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An iterative method of optimization with applications to optimal control problems with state space constraints

by

KAZIMIERZ MALANOWSKI,
ALEKSANDER WASILEWSKI

Institute for Organization,
Management and Control Sciences,
Warsaw

An iterative algorithm of finding a minimum of a convex functional on a closed convex and bounded set in a Hilbert space subject to linear constraints of state space type is proposed.

The algorithm is based on a combination of a convex programming method and a penalty function method. The penalty coefficient is modified in the process of iterations.

The proof of the convergence of the algorithm is given.

The application of the method to a problem of optimal heating subject to constraints of thermal stress is proposed and some numerical results are presented.

1. Introduction

An optimal control problem subject to constraints of control function and state space coordinates is considered.

The problem is formulated in an abstract form as the minimization of a convex functional on a closed, convex and bounded set in a Hilbert space subject to additional linear constraints.

An iterative procedure of solving this problem is proposed. It is based on a combination of two methods: a convex programming method and a penalty function method.

The penalty coefficient is modified in the process of iterations and thus it is not necessary to find a minimum of penalty functional with fixed penalty coefficient.

An algorithm of changes of the penalty coefficient is given, which provides the convergence of the iterative procedure to a solution of the initial optimization problem.

Each step of iterations requires solving of a finite — dimensional quadratic programming problem subject to linear constraints. To this end one of well known algorithms [7] can be applied.

The application of the method to the problem of optimal heating subject to constraints of thermal stress is proposed and some numerical results are presented.

2. Problem statement

Let V be a Hilbert space and U a closed, convex and bounded subset of V.

On the space V there is defined a non-negative convex functional $J(u)$ and n continuous linear operators $G_\nu(u)$ ($\nu=1, 2, ..., n$) mapping V into $L^2(0, T)$, where T is a given parameter.

We are going to consider the following problem of potimization (P): find an element $u_{opt} \in U$, called an optimal control, such that

$$J(u_{opt}) = \inf_{u \in U} J(u) \tag{1}$$

ubject to constraints

$$g_\nu(t) - G_\nu(u)(t) \geqslant 0, \quad \nu = 1, 2, ..., n, \quad \text{a.e. in } [0, T], \tag{1a}$$

where $g_\nu(\cdot) \in L^2(0, T)$ are given functions.

If the set

$$U' = \{u : u \in U;\ g_\nu(t) - G_\nu(u)(t) \geqslant 0,\ \nu = 1, 2, ..., n, \text{ a.e. in } [0, T]\} \tag{2}$$

is not empty, then an element u_{opt} exists. Indeed, since $G_\nu(u)$ are continuous and linear, the set U' is closed and convex; moreover it is bounded and hence weakly compact [4].

On the other hand $J(u)$ is weakly semicontinuous as a convex one [12]. Therefore it assumes its minimum on U' [12].

If in addition $J(u)$ is strictly convex the element u_{opt} is unique.

Instead of solving (P) directly we eliminate the constraints (1a) introducing a penalty for violating them.

Namely we introduce a well known penalty functional

$$J_{\varepsilon_i}(u) = J(u) + \frac{1}{\varepsilon_i} K(u) \triangleq J(u) + \frac{1}{\varepsilon_i} \sum_{\nu=1}^{n} [\max\{0, G_\nu(u)(t) - g_\nu(t)\}]^2 \, dt \tag{3}$$

depending on a positive parameter $\varepsilon_i > 0$, and we formulate an auxiliary problem (P_{ε_i}): find $u_{\varepsilon_i} \in U$ such that

$$J_{\varepsilon_i}(u_{\varepsilon_i}) = \inf_{u \in U} J_{\varepsilon_i}(u). \tag{4}$$

Since $G_\nu(u)$ are linear the penalty term

$$K(u) = \sum_{\nu=1}^{n} \int_0^T [\max\{0, G_\nu(u)(t) - g_\nu(t)\}]^2 \, dt$$

is convex. Hence the functional $J_{\varepsilon_i}(u)$ is convex and an element u_{ε_i} exists.

It is obvious that $K(u)$ is equal to zero iff the constraints (1a) are satisfied. Moreover it is known [1] that

$$J_{\varepsilon_i}(u_{\varepsilon_i}) \xrightarrow[\varepsilon_i \to 0]{} J(u_{opt}). \tag{5}$$

Therefore the solution of (P) can be approximated by a solution of P_{ε_i} with ε_i small enough.

However we do not know an a priori estimation of ε_i with which a required accuracy of the approximation is achieved.

Since the minimization of $J_{\varepsilon_i}(u)$ can be usually achieved only using an iterative procedure it seems reasonable to perform this minimization with the simultaneous decreasing of ε_i. In what follows such a method is presented. It assures the convergence to a solution of problem (P).

3. Iterative method of minimization

To minimize $J_{\varepsilon_i}(u)$ there will be applied a method introduced in [3] and generalized in [11].

In i-th iterative step of this method we find a minimum of a quadratic approximation of $J_{\varepsilon_i}(u)$ on a subset U_i or U. Where U_i is a convex hull of a finite number of elements u_i^j.

In the sequel we shall assume that $J(u)$ is twice weakly differentiable and the following condition is satisfied

$$(J''(u)\,v, v) \leqslant N(v, v) \qquad \forall u \in U, \qquad v \in V \tag{6}$$

where $N < \infty$.

The functional $K(u)$ is differentiable and

$$(K'(u), v) = 2 \sum_{\nu=1}^{n} \int_{M_\nu} [G_\nu(u)(t) - g(t)]\, G_\nu(v)(t)\, dt \tag{7}$$

where

$$M_\nu = \{t \in [0, T]: G_\nu(u)(t) - g_\nu(t) > 0\}. \tag{7a}$$

Denote

$$M_{\nu_i}^j = \{t \in [0, T]: G_\nu(u_i^j)(t) - g_\nu(t) > 0\} \tag{8}$$

and

$$\bar{M}_{\nu_i} = \bigcup_j M_{\nu_i}^j. \tag{8a}$$

For every $u \in U_i$, i.e. for every

$$u = \sum_j \alpha^j u_i^j, \ \alpha^j \geqslant 0, \ \sum_j \alpha^j = 1. \tag{9}$$

we have

$$M_\nu \subset \bar{M}_{\nu_i}. \tag{9a}$$

Let us denote

$$(\bar{K}_i v, v) = 2 \sum_{\nu=1}^{n} \int_{\bar{M}_{\nu_i}} G_\nu^2(v)(t)\, dt. \tag{9b}$$

Taking into account (7), (9) it is easy to check that

$$K(v) \leqslant K(u) + (K'(u), v-u) + (\bar{K}_i(v-u), v-u), \quad \forall u, v \in U_i. \tag{10}$$

Let u_i be an element of the set U_i. On the set U_i we define an auxiliary functional $J_{\varepsilon_i}(u)$ by

$$\bar{J}_{\varepsilon_i}(u) = \bar{J}_{\varepsilon_i}\Big(\sum_j \alpha^j u_i^j\Big) = J_{\varepsilon_i}(u_i) + \Big(J'_{\varepsilon_i}(u_i), \sum_j \alpha^j u_i^j - u_i\Big) +$$

$$+ \tfrac{1}{2}\Big(\Big(N + \frac{1}{\varepsilon_i}\bar{K}_i\Big)\Big(\sum_j \alpha^j u_i^j - u_i\Big), \sum_j \alpha^j u_i^j - u_i\Big). \tag{11}$$

As it is seen $\bar{J}_{\varepsilon_i}(u)$ is a quadratic functional of parameters α. Moreover it follows from (6) and (10) tht

$$J_{\varepsilon_i}(u) \leqslant \bar{J}_{\varepsilon_i}(u), \quad \forall u \in U_i \tag{12}$$

and

$$J_{\varepsilon_i}(u_i) = \bar{J}_{\varepsilon_i}(u_i). \tag{13}$$

Hence $\bar{J}_{\varepsilon_i}(u)$ is a quadratic approximation of $J_{\varepsilon_i}(u)$ in the neighbourhood of u_i, which majorizes this functional on U_i.

In each iterative step we find two elements — an element $u_i \in U_i$, which satisfies the condition

$$\bar{J}_{\varepsilon_{i-1}}(u_i) = \inf_{u \in U_i} \bar{J}_{\varepsilon_{i-1}}(u) \tag{14}$$

— an element $\bar{u}_{i+1} \in U$ such, that

$$(-J'_{\varepsilon_i}(u_i), u) \leqslant (-J'_{\varepsilon_i}(u_i), \bar{u}_{i+1}), \quad \forall u \in U \tag{15}$$

i.e. u_{i+1} is a point at which the hyperplane H_i ortogonal to $-J'_{\varepsilon_i}(u_i)$ supports the set U.

Having definitions (14) and (15) we are in the position to define the sets U_i.

As U_0 we choose any arbitrary point $u_0 \in U$. Then we find $\bar{u}_1$ satisfying (15) and as U_1 we take the segment joining u_0 and u_1, i.e.

$$U_1 = \mathrm{conv}\,\{u_0, \bar{u}_1\} \stackrel{\Delta}{=} \mathrm{conv}\,\{u_1^1, u_1^2\}.$$

In further iterations we construct the sets U_i in the same way putting

$$U_i = \mathrm{conv}\,\{u_0, \bar{u}_1, \ldots, \bar{u}_i\} \stackrel{\Delta}{=} \mathrm{conv}\,\{u_i^1, u_i^2, \ldots, u_i^{i+1}\}. \tag{16}$$

Remark. To find the element u_i we must solve a quadratic programming problem of minimization of functional (11) with respect to coefficients α^j, subject to linear constraints (9). It follows from the construction of U_i, that the number of coefficients α^j increases in each iteration. Hence increases also the time of computations necessary to find u_i. To avoid this difficulty a method proposed in [10] can be applied which allows to reduce the number of α^j to a given number.

As it has been already told the value of the coefficient ε_i is not constant but it is modified in the process of iterations. With the appropriate choice of the values of parameter ε_i the sequence $\{u_i\}$ can be used to find an approximation of the solution of problem (P).

Here we should point out the difficulty which can be encounter in such an approach.

For fixed value of ε_i in the described method of minimization of $J_{\varepsilon_i}(u)$ we obtain a sequence of non-increasing values $J_{\varepsilon_i}(u_i)$ convergent to $J_{\varepsilon_i}(u_{\varepsilon_i})$ from above.

On the other hand in penalty function method the optimal value $J(u_{opt})$ of the functional $J(u)$ is approximated from below (u_{ε_i} do not satisfy the constraints (1a)).

Hence if we change ε_i we can obtain an increase of the value of the functional $J_{\varepsilon_i}(u)$. Thus the sequence $\{J_{\varepsilon_i}(u_i)\}$ may lose its monotonicity, and may not be convergent to the optimal value.

However as it will be shown this difficulty can be overcome by an appropriate choice of ε_i.

Before it will have been shown we will prove the following

Lemma. For any arbitrary sequence $\{\varepsilon_i\}$ such that

$$0<\varepsilon_{i+1}\leqslant\varepsilon_i \tag{17}$$

we have

$$\lim_{i\to\infty}\left(-\varepsilon_i J'_{\varepsilon_i}(u_i),\ \bar{u}_{i+1}-u_i\right)\overset{\Delta}{=}\lim_{i\to\infty} x_i=0. \tag{18}$$

Proof. First note that the sequence $\{\varepsilon_i J_{\varepsilon_i}(u_i)\}$ is non-increasing. Indeed taking into account (12), (13), (14), (16) and (17) we obtain

$$\varepsilon_{i+1} J_{\varepsilon_{i+1}}(u_{i+1})\leqslant\varepsilon_i J_{\varepsilon_i}(u_{i+1})\leqslant\varepsilon_i \bar{J}_{\varepsilon_i}(u_{i+1})\leqslant\varepsilon_\varepsilon \bar{J}_{\varepsilon_i}(u_i)=\varepsilon_i J_{\varepsilon_i}(u_i).$$

On the other hand the sequence $\{\varepsilon_i J_{\varepsilon_i}(u_i)\}$ is bounded from below by zero, hence it is convergent.

Let us assume now that (18) does not hold. Hence in view of (15) we conclude that there exists a constant $\delta>0$, such that for every integer $Q>0$ there exist a subscript $\eta>Q$ such that

$$\left(-\varepsilon_\eta J'_{\varepsilon_\eta}(u_\eta),\ \bar{u}_{\eta+1}-u_\eta\right)\geqslant\delta. \tag{19}$$

Define

$$u^\alpha=u_\eta+\alpha(\bar{u}_{\eta+1}-u_\eta),\qquad \alpha\in(0,1). \tag{20}$$

It follows from (14) and (16) that

$$\varepsilon_\eta \bar{J}_{\varepsilon_\eta}(u^\alpha)\geqslant\varepsilon_\eta \bar{J}_{\varepsilon_\eta}(u_{\eta+1}),\qquad \forall\alpha\in(0,1) \tag{21}$$

From (11), (19), (20) and (21) we have

$$\begin{aligned}-\left(\varepsilon_\eta J'_{\varepsilon_\eta}(u_\eta),\ u_{\eta+1}-u_\eta\right)&\geqslant\tfrac{1}{2}\left((\varepsilon_\eta N+\bar{K}_\eta)\,u_{\eta+1}-u_\eta,\ u_{\eta+1}-u_\eta\right)+\\&+\alpha\left(-J'_{\varepsilon_\eta}(u_\eta),\ \bar{u}_{\eta+1}-u_\eta\right)-\tfrac{1}{2}\alpha^2\left((\varepsilon_\eta N+\bar{K}_\eta)(\bar{u}_{\eta+1}-u_\eta),\ \bar{u}_{\eta+1}-u_\eta\right)\geqslant\\&\geqslant\tfrac{1}{2}\left((\varepsilon_\eta N+\bar{K}_\eta)(u_{\eta+1}-u_\eta),\ u_{\eta+1}-u_\eta\right)+\alpha\delta-\tfrac{1}{2}\alpha^2\mu,\end{aligned}$$

where

$$\mu = \sup_{u, v \in U} \{\varepsilon_1 N(u-v, u-v) + (G(u-v), G(u-v))\} < \infty.$$

Substituting

$$\alpha = \min \{1, \delta/\mu\}$$

we obtain

$$(-\varepsilon_\eta J'_{\varepsilon_\eta}(u_\eta), u_{\eta+1} - u_\eta) \geq \tfrac{1}{2}((\varepsilon_\eta N + \bar{K}_\eta)(u_{\eta+1} - u_\eta), u_{\eta+1} - u_\eta) +$$
$$+ \min\left\{\tfrac{1}{2}\mu, \tfrac{1}{2}\frac{\delta^2}{\mu}\right\} = \tfrac{1}{2}((\varepsilon_\eta N + \bar{K}_\eta)(u_{\eta+1} - u_\eta), u_{\eta+1} - u_\eta) + \kappa \quad (22)$$

where $\kappa = \min\left\{\tfrac{1}{2}\mu, \tfrac{1}{2}\frac{\delta^2}{\mu}\right\} > 0$ does not depend on η.

Taking into consideration (12), (17) and (21) we get

$$\varepsilon_{\eta+1} J_{\varepsilon_{\eta+1}}(u_{\eta+1}) \leq \varepsilon_\eta \bar{J}_{\varepsilon_\eta}(u_{\eta+1}) = \varepsilon_\eta J_{\varepsilon_\eta}(u_\eta) + (\varepsilon_\eta J'_{\varepsilon_\eta}(u_\eta), u_{\eta+1} - u_\eta) +$$
$$+ \tfrac{1}{2}((\varepsilon_\eta N + \bar{K}_\eta)(u_{\eta+1} - u_\eta), u_{\eta+1} - u_\eta) \leq \varepsilon_\eta J_{\varepsilon_\eta}(u_\eta) - \kappa$$

or

$$\kappa \leq \varepsilon_\eta J_{\varepsilon_\eta}(u_\eta) - \varepsilon_{\eta+1} J_{\varepsilon_{\eta+1}}(u_{\eta+1})$$

which contradicts the convergence of $\{\varepsilon_i J_{\varepsilon_i}(u_i)\}$. This contradiction proves the lemma.

The sequence $\{\varepsilon_i\}$ is constructed in the following way

— choose $\varepsilon_0 > 0$

— put

$$\varepsilon_i = \begin{cases} \min\{\varepsilon_{i-1}, k_1 x_{i-1}^p\} & \text{if} \quad x_{i-1} > 0 \\ \dfrac{1}{k_2}\varepsilon_{i-1} & \text{if} \quad x_{i-1} = 0 \end{cases} \quad (23)$$

where $x_i = (-\varepsilon_i J'_{\varepsilon_i}(u_i), \bar{u}_{i+1} - u_i)$, and $k_1 > 0$, $k_2 > 1$ and $0 < p < 1$ are fixed constraints.

The sequence $\{\varepsilon_i\}$ constructed in such a way satisfies (17), hence it follows from Lemma that from the sequence $\{x_{v_i}\}$ can be substraced a subsequence $\{x_{(i)}\} \subset \{x_i\}$ monotonically decreasing to zero.

THEOREM. The following convergence takes place

$$J_{\varepsilon(i)}(u_{(i)}) \xrightarrow[(i)\to\infty]{} J(u_{opt}) \quad (24)$$

and in the case where u_{opt} is unique

$$u_{(i)} \xrightarrow[(i)\to\infty]{} u_{opt} \quad (25)$$

(if u_{opt} is not unique (25) takes place for some subsequence of $\{u_{(i)}\}$).

Proof. Let us consider first the case where $x_{i-1}=0$, i.e. where $\left(-J'_{\varepsilon_{i-1}}(u_{i-1}), \bar{u}_i - u_{i-1}\right)=0$. It means [9] that $J_{\varepsilon_{i-1}}(u)$ assumes at u_{i-1} its global minimum on U, i.e. $u_{i-1}=u_{\varepsilon_{i-1}}$. As it follows from (5) in this case any choice of $\varepsilon_i<\varepsilon_{i-1}$ is proper and in particular we can choose ε_i given by (23).

For the case where $x_{i-1}>0$ note that

$$\lim_{(i)\to\infty}\left(-J'_{\varepsilon(i)}(u_{(i)}), \bar{u}_{(i)+1}-u_{(i)}\right)=\lim_{(i)\to\infty}\frac{x_{(i)}}{\varepsilon_{(i)}}=0. \tag{26}$$

Indeed as it follows from Lemma and (23) for (i) large enough we have

$$0\leqslant\frac{x_{(i)}}{\varepsilon_{(i)}}=\frac{x_{(i)}}{k_1\, x^p_{(i-1)}}\leqslant\frac{1}{k_1}\, x^{1-p}_{(i-1)}$$

what in view of (18) proves the monotonic convergence to zero of the sequence $\left\{\frac{x_{(i)}}{\varepsilon_{(i)}}\right\}$.

Now taking into account that $J_{\varepsilon_{(i)}}(u)$ is convex we obtain from (4) and (15)

$$\frac{x_{(i)}}{\varepsilon_{(i)}}=\left(-J'_{\varepsilon(i)}(u_{(i)}), \bar{u}_{(i)+1}-u_{(i)}\right)\geqslant\left(-J'_{\varepsilon(i)}(u_{(i)}), u_{\varepsilon(i)}-u_{(i)}\right)\geqslant J_{\varepsilon(i)}(u_{(i)})+ \\ -J_{\varepsilon(i)}(u_{\varepsilon(i)})\geqslant 0. \tag{27}$$

(5) together with (26) and (27) prove (21).

To prove (25) let us note that from $\{u_{(i)}\}$ we can substract a weakly convergent subsequence $\{u_j\}\subset\{u_{(i)}\}$:

$$u_j \xrightarrow[j\to\infty]{} \bar{u}. \tag{28}$$

Since $J_{\varepsilon_i}(u)$ is convex we get from (15)

$$\frac{x_j}{\varepsilon_j}=\left(-J'_{\varepsilon_j}(u_j), \bar{u}_{j+1}-u_j\right)\geqslant\left(-J'_{\varepsilon_j}(u_j), u_{opt}-u_j\right)\geqslant J_{\varepsilon_j}(u_j)-J_{\varepsilon_j}(u_{opt})= \\ =J(u_j)+\frac{1}{\varepsilon_j}K(u_j)-J(u_{opt})$$

or

$$x_j+\varepsilon_j\left(J(u_{opt})-J(u_j)\right)\geqslant K(u_j). \tag{29}$$

Functional $K(u)$ is weakly lower semicontinuous as a convex one. Hence from (18), (28) and (19) we get

$$0=\lim_{j\to\infty}\left[x_j+\varepsilon_j\left(J(u_{opt})-J(u_j)\right)\right]\geqslant\overline{\lim_{j\to\infty}}\,K(u_j)\geqslant\underline{\lim_{j\to\infty}}\,K(u_j)\geqslant K(\bar{u})$$

which proves that $\bar{u}$ satisfies (1a).

On the other hand from (24) and (28) as well as from the fact that $J(u)$ is weakly lower semicontinuous it follows that

$$J(\bar{u})\leqslant\underline{\lim_{j\to\infty}}\,J(u_j)\leqslant\lim_{j\to\infty}J_{\varepsilon_j}(u_j)=J(u_{opt})$$

which proves that $\bar{u}=u_{opt}$. The uniqueness of u_{opt} assures [2] the convergence of the whole sequence $\{u_{(i)}\}$ to u_{opt} as in (25). Q. E. D.

COROLLARY. If the functional $J(u)$ satisfies

$$n(v, v) \leqslant (J''(u)\, v, v), \quad \forall u \in U, \ \forall v \in V \tag{30}$$

where $n>0$; then

$$u_{(i)} \xrightarrow[(i)\to\infty]{} u_{opt}. \tag{31}$$

Proof. Taking into account that $J(u)$ and $K(u)$ are convex from Taylor formula and (30) we obtain

$$J(u_{opt}) = J_{\varepsilon(i)}(u_{opt}) \geqslant J_{\varepsilon(i)}(u_{(i)}) + (J'_{\varepsilon(i)}(u_{(i)}), u_{opt} - u_{(i)}) + n(u_{opt} - u_{(i)}, u_{opt} - u_{(i)}).$$

Using (15) we get

$$(J(u_{opt}) - J_{\varepsilon(i)}(u_{(i)})) + (-J'_{\varepsilon(i)}(u_{(i)}), \bar{u}_{(i)+1} - u_{(i)}) \geqslant n(u_{opt} - u_{(i)}, u_{opt} - u_{(i)}). \tag{32}$$

Relations (24) and (26) together with (32) prove (31).

In the classical penalty function method the value of penalty term is usually taken as stop condition for iterative procedure, i.e. it is required that

$$K(u) < \delta_1, \tag{33}$$

where δ_1 is a given number.

In our case the condition (33) can be misleading. It follows from the fact that the value of penalty component $K(u_i)$ can be small, and at the same time the value of $J_{\varepsilon_i}(u_i)$ can be far from the minimal value of this functional on the whole set U. Hence we should additionally check if we are close enough to the minimum of $J_{\varepsilon_i}(u)$ on U.

To this end we can use the estimation (27) and besides (33) require that

$$(-J'_{\varepsilon(i)}\, u_{(i)}, \bar{u}_{(i)+1} - u_{(i)}) \leqslant \delta_2. \tag{34}$$

4. Problem of optimal heating subject to constraints of thermal stress

As an application of the iterative method described in Chapter 3 a problem of optimal heating of a homogenious plate subject to constraints of thermal stress is considered.

This problem was stated in [5] and [6].

We consider a system described by one — dimensional heat equation

$$\frac{\partial y(x,t)}{\partial t} = \frac{\partial^2 y(x,t)}{\partial x^2} \quad \text{for} \quad x \in (0,1),\ t \in (0,T) \tag{35}$$

along with initial condition

$$y(x, 0) = 0 \tag{35a}$$

and boundary conditions

$$\frac{\partial y(0,t)}{\partial x} = 0; \quad \frac{\partial y(1,t)}{\partial x} = \beta[v(t) - y(1,t)], \tag{35b}$$

where T is a fixed time of control, $\beta>0$ is the coefficient of heat exchange between the plate and the environment and $v(t)$ is the environment temperature the changes of which are governed by the equation

$$\frac{dv(t)}{dt}=-\frac{1}{\gamma}v(t)+\frac{1}{\gamma}u(t) \tag{36}$$

$$u(0)=0$$

where measurable control function u (the inflow of heating media) must satisfy the condition

$$0\leqslant u(t)\leqslant 1 \quad \text{a.e. in } [0, T]. \tag{36a}$$

Maximal thermal stress, which takes place in the system (35) at time t can be expressed by approximate formula

$$\sigma(t)=\lambda\int_0^1 [y(x, t)-y(0, t)]\,dx \tag{37}$$

where: $\lambda=Ep/(1-\Delta)$; E is Young's modulus; ρ is coefficient of linear thermal expansion; Δ is Poisson's ratio.

The problem of optimization is to find a control $u_{opt}(t)$, satisfying (36), which minimizes the functional,

$$J(u)=\big(z-y(T; u), z-y(T; u)\big)=\int_0^1 [z(x)-y(x, T; u)]^2\,dx \tag{38}$$

subject to the condition

$$\sigma(t)\leqslant\sigma_0 \quad \text{a.e. in } [0, T]. \tag{39}$$

where $z\in L^2(0, 1)$ is a given final distribution of temperature in the plate, which is to be approximate and given number σ_0 denotes the maximal admissible thermal stress.

It is known [8] that the mapping $u\to y(t; u)$ is linear and continuous from $L^2(0, t)$ to $L^2(0, 1)$, hence denoting

$$G(u)(t)=\sigma(t; u)=\lambda\int_0^1 [y(x, t; u)-y(0, t; u)]\,dx.$$

$$g(t)=\sigma_0$$

$$U=\{u\in L^2(0, T); 0\leqslant u(t)\leqslant 1 \quad \text{a.e. in } [0, T]\}$$

we find that our problem of optimization can be reduced to the scheme considered in Chapter 2.

Moreover the functional $J(u)$ is quadratic and its Hessian

$$\big(J''(u)v, v\big)=2\big(y(T; v); y(T; v)\big) \tag{40}$$

trivially satisfies (6). Therefore we can use the iterative procedure described in Chapter 3.

Note that in our case the condition (30) is not satisfied, hence we obtain only weak convergence of $\{u_{(i)}\}$ to u_{opt}. However if we use (40) and repeat the argument of Corollary we find that $\{y(T; u_{(i)})\}$ is strongly convergent to $y(T; u_{opt})$ in $L^2(0, 1)$.

To apply the procedure of minimization we must find elements u_i and $\bar{u}_{i+1}$ satisfying (14) and (15).

Functional $J_{\varepsilon_i}(u)$ is given by

$$J_{\varepsilon_i}(u)=J(u)+\frac{1}{\varepsilon_i}K(u)=\big(z-y(T;u), z-y(T;u)\big)+ \\ +\frac{1}{\varepsilon_i}\int_0^T [\max\{0, \sigma(t,u)-\sigma_0\}]^2\,dt \tag{41}$$

and $\tilde{J}_{\varepsilon_i}(u)$ has the form

$$\tilde{J}_{\varepsilon_i}(u)=J(u)+\frac{1}{\varepsilon_i}\Big[K(u_i)+\Big(K'(u_i), \sum_j \alpha^j u_i^j-u_i\Big)+ \\ +\frac{1}{2}\Big(\bar{K}_i\Big(\sum_j \alpha^j u_i^j-u_i\Big), \sum_j \alpha^j u_i^j-u_i\Big)\Big]=\Big(z-\sum_j \alpha^j y(u_i^j), z+ \\ -\sum_j \alpha^j y(u_i^j)\Big)+\frac{1}{\varepsilon_i}\Big[\int_{M_i}(\sigma(t;u_i)-\sigma_0)^2\,dt+2\int_{M_i}(\sigma(t;u_i)-\sigma_0)\times \\ \times\Big(\sum_j \alpha^j\sigma(t;u_i^j)-\sigma(t;u_i)\Big)dt+\int_{\bar{M}_i}\Big(\sum_j \alpha^j\sigma(t;u_i^j)-\sigma(t;u_i)\Big)^2 dt\Big], \tag{42}$$

where M_i and $\bar{M}_i$ are given by (7) and (8).

The element u_{i+1} is determined from the condition (14) using (42). To this and we must find coefficients α^j of convex combination, which minimize (42). This can be done using well known methods of quadratic programming in finite dimensional space. The element $\bar{u}_{i+1}$ must satisfy conditions (15) which in our case takes on the form

$$\big(z-y(T;u_i), y(T;\bar{u}_{i+1})\big)+\frac{1}{\varepsilon_i}\int_{M_i}(\sigma(t;u_i)-\sigma_0)\,\sigma(t;\bar{u}_{i+1})\,dt\geqslant \\ \geqslant\big(z-y(T;u_i), y(T;u)\big)+\frac{1}{\varepsilon_i}\int_{M_i}(\sigma(t;u_i)-\sigma_0)\,\sigma(t;u)\,dt. \tag{43}$$

If we introduce the system of the following equations adjoint to (35) and (36) respectively

$$-\frac{\partial p(x,t;u_i)}{\partial t}=\frac{\partial^2 p(x,t;u_i)}{\partial x^2}+\frac{\lambda}{\varepsilon_i}\int_{M_i}\max\{0, \sigma(t;u_i)-\sigma_0\}, \tag{44}$$

$$p(x,T;u_i)=y(x,T;u_i)-z,$$

$$\frac{\partial p(0,t;u_i)}{\partial x}=\frac{\lambda}{\varepsilon_i}\max\{0, \sigma(t;u_i)-\sigma_0\},$$

$$\frac{\partial p(1, t; u_i)}{\partial x} + \beta p(1, t; u_i) = 0,$$

$$\frac{dq(t; u_i)}{dt} = \frac{1}{\gamma}\,[q(t; u_i) + p(1, t; u_i)], \tag{45}$$

$$q(T; u_i) = 0,$$

then from (43) we obtain

$$\bar{u}_{i+1}(t) = \tfrac{1}{2}\,[1 + \operatorname{sgn} q(t; u_i)]. \tag{46}$$

As it follows from (42) and (46) in each step of iteration we must integrate two partial differential equations: the equation (35) of the system and the adjoint equation (44).

To obtain the aproximate solution of these equations a finite difference method was used. The time and space domains were devided into P and Q subintervals of the length $\Delta t = T/p$ and $\Delta x = 1/Q$ respectively. The solution of the quadratic programming problem necessary to find u_i was obtained using simplex method [7].

The computations were performed on the computer ODRA 1204. The following data were taken in computations:

$\beta = 10$; $\gamma = 0.003$; $T = 0.2$, $\lambda = 10^5$, $\sigma_0 = 2.7 \times 10^4$, $P = 40$; $Q = 10$; $\varepsilon_0 = 10^4$, $k_1 = 10$, $k_2 = 2$, $p = 1/2$.

Moreover it was assumed that the functions $z(x)$ and $y(x, 0)$ are constant and $z(x) \equiv 0.4$; $y(x, 0) \equiv 0$.

The total time of computations was 1091 sec. The obtained results are presented in figures 1 and 2.

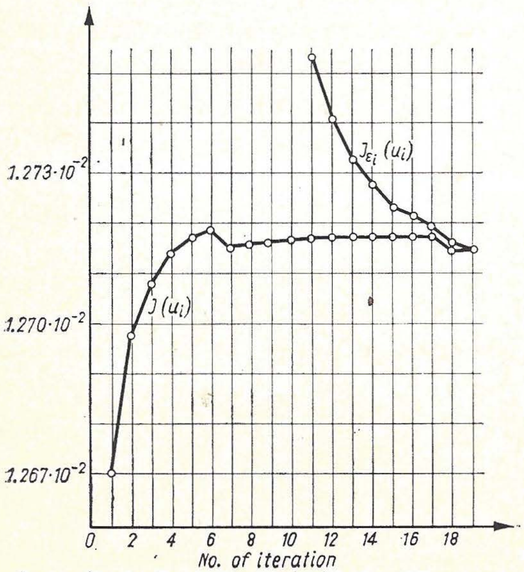

Fig. 1. Values of $J(u_i)$ and $J_{\varepsilon_i}(u_i)$ in the function of number of iteration

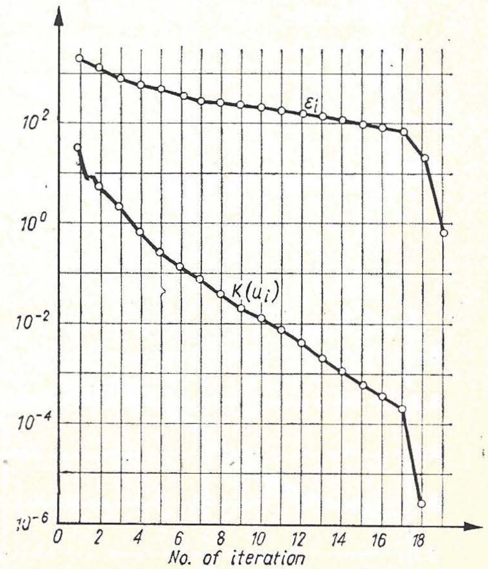

Fig. 2. Values of $K(u_i)$ and ε_i in the function of number of iteration

The plots of values of functionals $J(u)$, $J_{\varepsilon_i}(u)$ and $K(u)$ as well as the values of the parameter ε_i against the number of iterations are shown in figures 1 and 2.

The forms of obtained approximation of optimal control u_{opt}, thermal stress $\sigma(t; u_{opt})$ and final distribution of temperature are given in figures 3—5.

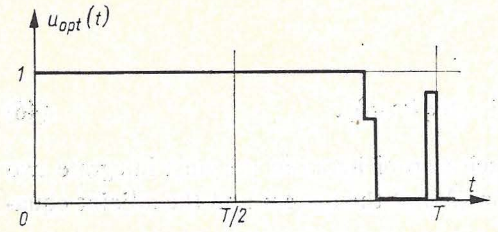

Fig. 3. Obtained approximation of optimal control

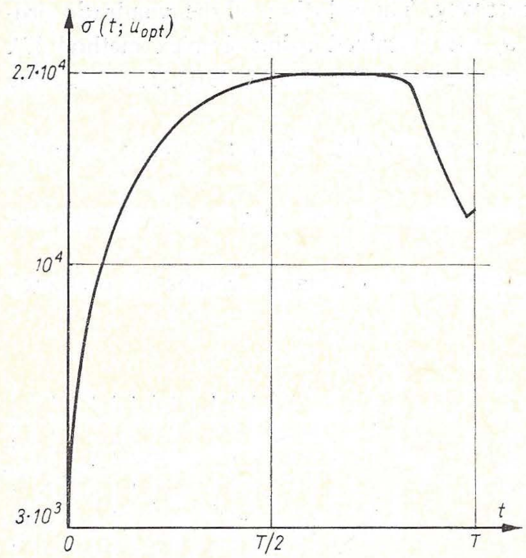

Fig. 4. Maximal thermal stress as a function of time

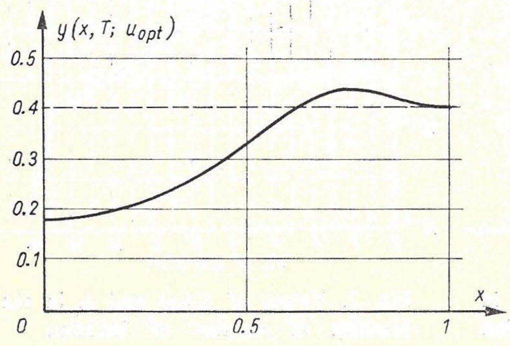

Fig. 5. Optimal final distribution of temperature

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Pewna metoda iteracyjna optymalizacji z zastosowaniem do zadań sterowania optymalnego z ograniczeniami w przestrzeni stanu

Zaproponowano algorytm iteracyjny wyznaczania minimum funkcjonału wypukłego na zbiorze domkniętym, wypukłym i ograniczonym w przestrzeni Hilberta przy ograniczeniach liniowych typu ograniczeń w przestrzeni stanu. Algorytm polega na połączeniu pewnej metody programowania wypukłego z metodą funkcji kary. Współczynnik kary jest modyfikowany w procesie iteracyjnym.

Podano dowód zbieżności algorytmu.

Jako przykład zastosowania tej metody podano zadanie wyznaczenia optymalnego nagrzewu pręta jednorodnego przy ograniczeniu naprężań cieplnych. Przedstawiono uzyskane wyniki numeryczne.

Некоторый итерационный метод оптимизации с применением к задачам оптимального управления с ограничениями в пространстве состояний

Предложен итеративный алгоритм определения минимума выпуклого функционала на замкнутом, выпуклом и ограниченном множестве в гильбертовом пространстве при линейных ограничениях типа ограничений в пространстве состояний. Алгоритм основан на сочетании некоторого метода выпуклого программирования с методом функции штрафа. Коэффициент штрафа модифицируется во время итерационного процесса.

Дано доказательство сходимости алгоритма.

В качестве иллюстрации применения этого метода приводится задача определения оптимального нагрева однородного стерженя, при ограничении по тепловой напряженности. Представлены полученные численные результаты.