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Solution of a three-dimensional bottleneck problem by means of Hurwicz's saddle-point conditions*)

by

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The problem of finding an optimal investment strategy for multistage production processes, studied previously by R. Bellman and R. Kulikowski is considered.

A complex of three industries integrated so as to produce a given product in a most efficient manner is investigated. In this formulation, the problem is that of finding three functions of time, representing investment strategies in the individual industries, that maximize a total amount of final product within a predetermined time interval $[0, T]$.

Hurwicz's saddle-point conditions have been formulated for the problem and solved numerically for hypothetical coefficients and all admissible initial conditions and positive values of T. Some of these results are presented and discussed.

1. Optimization problem in the dynamic Leontief system

The bottleneck problem as formulated by R. Bellman in his book on dynamic programming [1] is related to a problem of profit or net product maximization in the Leontief open dynamic model.

We shall first formulate the Leontief model based on a continuous version of the discrete Leontief model discussed in [2] and then indicate its relation to the model used in [1].

Suppose that we have a system of n industries each producing a single good. For $j=1, 2, \ldots, n$ let $x_i(t)$ be the activity level of i-th industry at time t measured by the flow rate of good i produced by the i-th industry and let $q_i(t)$ be the production capacity of industry i at time t.

For $i, j=1, 2, \ldots, n$ let a_{ij} be the minimum amount of good i needed as a row material to produce a unit amount of good j (or the minimum flow rate of good i necessary to produce good j at a unit rate) and let b_{ij} be the amount of good i

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needed as a capital stock to provide a unit production capacity (or the minimum flow rate of good i necessary for a unit increase in production capacity) of j-th industry.

Of course all a_{ij} and b_{ij} are nonnegative by definition but we distingnish those coefficients that are strictly positive. If a particular coefficient a_{ij} (or b_{ij}) is equal to zero it simply means that the i-th good is not used in production of good j (or as a capital stock in j-th industry). Let us introduce the following sets of subcripts.

$$I_{aj}=\{i\colon a_{ij}>0\};\ j=1, 2, \ldots, n,$$

$$I_{bj}=\{i\colon b_{ij}>0\};\ \ j=1, 2, \ldots, n.$$

Let $x_{ij}(t)$ be the flow rate of good i into j-th industry used as a raw material and let $z_{ij}(t)$ be the flow rate of good i into j-th industry used as a capital stock. We assume that the production function in each industry have the following form

$$x_j(t)=\min\left(\min_{i\in I_{aj}}\left(\frac{x_{ij}(t)}{a_{ij}}\right), q_j(t)\right);\quad j=1, 2, \ldots, n;\ \ t\in[0, T] \tag{1}$$

and

$$\dot{q}_j(t)=\min_{i\in I_{bj}}\left(\frac{z_{ij}(t)}{b_{ij}}\right);\quad j=1, 2, \ldots, n;\ \ t\in[0, T]. \tag{2}$$

where $\dot{q}_j(t)$ stands for $\frac{dq_j(t)}{dt}$ — a time dirivative of $q_j(t)$ and symbol $\min(a, b, \ldots, z)$ means the smallest of the numbers $a, b, \ldots, z$. The initial capacity of industry j will be denoted by c_j.

As we tend to maximize a net product or profit we can assume that for all $j=1, 2, \ldots, n$

$$\frac{x_{i'j}(t)}{a_{i'j}}=\frac{x_{i''j}(t)}{a_{i''j}}\quad \text{for all } i', i''\in I_{aj};\quad t\in[0, T] \tag{3}$$

$$\frac{z_{i'j}(t)}{b_{i'j}}=\frac{z_{i''j}(t)}{b_{i''j}}\quad \text{for all } i', i''\in I_{bj};\quad t\in[0, T] \tag{4}$$

Indeed, if for some j there existed $i'\in I_{aj}$ (or $i'\in I_{bj}$) such that $\frac{x_{i'j}(t)}{a_{i'j}}>$ $>\min_{i\in I_{aj}}\left(\frac{x_{ij}(t)}{a_{ij}}\right)\left(\text{or } \frac{z_{i'j}(t)}{b_{i'j}}>\min_{i\in I_{bj}}\left(\frac{z_{ij}(t)}{b_{ij}}\right)\right)$ on a finite interval, then some of the x_{ij} (or z_{ij}) would be wasted so that the net product or profit would not be greater than when all equations (3) and (4) hold.

The same reasoning would show that for all $i\in I_{aj}$ and $j=1, 2, \ldots, n$ $\frac{x_{ij}}{a_{ij}}$ should not exceed q_j. Under these assumptions equations (1) and (2) may be written in the following form

$$x_{ij}(t)=a_{ij}\,x_j(t);\ i\in I_{aj};\ j=1, 2, \ldots, n,\ t\in[0, T]; \tag{5}$$

$$x_j(t)\leqslant q_j(t);\ j=1, 2, \ldots, n;\ t\in[0, T]; \tag{6}$$

$$z_{ij}(t)=b_{ij}\,\dot{q}_j(t);\ i\in I_{bj};\ q_j(0)=c_j;\ j=1, 2, \ldots, n;\ t\in[0, T]. \tag{7}$$

Now we can write the usual ballance equations of the Leontief model

$$(I-A)\,x(t)-B\dot{q}(t)=p(t),\quad t\in[0,T], \tag{8}$$

where A and B are $n\times n$ matrices of coefficients a_{ij} and b_{ij} respectively, $x(t)$ and $\dot{q}(t)$ are n-dimensional vectors of components $x_i(t)$ and $\dot{q}_i(t)$, $i=1, 2, \dots, n$, and $p(t)$ represents n-dimensional vector of consumption rate at time t.

The optimization problem can now be formulated as that of finding functions x_{ij}, z_{ij}; $i, j=1, 2, \dots, n$ (decision variables) that give a maximal value of an integral

$$\int_0^T [a(t)]^T p(t)\,dt \tag{9}$$

where $a(t)$ is a predetermined n-dimensional vector function, subject to the constraints (5)—(8) and additional requirements of nonnegativitity of all functions x_{ij}, z_{ij}, p_i, $i, j=1, 2, \dots, n$.

Using equations (5), (7) and (8) and making an assumption that the system is productive [1]) we can reformulate the problem as follows.

Find vector functions x^o and q^o such that [2])

$$\int_0^T [a(t)]^T [(I-A)\,x^o(t)-B\dot{q}^o(t)]\,dt=\max_{x,q}\int_0^T [a(t)]^T [(I-A)\,x(t)-B\dot{q}(t)]\,dt \tag{10}$$

and that satisfy the following constraints

$$q\geqslant x; \tag{11}$$

$$(I-A)\,x\geqslant B\dot{q};\quad q(0)=c; \tag{12}$$

$$\dot{q}\geqslant 0. \tag{13}$$

It can be easily found that the present formulation is equivalent to the previous one.

2. Bottleneck problem — special case

Observe that since constant proportions between investments flows z_{ij} have been assumed there is actually only one independent variable function z_{ij} for each j. Let us distinguish one subscript i, say i^j for each j that may be arbitrarily chosen with the only requirement that $i^j\in I_{bj}$ when I_{bj} is nonempty. Furthermore, we shall assume that all I_{bj} are nonempty what means that each industry have a possibility of increasing its production capacity.

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[1]) The Leontief system is said to be productive if there exists a positive x for which some positive bill of goods $(I-A)\,x$ can be produced. It has been shown (for example in [3]) that the Leontief system is productive if and only if matrix $(I-A)$ has a nonnegative inverse. Owing to the productivity assumption inequality (12) guarantees x being nonnegative when q is nonnegative.

[2]) Throughout this paper the following notation is used. When y denotes a vector function it means $y{:}[0,T]\to R^\eta$, where η is appropriate natural number indicating dimensionality of this vector function. Sign $\geqslant$ (or $=$) between y' and y'' means that $y_i'(t)-y_i''(t)\geqslant 0$ (or $=0$) for all $1, 2, \dots, \eta$ and all $t\in[0,T]$. Symbol $y(t)$ denotes a value of the vector function y at time t.

Let us denote coefficients $b_{i_j j}$ by $1/b_j$ for $i=1, 2, ..., n$ and introduce a $n \times n$ diagonal matrix B of coefficients b_j and a vector function z defined as follows

$$\dot{q} = \bar{B} \cdot z. \tag{14}$$

It follows from the above definition and (7) that $z_j = z_{i_j j}$ for all $j=1, 2, ..., n$. If the system described is treated as a control system then z may be regarded as a control function and equation (14) together with the initial condition

$$q(0) = c \tag{15}$$

may be regarded as system's state equation.

We shall introduce another $n \times n$ matrix Ψ of technological coefficients that will be useful

$$\Psi = B \cdot \bar{B}. \tag{16}$$

It can be easily seen that the coefficients Ψ_{ij} of Ψ can be expressed in terms of coefficients b_{ij} of B

$$\Psi_{ij} = \frac{b_{ij}}{b_{i_j j}}. \tag{17}$$

Using this new notation we can write problem (10)—(13) as follows. Find vector functions x^o and z^o such that

$$\int_0^T [(I-A)\, x^o(t) - \Psi z^o(t)]^T a(t)\, dt = \max_{z, x} \int_0^T [(I - A\, x(t) - \Psi z(t)]^T \cdot a(t)\, dt \tag{18}$$

and subject to the constraints

$$(I-A)\, x - \Psi z \geqslant 0; \tag{19}$$

$$c + \int_0^t \bar{B} z(\tau)\, d\tau - x(t) \geqslant 0; \; t \in [0, T]; \tag{20}$$

$$z \geqslant 0. \tag{21}$$

Making two hypotheses that are satisfied by Bellman's three dimensional case we shall reduce the problem (18)—(21) to a strightforward generalization of the three-dimensional case called in this paper the bottleneck problem.

These two hypotheses are following:

(1) One of the industries uses row materials only from outside of the system and flows of these row materials are unlimited. Let us agree that it is industry number one.

(2) The whole system operates in order to produce net product in one industry only. This means that the vector function a in the integrand of the criterion function has only one nonzero component. Let us agree that it is industry number two.

The consequence of hypothesis (1) is that the production rate in the first industry is equal to the production capacity of that industry for all $t \in [0, T]$. This means that the first inequality in (20) becomes equality.

From the second hypothesis we can deduce that all inequalities except the second one may be replaced by equalities. This means that consumption rates in all but the second industry are equal to zero. Indeed, suppose that it is not true and that the optimal solution is such that there is a positive consumption of good $i \neq 2$. If this good, or raw materials used in its production, were used as raw materials and/or investments in industry 2 in appropriate proportions then some additional amount of good 2 would be produced and the criterion function's value would be increased.

Using n equations resulting from the above assumptions we can express x in terms of z and write equations (19) and (20) in the form

$$x(t) = A^* \left(c + \int_0^t \bar{B} z(\tau)\, d\tau \right) + B^* z(t): \ t \in [0, T]; \tag{22}$$

$$\mathscr{A} \left(c + \int_0^t \bar{B} z(\tau)\, d\tau \right) + \mathscr{B} z(t) \geqslant 0; \ t \in [0, T] \tag{23}$$

where $\mathscr{A}$, A^*, $\mathscr{B}$ and B^* are all $n \times n$ matrices whose coefficients are functions of coefficients of matrices $A, \bar{B}$ and Ψ.

Denoting the second rows of matrices $(I-A)$ and Ψ by $(I-A)_{(2)}$ and $\Psi_{(2)}$ respectively and introducing new vectors

$$a^* = [(I-A)_{(2)} A^*]^T; \tag{24}$$

$$b^* = [(I-A)_{(2)} B^* - \Psi_{(2)}]^T; \tag{25}$$

and assuming, for simplicity, that $a(t) = (0, 1, 0, \ldots, 0)$ for $t \in [0, T]$ we finaly state the bottleneck problem as follows.

Find vector function z^o (being an element of some functional space Z) such that

$$\int_0^T \left[a^{*T} \left(c + \int_0^t \bar{B} z^o(\tau)\, d\tau \right) + b^{*T} z^o(t) \right] dt = \max_{z \in U_z} \int_0^T \left[a^{*T} \left(c + \int_0^t \bar{B} z(\tau)\, d\tau \right) + b^{*T} z(t) \right] dt \tag{26}$$

where the set $U_z \subset Z$ consists of elements satisfying the following inequalities

$$\mathscr{A} \left(c + \int_0^t \bar{B} z(\tau)\, d\tau \right) + \mathscr{B} z(t) \geqslant 0, \ t \in [0, T], \tag{27}$$

$$z \geqslant 0. \tag{28}$$

3. Quasisaddle-point optimality conditions for the bottleneck problem

Hurwicz's quasi saddle-point conditions were applied to the bottleneck problem first by R. Kulikowski in [4] where two basic theorems providing necessary and sufficient conditions for optimality were used. These Theorems were first proved in [5] and then carefully studied in [6] and [7] in which their application to the bottleneck problem was considered in more detail and the two — dimensional case was solved.

In the present paper we use optimality conditions as formulated in [6] and [7]. Let $\bar{X}$ and Y be partially ordered Banach spaces. Let f and g be functions defined on $\bar{X}$, $f\colon \bar{X}\to R$ and $g\colon \bar{X}\to Y$. Let U be a set defined as follows

$$U=\{\bar{x}: \bar{x}\in \bar{X};\quad g(\bar{x})\geqslant 0;\quad \bar{x}\geqslant 0\} \tag{29}$$

We shall call P a problem of maximizing $f(\bar{x})$ over the set U. We shall call $\bar{x}^o$ a solution of problem P if

$$f(\bar{x}^o)=\max_{\bar{x}\in U} f(\bar{x})\,. \tag{30}$$

In order to formulate Hurwicz's theorem for the problem P we introduce a function $\Phi\colon \bar{X}\times Y^*\to R$ such that for $\bar{x}\in\bar{X}$ and $\lambda\in Y^*$

$$\Phi(\bar{x},\lambda)=f(\bar{x})+\lambda\,[g(\bar{x})] \tag{31}$$

where λ is a linear functional defined on Y, i.e. an element of the conjugate space Y^* of Y.

THEOREM. Let f and g be concave and both possess Fréchet differentials in $\bar{X}$. Let g satisfy conditions of regularity and regular convexity of certain set of functionals defined by means of g [3]). Then $\bar{x}^o$ is a solution of the problem P if and only if there exists a nonnegative functional [4]) $\lambda^o\in Y^*$ such that the following conditions hold.

$$d_{\bar{x}}\,\Phi\,[(\bar{x}^o,\lambda^o),\bar{x}^o]=0\,, \tag{32}$$

$$d_{\bar{x}}\,\Phi\,[(\bar{x}^o,\lambda^o),\bar{x}]\leqslant 0 \text{ for all } \bar{x}\geqslant 0;\ \bar{x}\in\bar{X}, \tag{33}$$

$$d_{\lambda}\,\Phi\,[(\bar{x}^o,\lambda^o),\lambda^o]=0\,, \tag{34}$$

$$d_{\lambda}\,\Phi\,[(\bar{x}^o,\lambda^o),\lambda]\geqslant 0 \text{ for all } \lambda\geqslant 0;\ \lambda\in Y^*. \tag{35}$$

Here $d_{\bar{x}}\,\Phi\,[(\bar{x}^o,\lambda^0),\cdot]\,(d_\lambda\,\Phi\,[(\bar{x}^o,\lambda^0),\cdot])$ denotes a Fréchet differential of a functional Φ with respect to $\bar{x}\,(\lambda)$ at point $(\bar{x}^o,\lambda^0)$. The point $(\bar{x}^o,\lambda^0)$ that satisfies the above conditions (32)—(35) is called a nonnegative quasisaddle-point of functional Φ.

If the functions f and g are not concave then (32)—(35) are necessary conditions for optimality. We shall assume in the following that f and g are concave and use the above theorem.

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[3]) The above theorem gathers the results of theorems V.O., V.3.3.2 and V.3.3.3 in [5] for the special case defined by the hypothesis of the present formulation.

The definitions of regularity of g and of the set of functionals mentioned above are given in section V.3.3.2 and in the hypothesis of the theorem V.3.3.2 in [5]. We shall not discuss them here as in the bottleneck problem defined in the preceding section f and g are linear and both hypotheses as well as that of concavity of f and g are satisfied.

[4]) We call a functional $\lambda\colon Y\to R$ nonnegative if $\lambda(y)\geqslant 0$ for all $y\geqslant 0$; $y\in Y$.

We assume that Z is a Cartesian product of n spaces $L^2(0, T)$ (that is $z_i \in L^2(0, T)$ for $i=1, 2, \ldots, n$) and we denote this space by $L_n^2(0, T)$.

The operator g is given by the left hand side of inequality (27)

$$[g(z)](t)=\mathscr{A}\left(c+\int_0^t \bar{B}z(\tau)\,d\tau\right)+\mathscr{B}z(t) \quad \text{for } t\in[0, T],\ z\in Z \tag{36}$$

and the set U is equal to U_z defined by (27) and (28).

The space Y is again $L_n^2(0, T)$ and hence every linear nonnegative functional $\bar{\lambda}$ defined on Y can be expressed as an integral

$$\bar{\lambda}(y)=\int_0^t [y(t)]^T\cdot\lambda(t)\,dt \quad \text{for } y\in Y \tag{37}$$

where λ is a nonnegative function belonging to $L_n^2(0, T)$. Using (37) we obtain the Lagrangean function in the form

$$\Phi(z, \lambda)=\int_0^T\left[a^{*T}\left(c+\int_0^t \bar{B}z(\tau)\,d\tau\right)+b^{*T}z(t)\right]dt+$$
$$+\int_0^T\left[\mathscr{A}\left(c+\int_0^t \bar{B}\,z(\tau)\,d\tau\right)+\mathscr{B}z(t)\right]^T\lambda(t)\,dt. \tag{38}$$

The Frechet differentials of Φ are following [5])

$$d_z\,\Phi[(z^o, \lambda^o), z]=\int_0^T\left[b^*+\mathscr{B}^T\lambda^0(t)+\int_t^T \bar{B}\left(a^*+\mathscr{A}^T\lambda^0(\tau)\right)d\tau\right]^T\cdot z(t)\,dt, \tag{39}$$

$$d_\lambda\,\Phi[(z^o, \lambda^o), \lambda]=\int_0^T\left[\mathscr{A}\left(c+\int_0^t \bar{B}z^o(\tau)\,d\tau\right)+\mathscr{B}z^o(t)\right]^T\lambda(t)\,dt. \tag{40}$$

Applying now (32) to (35) we obtain the folowing necessary and sufficient conditions for a vector function z to be an optimal solution of a problem (26)—(28)

$$\int_0^T\left[b^*+\mathscr{B}^T\lambda^o(t)+\int_t^T \bar{B}\left(a^*+\mathscr{A}^T\lambda^o(\tau)\right)d\tau\right]^T z^o(t)\,dt=0, \tag{41}$$

$$\int_0^T\left[b^*+\mathscr{B}^T\lambda^o(t)+\int_t^T \bar{B}\left(a^*+\mathscr{A}^T\lambda^o(\tau)\right)d\tau\right]^T z(t)\,dt\leqslant 0 \text{ for all } z\geqslant 0, \tag{42}$$

$$\int_0^T\left[\mathscr{A}\left(c+\int_0^t \bar{B}z^o(\tau)\,d\tau\right)+\mathscr{B}z^o(t)\right]^T\lambda^o(t)\,dt=0, \tag{43}$$

$$\int_0^T\left[\mathscr{A}\left(c+\int_0^t \bar{B}z^o(\tau)\,d\tau\right)+\mathscr{B}z^o(t)\right]^T\lambda(t)\,dt\geqslant 0 \text{ for all } \lambda\geqslant 0. \tag{44}$$

[5]) Usual in such cases (see [6] and [7]) integration by parts have been applied to obtain $d_z\,\Phi$ in the form of (39).

Observe that inequalities (42) and (44) can be satisfied for all nonnegative z and all nonnegative λ if an only if the first component of a scalar ptoduct under each integral (that is the term in square brackets) is respectively nonpositive and nonnegative almost everywhere in $[0, T]$. This implies that equations (41) and (43) are satisfied if and only if the integrands are equal to zero almost everywhere in $[0, T]$. Thus we can write equations (41)—(44) in the following equivalent form

$$\left[b^* + \mathscr{B}^T \lambda(t) + \int_t^T \bar{B}\left(a^* + \mathscr{A}^T \lambda(\tau)\right) d\tau\right]^T z(t) = 0 \text{ almost everywhere in } [0, T], \quad (45)$$

$$b^* + \mathscr{B}^T \lambda(t) + \int_t^T \bar{B}\left(a^* + \mathscr{A}^T \lambda(\tau)\right) d\tau \leqslant 0 \text{ almost everywhere in } [0, T], \quad (46)$$

$$\left[\mathscr{A}\left(c + \int_0^t \bar{B}z(\tau)\, d\tau\right) + \mathscr{B}z(t)\right]^T \lambda(t) = 0 \text{ almost everywhere in } [0, T], \quad (47)$$

$$\mathscr{A}\left(c + \int_0^t \bar{B}z(\tau)\, d\tau\right) + \mathscr{B}z(t) \geqslant 0 \text{ almost everywhere in } [0, T]. \quad (48)$$

The superscript o over x and λ has been neglected because all z and λ in (45)—(48) are z^o and λ^o.

4. Solution of the optimality conditions

To solve the above set of equations we assume that all the functions z_i (and consequently λ_i); $i = 1, 2, ..., n$ are piecewise continuous.

The scalar product on the left hand side of the equation (45) is a sum of products of the respective components of the two vectors. Since all first components of these products are nonpositive and all the second are nonnegative, the equation (45) is equivalent to the set of n similar equations — one for each component of the scalar product. The same is true for equation (47) and we can write equations (45) and (47) in the form

$$\left[b_j^* + \mathscr{B}^T(j)\, \lambda(t) + \int_t^T b_j\left(a_j^* + \mathscr{A}^T(j)\, \lambda(\tau)\right) d\tau\right] z_j(t) = 0; \quad j = 1, 2, ..., n$$

$$\text{almost everywhere in } [0, T], \quad (49)$$

$$\left[\mathscr{A}(j)\left(c + \int_0^t \bar{B}z(\tau)\, d\tau\right) + \mathscr{B}(j)\, z(t)\right] \lambda_j(t) = 0; \quad j = 1, 2, ..., n$$

$$\text{almost everywhere in } [0, T], \quad (50)$$

where $\mathscr{A}(j)$ denotes j-th row of matrix A, z_j—j-th components of vector z and so on.

It follows from (49) that whenever $z_j(t) > 0$ in some interval $[T', T'']$; $0 \leqslant T' < T'' \leqslant T$ then for $t \in [T', T'']$ we have

$$b_j^* + \mathscr{B}^T(j)\, \lambda(t) + \int_t^T b_j\left(a_j^* + \mathscr{A}^T(j)\, \lambda(\tau)\right) d\tau = 0; \quad j = 1, 2, ..., n. \quad (51)$$

Thus if in some interval there are some components of z (say in a number i; $i=0, 1, 2, \ldots, n$) that are strictly positive then the appropriate inequalities in (46) become equalities so we have i equations and $n-i$ inequalities in (46). The same applies to (47) and (48). It is convenient to introduce a concept of a state of a vector.

Let $s(t)$ be a $2n$-component vector $(s_1(t), s_2(t), \ldots, s_n(t), \ldots, s_{n+1}(t), \ldots, s_{2n}(t))$ such that

$$s_i(t)=\begin{cases} 0 \text{ if } z_i(t)=0 \\ 1 \text{ if } z_i(t)>0 \end{cases} \quad i=1, 2, \ldots, n; \; t \in [0, T], \tag{52}$$

$$s_{n+i}(t)=\begin{cases} 0 \text{ if } \lambda_i(t)=0 \\ 1 \text{ if } \lambda_i(t)>0 \end{cases} \quad i=1, 2, \ldots, n; \; t \in [0, T]. \tag{53}$$

We say that value $s(t)$ of the function $s:[0, T]\to S$ defined above (where S is the set of all $2n$-component binary vectors) defines state of vector function (z, λ) at time t.

Since z and λ are piecewise continuous s is piecewise constant. In the finite interval $[0, T]$ there is a finite number of points in which s changes its value and a finite number of intervals (say N) in which (z, λ) is continuous and does not change its state.

Let us number these intervals backword from 1 to N and denote their boundary points by T_i, $i=0, 1, \ldots, N$ in such a way that

$$0=T_N<T_{N-1}<\ldots<T_1<T_0=T \tag{54}$$

and ith interval I_i is the interval $(T_i, T_{i-1}]$; $i=1, 2, \ldots, N$.

We can write equations (45)—(48) for each interval separately in the following form.

$$\left[b^*+\mathcal{B}^T\lambda(t)+\int_t^{T_{i-1}} \bar{B}\left(a^*+\mathcal{A}^T\lambda(\tau)\right)d\tau+Q(T_{i-1})\right]^T z(t)=0;$$

$$t \in [T_i, T_{i-1}]; \; i=1, 2, \ldots, N; \tag{55}$$

$$b^*+\mathcal{B}^T\lambda(t)+\int_t^{T_{i-1}} \bar{B}\left(a^*+\mathcal{A}^T\lambda(\tau)\right)d\tau+Q(T_{i-1})\leqslant 0; \; t \in [T_i, T_{i-1}];$$

$$i=1, 2, \ldots, N; \tag{56}$$

$$\left[\mathcal{A}\left(q(T_i)+\int_{T_i}^{t} \bar{B}z(\tau)\,d\tau\right)+\mathcal{B}z(t)\right]^T \lambda(t)=0; \; t \in [T_i, T_{i-1}]; \; i=1, 2, \ldots, N; \tag{57}$$

$$\mathcal{A}\left(q(T_i)+\int_{T_i}^{t} \bar{B}z(\tau)\,d\tau\right)+\mathcal{B}z(t)\geqslant 0; \; t \in [T_i, T_{i-1}]; \; i=1, 2, \ldots, N; \tag{58}$$

where vectors $q(T_i)$ and $Q(T_{i-1})$ are constant in the interval i and are given by the equations

$$q(T_i)=c+\int_0^{T_i} \bar{B}z(\tau)\,d\tau; \; i=1, 2, \ldots, N, \tag{59}$$

$$Q(T_{i-1})=\int_{T_{i-1}}^{T} \bar{B}\left(a^*+\mathcal{A}^T\lambda(\tau)\,d\tau\right); \; i=1, 2, \ldots, N. \tag{60}$$

For given matrices $\mathscr{A}$, $\mathscr{B}$ and vectors a^* and b^* let $\Omega \subset R^n$ be a set of values of vector c for which piecewisecontinuous solutions of inequalities (27) and (28) exist.

Any nonnegative piecewise-continuous function $z \in L_n^2(0, T)$ for which there exists a nonnegative piecewise-continuous function $\lambda \in L_n^2(0, T)$ such that (z, λ) is a solution of (45)—(48), is an optimal solution of the problem (26)—(28). From now on, we shall refere to such a function (z, λ) as an optimal solution.

In what follows, we shall present a method of obtaining optimal solutions for all $c \in \Omega$.

Let i be any of the numbers 1, 2, ..., N and let s^j be any element of S.

Suppose that the solution of (55)—(58) is in state s^j, i.e. for $t \in I_i = (T_i, T_{i-1}] \subset \subset [0, T]$ $s(t) = s^j$. Denote this solution $(z, \lambda)_i^j$. It depends on parameters vectors $q(T_i)$ and $Q(T_{i-1})$. A nonnegative solution of (55)—(58), $(z, \lambda)_i^j$ exists, in general, only for certain values of $q(T_i)$ and $Q(T_{i-1})$. We can say that this solution exists under certain existance conditions which can be written in the form

$$\varphi_i^j(q(T_i), Q(T_{i-1}), \Delta_i) \geq 0 \tag{61}$$

where

$$\Delta_i = T_{i-1} - T_i \tag{62}$$

and φ_i^j is n_j-dimensional vector function what means that for each s^j existance conditions are given by n_j inequalities.

Let I_s be a set of all indeces j for which a nonnegative solutions exist for some i. Let us construct a sequence

$$(z, \lambda)_r^{j_r}, (z, \lambda)_{r-1}^{j_{r-1}}, \ldots, (z, \lambda)_1^{j_1} \tag{63}$$

where r is natural number and $j_k \in I_s$ for $k = 1, 2, \ldots, r$.

Let us consider the following set of inequalities connected with this sequence

$$\varphi_k^{j_k}(q(T_k), Q(T_{k-1}), \Delta_k) \geq 0, \quad k = 1, 2, \ldots, r. \tag{64}$$

These inequalities can be expressed in terms of $q(T_r)$ and $Q(T_r)$. Indeed, introducing notation

$$\Delta_i\, q = \int_{T_i}^{T_{i-1}} \bar{B} z(t)\, dt; \quad i = 1, 2, \ldots, N, \tag{65}$$

$$\Delta_i\, Q = \int_{T_i}^{T_{i-1}} \bar{B}(a^* + \mathscr{A}^T \lambda(t))\, dt; \quad i = 1, 2, \ldots, N, \tag{66}$$

we can write equations (59) and (60) in the form

$$q(T_i) = q(T_{i-1}) - \Delta_i\, q; \; i = 1, 2, \ldots, N; \tag{67}$$

$$Q(T_i) = Q(T_{i-1}) + \Delta_i\, Q; \; i = 1, 2, \ldots, N. \tag{68}$$

Using the above equations we can express $q(T_i)$ and $Q(T_{i-1})$ in terms of $q(T_r)$ and $Q(T_r)$ and write inequalities (64) in the form

$$\tilde{\varphi}_k^{j_k}(q(T_r), Q(T_r), \Delta_k, \Delta_{k+1}, \ldots, \Delta_r) \geq 0; \quad k = 1, 2, \ldots, r. \tag{69}$$

Observe that for the last element (i.e. element number 1) of any sequence must be

$$Q(T)=0. \tag{70}$$

Using (67) and (68) we write this equation in the form

$$\tilde{\varphi}_0^{j_0}(Q(T_r), \Delta_1, \Delta_2, \ldots, \Delta_r)=0. \tag{71}$$

For $m=1, 2, \ldots, m_r$ and $r=1, 2, \ldots$ let $r\cdot m$ be a number of a r-element sequence.

We shall say that sequence $r\cdot m$ is permissible if there exists a set $\Omega_{r\cdot m}(T_r)$ of pairs $(q(T_r), Q(T_r))$ such that $q(T_r)\in\Omega$, $Q(T_r)\in R^n$ and inequality (69) and equation (71) are satisfied for $\Delta_k>0$; $k=1, 2, \ldots, r$.

For each pair $(q(T_r), Q(T_r))\in\Omega r\cdot m(T_r)$ there exists a nonempty set of vectors $(\Delta_1, \Delta_2, \ldots, \Delta_r)$ such that $\Delta_k>0$ for $k=1, 2, \ldots, r$.

Denote this set $\Delta_{qQ}^{r\cdot m}$ and define a set

$$\mathcal{T}_{r\cdot m}(q(T_r), Q(T_r))=\{v: v=\sum_{k=1}^{r}\Delta_k; [(\Delta_1, \Delta_2, \ldots, \Delta_r)\in\Delta_{qQ}^{r\cdot m}\}. \tag{72}$$

Vector $Q(T_r)$ may be calculated by means of equations (66) and (68) starting with $Q(T_0)=Q(T)=0$ as an initial value. Thus actually the set $\mathcal{T}_{r\cdot m}$ is a function of $q(T_r)$ only.

It follows from the definition of a permissible sequence of interval solutions that every such a sequence determines an optimal solution for certain values of c and T in the following way. If there exists an element $(q(T_r), Q(T_r))\in\Omega_{r\cdot m}(T_r)$ such that $q(T_r)=c$ and if T is an element of $\mathcal{T}_{r\cdot m}|_{q(T_r)=c}$ then a function

$$(z(t), \lambda(t))=(z(t), \lambda(t))_k^{j_k} \text{ for } t\in(T_k, T_{k-1}) \quad k=1, 2, \ldots, r \tag{73}$$

is an optimal solution for the pair (c, T).

Observe that $T\in\mathcal{T}_{r\cdot m}(c)$ means that there is an element $(\Delta_1, \Delta_2, \ldots, \Delta_r)\in\Delta_{cQ}^{r\cdot m}$ such that

$$\sum_{k=1}^{r}\Delta_k=T \tag{74}$$

what means that $T_r=0$.

On the other hand, for every optimal solution there exists r and a sequence $r\cdot m$ $(m=1, 2, \ldots, m_r)$ such that the optimal solution can be expressed in the form (73) as there are no other solutions of (45)—(48).

Constructing all permissible sequences we obtain all optimal solutions.

Note that if $r\cdot m'$ is a permissible sequence then it is easy to construct a sequence $r+1\cdot m''$ $(m''=1, 2, \ldots, m_{r+1})$ by adjoining $(z, \lambda)_{r+1}^{j_{r+1}}$ for $j_{r+1}\in I_s$. In order to check whether this sequence is permissible or not it is enough to express set $\Omega_{r\cdot m'}(T_r)$ in terms of $q(T_{r+1}), Q(T_{r+1})$ and by adjoining inequality

$$\varphi_{r+1}^{jr+1}(q(T_{r+1}), Q(T_{r+1}), \Delta_{r+1})\geqslant 0 \tag{75}$$

construct set $\Omega_{r+1\cdot m''}(T_{r+1})$. If $\Omega_{r+1\cdot m''}$ is not empty then sequence $r+1\cdot m''$ is permissible and we determin set $\mathcal{T}_{qQ}^{r+1\cdot m''}(q(T_{r+1}))$ to define a range of parameters (c, T) for which an optimal solution is determined by sequence $r+1\cdot m''$.

The technique described above has been used for obtaining solution of the particular three-dimensional example formulated in [1], in which the three industries are steel, auto and tool industry with the goal of producing autos.

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Rozwiązanie trójwymiarowego problemu wąskich gardeł przy użyciu warunków punktu siodłowego Hurwicza

Rozważono zagadnienie poszukiwania optymalnej strategii inwestycji dla zespołu kilku procesów produkcyjnych, rozważane uprzednio przez R. Bellmana i R. Kulikowskiego.

Zagadnienie dotyczy zespołu trzech przemysłów zorganizowanego tak, aby uzyskać określony produkt w sposób najbardziej efektywny. W niniejszym sformułowaniu zagadnienie polega nanalezieniu trzech funkcji czasu reprezentujących strategie inwestycyjne w poszczególnych przemysłach, maksymalizujących całkowity produkt końcowy jednego przemysłu w zadanym przedziale czasu $[0, T]$.

Przedstawiono warunki punktu siodłowego Hurwicza dla rozważanego zagadnienia i rozwiązano je dla założonych współczynników technicznych produkcji i inwestycyjnych w całym dopuszczalnym obszarze warunków początkowych, dla dodatnich wartości horyzontu planowania T.

Решение трехмерной задачи узких мест при использовании седловой точки Гурвича

Рассмотрена задача поиска оптимальной стратегии капиталовложений для комплекса нескольких производственных процессов, ранее рассмотренная Р. Беллманом и Р. Куликовским.

Задача касается комплекса состоящего из трех отраслей промышленности, организованного таким образом, чтобы наиболее эффективно получить определенный продукт. В данной формулировке задача состоит в нахождении трех функций времени, отражающих стратегии капиталовложений в отдельных отраслях промышленности, максимизирующих полный конечный продукт одной отрасли промышленности за заданный период времени $[0, T]$.

Представлены условия седловой точки Гурвина для рассматриваемой задачи и получены их решения при предполагаемых технических коэффициентах производства и капиталовложениях, для всей допускаемой области начальных условий и для положительных значений горизонта планирования T.