting matrix along $\mathbf{M}_0 \mathbf{B}\mathcal{U}$, so that $\mathcal{N}(\mathbf{P}_0) = \mathbf{M}_0 \mathbf{B}\mathcal{U}$ and in the differential equation

$$\mathbf{P}_0 \dot{\mathbf{q}}_0 = \mathbf{P}_0 \mathbf{M}_0 \mathbf{A}\mathbf{x} + \mathbf{P}_0 \mathbf{M}_0 \mathbf{B}\mathbf{u} \tag{32}$$

the unknown inputs are not present. Equations (29) and (32) can be written together as

$$\begin{bmatrix} \mathbf{q}_0 \\ \mathbf{P}_0 \dot{\mathbf{q}}_0 \end{bmatrix} = \begin{bmatrix} \mathbf{M}_0 \\ \mathbf{P}_0 \mathbf{M}_0 \mathbf{A} \end{bmatrix} \mathbf{x} + \begin{bmatrix} \mathbf{0} \\ \mathbf{P}_0 \mathbf{M}_0 \mathbf{B} \end{bmatrix} \mathbf{u}, \tag{33}$$

or, in more compact form,

$$\mathbf{q}_1 = \mathbf{M}_1 \mathbf{x} + \mathbf{f}_1, \tag{34}$$

where the vector $\mathbf{q}_1$ is a known linear function of the output and its first derivatives whereas the vector $\mathbf{f}_1$ is a known linear function of the accessible part of the input

From (34) the state can be recognized except for the subspace

$$\mathscr{X}_1'' = \mathscr{N}(\mathbf{M}_1) = \mathscr{N}(\mathbf{M}_0) = \mathscr{N}(\mathbf{M}_0) \cap \mathscr{N}(\mathbf{P}_0\mathbf{M}_0\mathbf{A}) =$$
$$= \mathscr{N}(\mathbf{M}_0) \cap \mathbf{A}^{-1*}\mathbf{M}_0^{-1*}\mathscr{N}(\mathbf{P}_0) = \mathscr{N}(\mathbf{M}_0) \cap \mathbf{A}^{-1*}\mathbf{M}_0^{-1*}\mathbf{M}_0\mathbf{B}\mathscr{U} =$$
$$= \mathscr{N}(\mathbf{M}_0) \cap \mathbf{A}^{-1*}(\mathscr{N}(\mathbf{M}_0)+\mathbf{B}\mathscr{U}). \qquad (35)$$

Starting from (34) instead of (29) and applying k times the same procedure, the sequence of subspaces

$$\mathscr{N}(\mathbf{M}_i) = \mathscr{N}(\mathbf{M}_{i-1}) \cap \mathbf{A}^{-1*}(\mathscr{N}(\mathbf{M}_{i-1})+\mathbf{B}\mathscr{U}), \qquad i = 1, 2, \ldots, k \qquad (36)$$

is obtained. Being $\mathscr{N}(\mathbf{M}_i) \subseteq \mathscr{N}(\mathbf{M}_{i-1})$, hence $\mathbf{A}^{-1*}(\mathscr{N}(\mathbf{M}_i)+\mathbf{B}\mathscr{U}) \subseteq \mathbf{A}^{-1*}(\mathscr{N}(\mathbf{M}_{i-1})+\mathbf{B}\mathscr{U})$, equations (36) by recursive substitution of the first term in the right side members are easily proved to be equivalent to (27). Q.E.D.

6. Unknown-input observers and inverse systems for non purely dynamical plants

Theorem 7 and Theorem 8 of the previous section provide complete algorithms for testing unknown-input observability of purely dynamic system by means of dynamic devices including differentiators in presence of lack of information on the input.

These results are of primary importance since a very trivial mathematical trick makes possible their application also to the solution of a much more general problem, that is the combined state and input observation of a non purely dynamical system. This problem includes as a special case that of the system in version, without or with the stability constraint.

Consider the non purely dynamic system

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}+\mathbf{B}\mathbf{u}, \qquad (37a)$$

$$\mathbf{y} = \mathbf{C}\mathbf{x}+\mathbf{D}\mathbf{u}. \qquad (37b)$$

Assume that a block of integrators is connected to the input a shown in Fig. 10, so that an overall non purely dynamical system with state dimension $n+m$ is obtained.

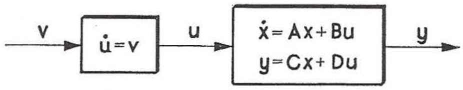

Fig. 10. Augmentig the state of the plant

Let $\hat{\mathbf{x}} = [\mathbf{x}, \mathbf{u}]$ be the augmented state, so that the equations of the overall system are

$$\dot{\hat{\mathbf{x}}} = \hat{\mathbf{A}}\hat{\mathbf{x}}+\hat{\mathbf{B}}\mathbf{x}, \qquad (38a)$$

$$\mathbf{y} = \hat{\mathbf{C}}\hat{\mathbf{x}}, \qquad (38b)$$

where

$$\hat{\mathbf{A}} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}, \quad \hat{\mathbf{B}} = \begin{bmatrix} \mathbf{0} \\ \mathbf{I}_m \end{bmatrix}, \quad \hat{\mathbf{C}} = [\mathbf{C}\ \mathbf{D}]. \tag{39}$$

The most general observer for the case where the input is completely unknown is implemented as shown in Fig. 11, i.e. by an algebraic unit including differentiatots

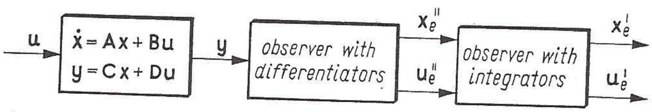

Fig. 11. The implementation of a dynamic observer or an inverse system employing differentiators

followed by a dynamic unit[6]. According to Theorem 8, the observer with differentiators provides at most an estimate of the augmented state except for the subspace:

$$\hat{\mathscr{X}}'' = \mathrm{MCI}(\hat{\mathbf{A}}, \mathscr{R}(\hat{\mathbf{B}}), \mathscr{N}(\hat{\mathbf{C}})). \tag{40}$$

The unobservability subspace (40) is pertinent to the case where the numbe of cascaded differentiatot stages is not restricted. If it is bounded to be at most k, according to Theorem 8 the recursive procedure for the computation of the controlled invariant appearing in (40) must be stopped at the k-th step and the corresponding subspace $\hat{\mathscr{X}}''_k$ must be considered instead of (40).

Then, taking information from the observer with differentiators, a dynamic observer can provide an estimate of the augmented state except for the subspace $\hat{\mathscr{X}}' \cap \hat{\mathscr{X}}''$ or, if stability is required, $\hat{\mathscr{X}}'_s \cap \hat{\mathscr{X}}''$, being

$$\hat{\mathscr{X}}' = \mathrm{mic}(\hat{\mathbf{A}}, \hat{\mathscr{X}}'', \mathscr{R}(\hat{\mathbf{B}})), \tag{41}$$

$$\hat{\mathscr{X}}'_s = \mathrm{MI}(\hat{\mathbf{A}}+\hat{\mathbf{K}}\hat{\mathbf{C}}, \hat{\mathscr{X}}'+\hat{\mathscr{X}}'') \cap (\mathscr{S}_+(\hat{\mathbf{A}}+\hat{\mathbf{K}}\hat{\mathbf{C}})+\hat{\mathscr{X}}'), \tag{42}$$

where $\hat{\mathbf{K}}$ is any matrix such that $(\hat{\mathbf{A}}+\hat{\mathbf{K}}\hat{\mathbf{C}})\hat{\mathscr{X}}' \subseteq \hat{\mathscr{X}}'$.

The general observer represented in Fig. 11 provides estimates of the state and the input as a unique vector. In other words it reproduces a possible correlation between state and input unobservable components. Criteria for testing the complete state observability and the invertibility of the dynamical system (37) are provided by the following corollaries, which can be derived as straightforward applications of the previously developed theory.

COROLLARY 5. The state of the plant (37) is completely unknown-input observable by means of a dynamic device including differentiators if and only if the subspace $\hat{\mathscr{X}}' \cap \hat{\mathscr{X}}''$ (or $\hat{\mathscr{X}}'_s \cap \hat{\mathscr{X}}''$ if stability is required) is contained in $\mathscr{R}(\hat{\mathbf{B}})$.

COROLLARY 6. The plant (37) is completely invertible by means of a dynamic device including differentiators if and only if the subspace $\hat{\mathscr{X}}' \cap \hat{\mathscr{X}}''$ (or $\hat{\mathscr{X}}'_s \cap \hat{\mathscr{X}}''$ if stability is required) is contained in $\mathscr{N}(\hat{\mathbf{B}}^{\mathrm{T}})$.

[6] The approach can be easily extended to the more general case where the input is known except for a subspace, which is not treated here for the sake of notational simplicity.

7. Conclusions

It has been shown how the well known theory of dynamic observers, which in the last few years has been the object of many interesting researches in the field of system theory, can be extended to the case in which some of the inputs of the observed system are unaccessible for measurement.

The approach herein developed has been based on the concepts of controlled and conditioned invariance, which seem to be very efficient tools for simplifying the mathematical treatment of many control problems which usually involve very complex matrix operations. In particular, also the problem of stability of the unknown-input observers, which has not been previously considered, has been stated and approached in geometrical terms.

Furthmermore the convenience of considering together the problems of observing the state and the input has been pointed out and a synthesis procedure for inverse systems, which, taking into account also constraints, is much more complete and exhaustive of those previously treated in the literature, has been presented.

References

1. Basile G., Laschi R., Marro G., Invarianza controllata e noninterazione nello spazio degli stati. *L'Elettronica* **56**, 1 (1969).
2. Basile G., Marro G., Controlled and conditioned invariant subspaces in linear system theory. *J. Optimiz. Theory and Appl.* **3**, 5 (1969).
3. Basile G., Marro G., On the observability of linear time-invariant systems with unknown inputs. *J. Optimiz. Theory and Appl.* **3**, 6 (1969).
4. Laschi R., Marro G., Alcune considerazioni sull'osservabilitá dei sistemi dinamici con ingressi inaccessibili. Proc. 70 Riunione Annuale AEI, Rimini 1969. Paper 1.1.06.
5. Basile G., Marro G., A state space approach to non-interacting controls. *Ric. Autom.* **1**, 1 (1970).
6. Basile G., Marro G., On the perfect output controllability of linear dynamic systems. *Ric. Autom.* **2**, 1 (1971).
7. Basile G., Marro G., Structural problems of regulators. In: Theory and applications of variable-structure systems with emphasis on modeling and identification. Eds. A. Ruberti and R. R. Mohler. New York 1972.
8. Wonham W. M., Morse A. S., Decoupling and pole assignment in multivariable systems: a geometric approach. *SIAM J. Contr.* **8**, 1 (1970).
9. Morse A. S., Wonham W. N., Decoupling and pole assignment by dynamic componsation. *SIAM J. Contr.* **8**, 3 (1970).
10. Brockett R. W., Poles, zeros and feedback: state space interpretation. *IEEE Trans. Autom. Contr.* **AC-10**, 2 (1965).
11. Sain M. K., Massey J. L., Invertibility of linear time in variant dynamical systems. *IEEE Trans. Autom. Contr.* **AC-14**, 2 (1969).
12. Dorato P., On the inverse of linear dynamic systems. *IEEE Trans. Syst. Scie. and Cyber.* **SSC-5**, 1 (1969).
13. Silverman L. M., Inversion of multivariable linear systems. *IEEE Trans. Autom. Contr.* **AC-14**, 3, (1969).
14. Pal Singh S., A note on iversion of linear systems. *IEEE Trans. Autom. Contr.* **AC-15**, 4 (1970).

15. Panda S. P., A direct proof of equivalence of invertibility criteria of multivariable linear system. *Int. J. Contr.* **13**, 5 (1971).
16. Panda S. P., Silverman L. M., Massey J. L., Comments of inversion of multivariable linear systems. *IEEE Trans. Autom. Contr.* **AC-15** 4 (1970).
17. Luenberger D. G., Observing the state of a linear systems. *IEEE Trans. Milit. Electron.* **MIL-8**. (1964) 74–80.
18. Luenberger D. G., Observers for multivariable systems. *IEEE Trans. Autom. Contr.* **AC-11**, **2** (1966).
19. Luenberger D. G., An introduction to observers. *IEEE Truns. Autom. Contr.* **AC-16**, 6 (1971).
20. Langenhop C. E., On the stabilization of linear systems. *Proc. Amer. Math. Soc.* **15**, 5 (1964).
21. Wonham W. M., On pole assignment in multi-input controllable linear systems. *IEEE Trans. Autom. Contr.* **AC-12**, 6 (1967).
22. Basile G., Marro G., Relazione fra la stabilità e la controllabilità dei sistemi su alcuni sottospazi caratteristici. *L'Elettrotecnica* **57**, 2 (1970).
23. Kalman R. E., On the general theory of control systems. Proc. I IFAC Congr. Vol. 1. London 1961.
24. Sarachik P. E., State estimation from measurements containing correlated noise using differentiators. *Int. J. Contr.* **13**, 1 (1971).

Niezmienniczość podprzestrzeni sterowalności i podprzestrzeni nieobserwowalności w syntezie obserwatorów nieznanych wejść i układów odwrotnych

Wyprowadzono na podstawie pojęcia niezmienniczości podprzestrzeni sterowalności i podprzestrzeni nieobserwowalności ogólną systematyczną procedurę syntezy urządzeń odtwarzających stan lub nieznane wejścia dla liniowych układów synamicznych stacjonarnych. Zasadniczymi cechami tego podejścia, odróżniającymi je od dotychczas stosowanych, są prosta i jednolita metoda geometryczna oraz możliwość uwzględnienia warunku stabilności urządzeń syntezowanych.

Инвариантность подпространства управляемости и подпространства ненаблюдаемости при синтезе наблюдателя неизвестных входов и обратных систем

На основе понятия инвариантности подпространства управляемости и подпространства ненаблюдаемости разработана общая систематическая процедура синтеза устройств воспроизводящих состояние или неизвестные входы для линейных динамических стационарных систем. К преимуществам такого подхода, отличающими его от ранее применяемых, относятся простота и единый геометрический метод а также возможность учитывания условия устойчивости синтезируемых устройств.