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On the reachability of a given set under disturbances*

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For a control system subject to disturbances, the problem is considered of finding the set in state space starting from which an initial state can be brought into a given target set, when the target is to be reached "for sure" and the disturbances are bound to belong to a given set.

The problem is considered for a different information structures available to the controller, and the key points for its solution are analized using the concept of geometrical difference of sets. For the special case of unbounded controls and disturbances, and of target set given by a linear subspace, the determination of the "starting set" is carried out for two among the possible information structures, and interesting properties are shown which are valid in this case.

1. Introduction

Given a control system subject to partialy unknown perturbations, a problem which arises naturally is to find the set X_0 of initial states x_0 which may be transfered "for sure" into a given target set X_N by an admissible control.

Many practical examples of such a problem may be found in various fields of system science, of operational research and managment science, when a "worst case" or conservative phylosophy is to be adopted, that is when the aim is to reach a goal for sure, not to maximize the expectation of a given event; an intuitive case is for example the problem of determining the zone starting from which a given vehicle can reach for sure a target in a noisy environment. This kind of problems, as well as the determination of the best control in the worst possible conditions, has been formally treated only recently, either in itself or in the framework of differential game theory, in which it may be embedded if the partially unknown disturbances or environmental situations are conservatively treated as the adversary of game theory.

The algorithms which are presently available for the solution of the problem above, even for linear systems with admissible controls and disturbances belonging

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to convex compact sets, are generally quite difficult to be implemented and it seems that further research is needed in order to furnish flexible algorithms for facing practical situations. The purpose of this paper is to sketch the key points of the methods of solution proposed until now for different information structures available both to the controller and to the adversary disturbances, and to show some interesting simplifications which are consequence of taking a subspace as a target set and any control and disturbance as admissible; for this case it is possible to state some interesting results in a compact form, as we shall do.

2. Statement of some problems

As an example of the mentioned problems, let us consider the following linear discrete time system

$$x_{K+1} = Ax_K + Bu_K + Cw_K, \qquad K = 0, 1, \dots, \tag{1}$$

where the states x_K belong to R^n, controls u_K and disturbances w_K to closed compact sets U of R^m and W of R^r respectively, and the target is a given closed compact set X_N of R^n (many types of extensions, like the one to time-varying systems with sets U and W varying also with time instant are possible but inessential to our purposes).

We formulate the problem of reachability of X_N from X_0 under disturbances by giving two among the main possible information structures.

PROBLEM 1. Determine the set X_0^1 of initial states x_0 which may be transferred into X_N at time instant N by an admissible sequence $\{u_0(x_0), u_1(x_1), \dots, u_{N-1}(x_{N-1})\}$ of controls, for any admissible sequence $\{w_0, w_1, \dots, w_{N-1}\}$ of disturbances. Clearly, for each of its "moves" u_i the controller may take into account his perfect information about the present state x_i, but is ignorant about the "move" w_i of the disturbances.

PROBLEM 2. Determine the set X_0^2 of initial states x_0 which may be transferred into X_N at time instant N by an admissible sequence $\{u_0(x_0, w_0), \dots, u_{N-1}(x_{N-1}, w_{N-1})\}$ of controls for any admissible sequence $\{w_0, \dots, w_{N-1}\}$ of disturbances.

Obviously, the information available to the controller is in this case "larger" than in the previous case, since in each step he knows the "move" of the disturbances.

Even for these simple examples of the general problem, with the mentioned clearly defined information structures, the used methods for finding X_0^1 or X_0^2 have required the use of the so called operation of geometric difference of sets[1] and either the use of separation theorems for convex compact sets [1, 2], or the use of support

[1] Given two sets S and T, their geometric difference $Z = S - T$ is defined as $Z = \{z : z + T \subset S\}$ ($Z = \emptyset$ if $S \not\supset T$).

functions to describe sets and set—inclusion [3, 4], or some ellipsoid—type or polyedrical-type approximation of sets [5, 6], which have the advantage of describing sets with a finite number of parameters but can give only sufficient conditions for a point x_0 to be transferrable into X_N, that is can give only subsets of X_0^1 or X_0^2.

When the instant of time at which reachability occurs is of interest, game theoretical approaches to the above kind of problems are interesting, but computational results are rather involved indeed [7, 8].

3. Formal solution of problems 1 and 2; definition of set regularity

Referring to problem 1, and using equation (1), we see that at $(N-1)^{\text{th}}$ step the state x_{N-1} is "transferrable" into X_N iff there exists $u_{N-1}(x_{N-1}) \in U$ such that

$$Ax_{N-1}+Bu_{N-1}+Cw_{N-1} \in X_N,$$

for every $w_{N-1} \in W$, that is iff there exists $u_{N-1}(x_{N-1}) \in U$ such that

$$Ax_{N-1}+Bu_{N-1}+CW \in X_N$$

or

$$Ax_{N-1}+Bu_{N-1} \in (X_N-CW),$$

that is iff

$$Ax_{N-1} \in (X_N-CW)-BU.$$

Therefore the set X_{N-1} of states x_{N-1} which may be transferred into X_N in one step satisfies the following equation

$$Y_{N-1}^1 = AX_{N-1}^1 = (X_N-CW)-BU$$

and similarly the set X_{N-i} $(i = 1, \ldots, N)$ of states which may be transferred into X_N in i steps satisfies:

$$Y_{N-i}^1 = A^iX_{N-i}^1 = (Y_{N-i+1}^1-A^{i-1}CW)-A^{i-1}BU, \quad i = 1, \ldots, N. \tag{2}$$

Equations (2), together with the obvious equality

$$Y_N^1 = X_N, \tag{3}$$

is a recursive algorithm for building $Y_{N-1}, \ldots, Y_0$ from U, W, X_N and therefore X_0^1 which is characterized by

$$A^NX_0^1 = Y_0^1. \tag{4}$$

Proceeding in an analogous manner for problem 2, and taking into account the different information structure, we have that in this case x_{N-1} may be transferred into X_N in one step iff for every $w_{N-1} \in W$ there exists $u_{N-1}(x_{N-1}, w_{N-1}) \in U$ such that

$$Ax_{N-1}+Bu_{N-1}+Cw_{N-1} \in X_N,$$

that is iff

$$Ax_{N-1}+CW \in X_N-BU$$

Therefore the set X_{N-1}^2 of states x_{N-1} of this type is given by

$$Y_{N-1}^2 = AX_{N-1}^2 = (X_N - BU) - CW$$

and similarly the sets X_{N-i}^2, $i = 1, \ldots, N$, of states which may be transferred into X_N in i steps by admissible controls and for every disturbance satisfy:

$$Y_{N-i}^2 = A^i X_{N-i}^2 = (Y_{N-i+1}^2 - A^{i-1}BU) - A^{i-1}CW, \quad i = 1, \ldots, N. \tag{2'}$$

We have again found a recursive algorithm which, starting again backward from

$$Y_N^2 = X_N \tag{3'}$$

gives the following characterization of X_0^2:

$$A^N X_0^2 = Y_0^2. \tag{4'}$$

As it is apparent from (2), (3), (4) or from (2'), (3'), (4'), the determination of X_0^1 or of X_0^2 involves essentially the computation of N geometric differences and N additions of sets. The analogous of problem 1 and problem 2 for continuous systems considering only open loop controls would bring to a similar computation for only one step, but the matrices $A^{i-1}B$ and $A^{i-1}C$, $i = 0, \ldots, N-1$, would be substituted by linear integral operators from sets U and W of control and disturbance functions into R^n; separation theorems techniques for convex compact sets have been used (see [1, 2]) for this case. For continuous systems and closed loop controls see the approach of [8].

Going back to consider equations (2), (3), (4) or (2'), (3'), (4'), we see that the key difficulty is the description of sets resulting from operations on sets. A natural tool to be used for this purpose is the one given by support functions (see for example [3, 9]), when sets X_N, U, W, and therefore also all other sets involved, are convex and closed. Support functions are linear with respect to the addition of such a kind of sets; unfortunately, such a property is not enjoyed by the operation of geometric difference of sets: if h is the support function of the set $X: h_X(p) = \sup_{x \in X} \langle p, x \rangle$, $\forall\, p \in X^*$, when X^* is the dual of the space X, then for every set S, T

$$h_{S-T}(p) \leqslant h_S(p) - h_T(p). \tag{5}$$

It is therefore important to find conditions on the sets involved in (2), (3), (4) $\big($or (2'), (3'), (4')$\big)$ for (5) to be hold as an equality (we suppose for simplicity that the geometric differences are never the empty set). In such a case, the construction of the sets $Y_{N-1}^1, \ldots, Y_0^1$ from Y_N^1 and of $Y_{N-1}^2, \ldots, Y_0^2$ from Y_N^2 would be straightforward and we would also have the interesting consequence that these two sequences of sets would coincide if $Y_N^1 = Y_N^2$; therefore X_0^1 would also coincide with X_0^2 and the difference in the information structures of problems 1 and 2 would not have any effect.

Defining $S_T = (S-T)+S$, $(S_T \subset S)$ as the "regular part" of S with respect to T, we define the set S to be regular with respect to T iff $S_T = S$. It is easy to prove that (5) holds as an equality iff S is regular with respect to T: indeed, from the regularity it follows that

$$h_{S-T}(p) + h_T(p) = h_S(p), \ \forall p,$$

that is

$$h_{S-T}(p) = h_S(p)+h_T(p), \quad \forall p, \tag{6}$$

and the regularity follows from (6) for any couple of sets S and T completly described by the support functions.

These last remarks seem an important reason for further research for finding conditions easy to be expressed on couples of sets (representing respectively target sets and reachable regions) in order the first one to be regular with respect to the second one.

4. Unbounded controls and distarbances, and subspaces as target sets

A case in which the regularity conditions are trivially satisfied is when the sets under consideration are linear subspaces. Sets of this kind enjoy the further following absorption property:

$$S-T = \begin{cases} S \text{ if } S \supset T \\ \emptyset \text{ if } S \not\supset T \end{cases}. \tag{7}$$

An interesting consequence of this property applies to our problems 1 and 2 when U and W are R^m and R^r respectively, and the target set X_N is a linear subspace M of R^n. In this case we can define the orthogonal complement N of M in R^n: R^n $N+M$ and the projection operator Π of R^n onto N. We have

$$\Pi x \in N, \quad \forall x \in R^n, \quad \Pi x = \{0\}, \quad \forall x \in M.$$

The target set X_N is therefore characterized by

$$M_N = \Pi X_N = \Pi M = \{0\}. \tag{8}$$

Referring to problem 1 and using equation (1) we have therefore, for x_{N-1} to be transferrable into M in one step:

$$\Pi(Ax_{N-1}+Bu_{N-1}+Cw_{N-1}) = M_N = \{0\}, \quad \forall w_{N-1} \in W,$$

that is

$$\Pi AX^1_{N-1} = (M_N-CW)+BU = M^1_{N-1}$$

and using again equation (1)

$$\underset{(i=1,\dots N)}{M^1_{N-i}} = \Pi A^i X^1_{N-i} = (M^1_{N-i+1}-\Pi A^{i-1}CW)-\Pi A^{i-1}BU,$$

which gives an iterative algorithm for the construction of sets X^1_{N-i} of states which may be transferred into X_N in i steps, until

$$M^1_0 = \Pi A^N X_0 = (M^1_1-A^{N-1}CW)-\Pi A^{N-1}BU$$

which characterizes X^1_0.

Recalling $U = R^m$ and $W = R^r$, and defining

$$P_i = \Pi(\operatorname{span} A^{i-1}B), \quad Q_i = \Pi(\operatorname{span} A^{i-1}C), \tag{9}$$

we have

$$M^1_{N-i} = \Pi A^i X_{N-i} = (M^1_{N-i+1} - Q_i) + P_i, \quad i = 1, \ldots, N, \tag{10}$$

that is, using the absorption property of geometrical difference for subspaces

$$\begin{aligned} M^1_{N-i} &= \Pi A^i X^1_{N-i} = M_{N-i+1} + P_i, && \text{if} \quad M^1_{N-i+1} \supset P_i. \\ M_{N-1} &= \emptyset, && \text{if} \quad M^1_{N-i+1} \not\supset P_i \end{aligned} \tag{11}$$

From (11) we see that M_{i-1} is not empty iff

$$Q_1 = \{0\}, \quad \text{and} \quad P_1 \supset Q_2, (P_1+P_2) \supset Q_3, \ldots, \sum_1^{i-1} jP_j \supset Q_i. \tag{12}$$

Proceeding in an analogous way for problem 2 in this case, we have again (8) and

$$M^2_{N-i} = \Pi A^i X^2_{N-i} = (M^2_{N-i+1} + P_i) - Q_i \tag{10'}$$

until

$$M^2_0 = \Pi A^N X^2_0 = (M^2_1 + P_N) - Q_N.$$

Using again the absorption property we have

$$\begin{aligned} M^2_{N-i} &= M^2_{N-i+1} + P_i, && \text{if} \quad M^2_{N-i+1} + P_i \supset Q_i \\ M^2_{N-1} &= \emptyset, && \text{if} \quad M^2_{N-i+1} + P_i \not\supset Q_i. \end{aligned} \tag{11'}$$

From (11′) we see that M^2_{N-i} is not empty iff

$$P_1 \supset Q_1, (P_1+P_2) \supset Q_2, \ldots, \sum_1^{i} jP_j \supset Q_i. \tag{12'}$$

As it is intuitive condition (12) for the existance of some x_0 which can be brought into X_N in i steps (for example in $i = N$ steps) is more strict than condition (12′), which corresponds to an information structure more favorable for the controller. Nevertheless, it is very interesting to notice that when (12) is satisfied, then

$$M^1_0 = \sum_1^N jP_j = M^2_0.$$

We may summarize the proceeding results in the following:

THEOREM 1. The set X^1_0 solution of the problem 1 for $U = R^m$, $W = R^r$ and X_N given by a subspace M of R^n is not empty iff equations (12) are satisfied, and is characterized by

$$\Pi A^N X^1_0 = M^1_0,$$

where M^1_0 is given by the recursive algorithm (11) starting from $M^1_N = \{0\}$. Analogously, the set X^2_0 is not empty iff equation (12′) are satisfied, and is characterized by

$$\Pi A^N X^2_0 = M^2_0,$$

where M_0^2 is given by the recursive algorithm (11′) starting from $M_N^2 = \{0\}$. Furthermore, if X_0^1 is not empty, then $X_0^1 = X_0^2$, with

$$M_0^1 = M_0^2 = \sum_1^N jP_j.$$

For the particular system

$$x_{K+1} = Ax_K + Bu_K + w_K, \qquad K = 0, 1, \ldots, \tag{13}$$

we have the following

COROLLARY. Considering problem 2, if X_0^2 is not empty for some N, then
(a) any $x_0 \in X_0^2$ can be brought into M in an arbitrary number of steps,
(b) $X_0^2 = R^n$.

Proof. Since (12′) is satisfied with $Q_1 = \Pi I = L$, we have $P_1 \supset Q_1 = L$; on the other hand $\sum_1^i {}_j P_j \supset \sum_1^{i-1} {}_j P_j, \ldots, \supset P_1$; therefore, since all P_i are in L, $\sum_1^i {}_j P_j = L$ for any i, that is $M_0^2 = M_1^2 = ,\ldots, = M_N^2 = L$. Recalling $M_i^2 = \Pi\, A^i X_0^2$, part (a) is proved.

Part (b) follows from the characterization of X_0^2:

$$X_0^2 = \{x : \Pi A^N x \in M_0\},$$

from the definition of Π and from $M_0 = L$.

We may observe that in our case, since $C = I$, $\sum_1^{i-1} {}_j P_j \supset ,\ldots, \supset P_1 \supset Q_1 = \Pi R^n$, that is Π span $[B, AB, \ldots, A^{N-1}B^n] \supset \Pi R$. This means the system was controllable (in classical sense) at least in the subspace L.

References

1. MARZOLLO A., An algorithm for the determination of ε controllability conditions in the presence of noise with unknown statistics. In: Automate and Remote Control. München 1972.
2. DELFOUR H. C., MITTER S. K., Reachability of perturbed systems and mini-sup problems. *SIAM J. Autom. Contr.* (1969).
3. WITSENHAUSEN H., A minimax control problem for sampled linear systems. *Trans. IEEE Autom. Contr.* (1968).
4. WITSENHAUSEN H., Sets of possible states given perturbed observations. *Trans. IEEE Autom. Contr.* (1968).
5. GLOVER J. D., Schweppe, Control of linear dynamic systems with set constrained disturbances. *Trans. IEEE Autom. Contr.* (1971).
6. BERTZEKAS D. P., RHODES I. B., On the minimax reachability of target sets and target tubes. *Automatica* 7 (1971).
7. PONTRYAGIN L., Linear differential games. Pt. 1–2. *Sov. Math. Dokl.* **8**, 3, 4 (1967).
8. BORGEST W., VARAIYA P., Target function approach to linear pursuit problems. *Trans. IEEE Autom. Contr.* (1971).
9. KARLIN S., Matehmatical methods and theory in games, programming and economics. New York 1959.

O osiągalności zbioru docelowego przy zakłóceniach

Rozważono zadanie znalezienia dla układu sterowania podlegającego zakłóceniom takiego zbioru w przestrzeni stanów, że należący do niego stan początkowy może być doprowadzony do zbioru docelowego, przy czym zbiór docelowy ma być osiągnięty „z pewnością" a zakłócenia należą do pewnego danego zbioru.

Powyższe zadanie rozważono dla różnych struktur informacyjnych regulatora i zanalizowano zasadnicze punkty rozwiązania na podstawie pojęcia geometrycznej różnicy zbiorów. Interesujące własności rozwiązania pokazano dla przypadku szczególnego, w którym sterowania i zakłócenia są nieograniczone, zbiór docelowy jest podprzestrzenią liniową, a wyznaczanie „zbioru początkowego" przeprowadza się dla dwóch możliwych struktur regulatora.

О достигаемости целевого множества при помехах

Рассмотрена задача нахождения для системы управления, подвергаемой помехам, такого множества в пространстве состояний, чтобы принадлежащее к нему начальное состояние могло бы быть доведено до целевого множества, причем целевое множество должно быть достигнуто ,,с уверенностью'', а помехи принадлежат к определенному данному множеству.

Выше указанная задача рассмотрена для разных информационных структур регулятора и проанализированы главные точки решения на основе понятия геометрической разности множеств. Интересные свойства решения показаны для частного случая, когда управление и помехи не ограничены, целевое множество является линейным подпространством, а определение ,,начального множества'' проводится для двух возможных структур регулятора.