Control
and Cybernetics

VOL. 3 (1974) No. 3/4

Sensitivity analysis of linear infinite-dimensional optimal control systems under changes of system order

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This paper is concerned with the real sensitivity analysis of singular pertirbed optimal control systems in various structures. The basic formulation of the sensitivity problem is presented. The properties of thesolution of linear differential equations in Banach space with a small parameter in the derivative are considered. The results are applied to the λ-sensitivity analysis. the method consists in an approximation of the sensitivity measure by its first- and second-order derivatives. An example ilustrates the application of the method.

1. Introduction

Realistic description of a physical, economical etc. process to be controlled usually results in a complicated mathematical model. Applications of control and optimization technique to this model often lead to enormous analytical and computational difficulties. Therefore simpler but more relevant practical models may be considered. To accomplish this it is necessary to analyse the effectiveness of the optimal control, defined on the basis of the model and applied to the real process.

In this paper we extend the concept of sensitivity to changes of the system order, i.e. so-called λ-sensitivity or structural sensitivity of optimal control systems. The finite dimensional structural sensitivity problem has been studied by many authors — see [4], [6]. The mathematical basis of this problem has been given by Tichonov [7], Tuptchiev [8] and Vasil'eva [9], [10]. The continuonity of the sensitivity measure for the processes described by differential equations in Hilbert space has been proven [2]. We consider now a more general class of models and apply the variational method to the real sensitivity problem.

Consider a process described by the following state equations:

$$\dot{x}_1=A_{11}x_1+A_{12}x_2+B_1u, \qquad x_1(0)=x_{10}, \qquad \text{(1a}$$

$$\lambda\dot{x}_2=A_{21}x_1+A_{22}x_2+B_2u, \qquad x_2(0)=x_{20}, \qquad \text{(1b)}$$

where the state $x_i(t) \in B_x^i$ is an absolute continuous function of time t; $u(t) \in B_u$ is a measurable and essentially bounded function of t; B_x^i, B_u are Banach spaces where B_x^1, B_u are reflexive; $\lambda \in [0, \lambda_1]$ is a small[1]) positive parameter. We assume the operators A_{ij} are linear and bounded.

The λ-sensitivity analysis consists in determining the quantitive effects due to reducing the order of the system, i.e., defining the following low-order model:

$$\left| \begin{array}{ll} \dot{\bar{x}}_1 = A_{11}\bar{x}_1 + A_{12}\bar{x}_2 + B_1\bar{u}, & \bar{x}_1(0) = x_{10}, \\ 0 = A_{21}\bar{x}_1 + A_{22}\bar{x}_2 + B_2\bar{u}, & \end{array} \right| \quad \begin{array}{r} (2a) \\ (2b) \end{array}$$

by setting $\lambda=0$. We assume the model is well-defined, i.e., there exists A_{22}^{-1}. We say that the model is degenerated if the initial condition x_{20} does not satisfy the equation (2b) for $t=0$.

The performance index is assumed to have the form

$$J(x_1, u) = 0.5\langle x_1(t_1), Fx_1(t_1)\rangle + 0.5\int_0^{t_1} (\langle x_1, Qx_1\rangle + \langle u, Ru\rangle)\, dt \tag{3}$$

where R is a linear bounded positive definite and selfadjoint operator: F, Q are linear and bounded positive semidefinite and selfadjoint operators: t_1 is a fixed final time.

The maximum principle — see [3] — implies that along the optimal trajectory

$$\hat{\bar{u}}(t) = R^{-1}\mathfrak{B}^*\hat{\bar{\psi}}_1(t) \tag{4}$$

where $\mathfrak{B} = B_1 - A_{12}A_{22}^{-1}B_2$ and $\hat{\bar{\psi}}_1 \in B_x^{1*}$ represents the constate variable, which satisfies the canonical equations:

$$\left| \begin{array}{l} \dot{\bar{x}}_1 = \mathfrak{A}\bar{x}_1 + \varphi\bar{\psi}_1, \\ \dot{\bar{\psi}}_1 = -\mathfrak{A}^*\bar{\psi}_1 + Q\bar{x}_1, \end{array} \right| \quad \begin{array}{r} (5a) \\ (5b) \end{array}$$

with the boundary conditions

$$\bar{x}_1(0) = x_{10}, \qquad \bar{\psi}_1(t_1) = -F\bar{x}_1(t_1),$$

where $\mathfrak{A} = A_{11} - A_{12}A_{22}^{-1}A_{21}$, $\varphi = \mathfrak{B}R^{-1}\mathfrak{B}^*$. Setting

$$\hat{\bar{\psi}}_1(t) = K(t)\hat{\bar{x}}_1(t) \tag{6}$$

we obtain the Riccati operator equation

$$-\dot{K} = K^*\mathfrak{A} + \mathfrak{A}^*K + K^*\varphi K - Q, \tag{7}$$

$$K(t_1) = -F.$$

Thus, the optimal control can be defined in a form of the closed-loop controller

$$\hat{\bar{u}}(t) = R^{-1}\mathfrak{B}^*K(t)\hat{\bar{x}}_1(t). \tag{8}$$

[1]) Actually, the "smallness" of the parameter λ can be determined on the basis of the sensitivity analysis which is considered in this paper.

The optimal control (4) for the model (2a, b) can be applied to the real process (1a, b) in various structuressee [12]. Since the model deviates from reality, the real state (and control) will be not optimal for the process nor for the model. Therefore, it is necessary to estimate the performance losses in (3) when the model order changes [2]). These losses will determine, in some sense, the effectiveness of the model and of the control structure.

2. Real sensitivity analysis

We present now some basic notions of the sensitivity problem, which was broadly investigated in [12]. Generally, the optimal control problem under equality constraints can be defined as follows

$$\min_{x, u} J(x, u) = \min_{u} J\big(X(u, a), u\big) = \hat{J}(a), \tag{9}$$

where $x = X(u, a)$ represents the state equation, the state $x \in B_x$; $u \in B_u$ is the control; $a \in B_a$ is a parameter. We assume the real process is represented by another state equation, which differs from the original one in the value of the parameter (e.g. $x = X(u, \alpha)$). Suppose the optimal control law is represented by the operator equation $u^i = R^i(x, a, \alpha)$ where i denotes the i — the structure of the control system. The real state x^i and the real value of the performance functional are determined as

$$x^i = X\big(R^i(x^i, a, \alpha), \alpha\big), \tag{10a}$$

$$J^i(a, \alpha) \overset{\mathrm{df}}{=} J(x^i, u^i). \tag{10b}$$

We call the operators $X^i(a, \alpha) = x^i$, which is a solution of (10a), and $U^i(a, \alpha) = u^i$ (if they exist) *the structural state and control characteristics.*

Assume there exists the optimal solution $\hat{x} = X^i(\alpha, \alpha) \overset{\mathrm{df}}{=} \hat{X}(\alpha)$, $u = U^i(\alpha, \alpha) \overset{\mathrm{df}}{=} \hat{U}(\alpha)$. The operators $\hat{X}(\alpha)$, $\hat{U}(\alpha)$ are called *the basis state and control characteristics.* The functional

$$S^i(a, \alpha) \overset{\mathrm{df}}{=} J^i(a, \alpha) - \hat{J}(\alpha) \tag{11}$$

evaluates the performance losses due to an imperfect knowledge of the process parameter α, and is called *the sensitivity measure* — see [12].

Suppose the state operator X and the performance J are twice differentiable with respect to their arguments. The following lemma results from [12].

LEMMA 2.1. Suppose the basic and structural characteristics are twice strongly and continuously differentiable with respect to a, α in an open set containing $a = \alpha$. Then the sensitivity measure is twice differentiable with respect to a, α and its derivatives satisfy the relations:

(i) $$S^i_a(a, a) = S^i_\alpha(a, a) = 0, \tag{12a}$$

(ii) $$S^i_{aa}(a, a) = -S^i_{a,\alpha}(a, a) = S^i_{\alpha\alpha}(a, a). \tag{12b}$$

[2]) Note that if $A_{21} = 0$, the sensitivity problem is trivial.

Thus, the sensitivity measure can be approximated by the Taylor expansion

$$S^i(a, \alpha)=0.5\,\langle S^i_{aa}(a, a), \delta a, \delta a\rangle+o(\|\delta a\|^2), \tag{13}$$

where $\delta a=\alpha-a$ and

$$S^i_{aa}(a, a)=X^{i*}_a\,\hat{L}_{xx}\,X^i_a+2X^{i*}_a\,\hat{L}_{xu}\,U^i_a+U^{i*}_a\,\hat{L}_{uu}\,U^i_a, \tag{14}$$

where X^i_a, U^i_a are the Frechet derivatives of the structural characteristics with respect to a at (a, a) and $L(\eta, x, u, a)$ is the Lagrange functional of the optimization problem; η represents the adjoint variable.

Now suppose $a, \alpha\in[a_0, a_1]\subset R^1$. Let the basic sensitivity characteristics have continuous second-order derivatives at $(a_0, a_1]$ and one-side first- and second-order derivatives at $a=a_0$. Suppose the basic characteristics and their derivatives are continuous when $a\to a_0^+$, i.e., for instance

$$\lim_{a\to a_0^+}\hat{X}_a(a)=\hat{X}_{a+}(a_0)\ \text{etc.} \tag{15}$$

where $\hat{X}_{a+}(a_0)$ denotes the one-side derivatives.

Suppose the structural characteristics have continuous second-order derivatives with respect to $a, \alpha\in(a_0, a_1]$ and one-side first- and second-order derivatives with respect to a, α at $a=\alpha=a_0$. Suppose the derivatives are continuous when $\alpha=a\to a_0^+$, i.e.

$$\lim_{a\to a_0^+}X^i_a(a, a)=X^i_{a+}(a_0, a_0)\ \text{etc.} \tag{16}$$

Then the following lemma holds:

LEMMA 2.2. The sensitivity measure has first- and second-order one-side derivatives with respect to a, α at (a_0, a_0) and these derivatives satisfy the relations (12a, b).

The proof follows from the properties of the composite function (e.g. $S^i(a, \alpha)=$ $=J(X^i(a, \alpha), U^i(a, \alpha))-J(\hat{X}(\alpha), \hat{U}(\alpha))$. Because of the continuity

$$S^i_{a+}(a_0, a_0)=\lim_{a\to a_0^+}S^i_a(a, a)=0.$$

Analogously $S^i_{\alpha+}(a_0, a_0)=0$ and

$$S^i_{aa+}(a_0, a_0)-S^i_{\alpha\alpha+}(a_0, a_0)=\lim_{a\to a_0^+}\left(S^i_{aa}(a, a)-S^i_{\alpha\alpha}(a, a)\right)=0.$$

This lemma implies that the sensitivity measure $S^i(a_0, \alpha)$, $\alpha\in[a_0, a_1]$ can be approximated locally as in (13), (14) where X^i_a, U^i_a will denote the one-side derivatives at point (a_0, a_0).

This result can be also stated in a form of a relativity principle of the local sensitivity analysis. Namely, it is not important which of the parameters has changed, e.g., to approximate $S^i(a_0, \alpha)$ we can compute the one-side derivatives of $S^i(\alpha, a_0)$ with respect to a at point (a_0, a_0) — see [12].

The derivatives $\hat{X}_a$, $\hat{U}_a$ are called the basic sensitivity functions (operators); X^i_a, U^i_a are called the structural sensitivity functions.

3. Basic theorems — the homogeneous equations

Consider first the singular perturbed homogeneous equation (1a, b), setting $u=0$. Assume the operators A_{ii} are linear, but not necessarily bounded[3]). Let the Chauchy problem (1a, b) be well-defined for $\lambda \in [0, \lambda_1]$, that is, let A_{ii} be infinitezimal generators and let there exists the operator $\mathfrak{A}$, defined by (5a, b) which is also an infinitezimal generator. Let $T_1(t)$ denote the stringly continuous semigroup generated by A_{11} $[T_2(t, \lambda), \lambda \in (0, \lambda_1]$ denote the semigroup (strongly continuous) generated by $\frac{1}{\lambda} A_{22}$; $\bar{T}(t)$ denote the semigroup generated by $\mathfrak{A}$.

THEOREM 3.1. If there exist $\varepsilon \in (0, \lambda_1]$, $a_1, a_2 < 0$, N_1, N_2 such that for every $t \in [0, t_1]$ and for every $\lambda \in (0, \varepsilon)$[4])

$$\|T_1(t)\| \leqslant N_1 e^{a_1 t}, \qquad \|T_2(t, \lambda)\| \leqslant N_2 e^{\frac{a_2 t}{\lambda}} \tag{17}$$

then for every

$$\left| \begin{array}{ll} t \in [0, t_1], & \lim\limits_{\lambda \to 0+} x_1(t, \lambda) = \bar{x}_1(t), \\ t \in (0, t_1], & \lim\limits_{\lambda \to 0+} x_2(t, \lambda) = \bar{x}_2(t), \end{array} \right| \qquad \begin{array}{r} (18a) \\ (18b) \end{array}$$

where $x_1(t, \lambda)$ are the solutions of the perturbed system, and $\bar{x}_i(t)$ are determined by (2a, b), with $u=0$.

This theorem is a generalization of the analogous result presented in [2], and can be similarly proven. In order to show the uniform boundnes of $x_i(t, \lambda)$, the Gronwall inequality and the Tichonov's theorem [7] can be used. The continuity follows from an application of the Green formula.

The above theorem implies that if the subprocess (1b) is stable, the limit trajectory of $\bar{x}_1(t)$ does not depend on the degeneracy of the model. This result is often interpreted as an effect of "boundary layer" — see [6], [10].

We assume that the assumptions of the Theorem 3.1 are fulfilled in further analysis. In Appendix 1 the following theorem is proven.

THEOREM 3.2. The solution $x_1(t, \lambda)$ of the perturbed system (1a, b) has a one-side derivative with respect to λ at $\lambda = 0^+$, which is defined by the equation

$$\dot{\bar{\xi}}_1 = \mathfrak{A}\bar{\xi}_1 - A_{12} A_{22}^{-2} A_{21} \dot{\bar{x}}_1 \tag{19a}$$

with the initial condition

$$\bar{\xi}_1(0) = A_{12} A_{22}^{-1} (x_{20} + A_{22}^{-1} A_{21} x_{10}). \tag{19b}$$

A direct consequence of this theorem is the following corollary.

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3) Obviously, an additional assumption $x_{0i} \in \mathscr{D}(A_{ii})$ is needed.

4) This condition can be stated in an another form: the spectrum of operator A_{22} is situated in the left half of the complex plane. Note that this condition is sufficient for the existence of A_{22}^{-1}.

COROLLARY 3.1. The derivative $\xi_1(t, \lambda)$ of $x_1(t, \lambda)$ with respect to λ for $\lambda > 0$ tends continuously to the one-side derivative $\bar{\xi}_1(t)$, determined by (19a, b), when $\lambda \to 0^+$.

NOTE 3.1. Assume that the operators A_{ij} depend on t, i.e., the process is not stationary. Then, the analogous theorems can be proven (see Appendix 1), with the following assumptions.

(i) $A_{ij}(t)$ have domains independent of t and are strongly continuously twice differentiable with respect to t in their domains.

(ii) The strongly continuous semigroups $T_1(t, \tau)$, $T_2(t, \tau, \lambda)$ satisfy the inequalities

$$\|T_1(t, \tau)\| \leqslant N_1 e^{a_1(t-\tau)}, \qquad \|T_2(t, \tau, \lambda)\| \leqslant N_2 e^{\frac{a_2(t-\tau)}{\lambda}}, \tag{20}$$

where $a_2 < 0$. The initial condition for the derivative $\bar{\xi}_1(t)$ is

$$\bar{\xi}_1(0) = A_{12}(0) A_{22}^{-1}(0) \left(x_{20} + A_{22}^{-1}(0) A_{21}(0) x_{10}\right). \tag{21}$$

NOTE 3.2. Is the operators A_{ij} depend on λ and are strongly continuous with respect to λ when $\lambda \to 0^+$, whereby $A_{22}^{-1}(\lambda) A_{21}(\lambda)$ strongly converges to $A_{22}^{-1}(0) A_{21}(0)$; and if for $\lambda \in (0, \varepsilon]$ the inequalities (17) are fulfiled, then Theorem 3.1 holds — see [2].

Moreover, if the operators $A_{ij}(\lambda)$ are strongly continuously differentiable with respect to $\lambda \in [0, \varepsilon]$, whereby $A_{22}^{-1}(\lambda) A_{21}(\lambda)$ is strongly continuously differentiable, then the one-side derivative $\bar{\xi}_1(t)$ exists and satisfies the equation (see Appendix 1)

$$\dot{\bar{\xi}}_1 = \mathfrak{A}\bar{\xi}_1 + \left(\frac{dA_{11}}{d\lambda}(0) - A_{12}(0) \frac{d}{d\lambda}(A_{22}^{-1} A_{21})(0) - \right.$$
$$\left. - \frac{dA_{12}}{d\lambda}(0)(A_{22}^{-1} A_{21})(0)\right) \bar{x}_1 - A_{12}(0) A_{22}^{-2}(0) A_{21}(0) \dot{\bar{x}}_1 \tag{22}$$

with the initial condition (21).

NOTE 3.3. In the operators A_{ij} depend on t and λ, and $A_{ij}(t, \lambda)$ strongly converges to $A_{ij}(t, 0)$ whereby $A_{22}^{-1}(t, \lambda) A_{21}(r, \lambda)$ strongly and uniformly converges to $A_{22}^{-1}(t, 0) A_{21}(t, 0)$, and the semigroup $T_1(t, \tau, \lambda)$, $T_2(t, \tau, \lambda)$ satisfies (20), then Theorem 3.1 holds.

If the operators $A_{ij}(\mathrm{t}, \lambda)$ are strongly continuously and uniformly differentiable with respect to λ and twice differentiable (strongly continuously, uniformly) with respect to t whereby $A_{22}^{-1}(\mathrm{t}, \lambda) A_{21}(\mathrm{t}, \lambda)$ is differentiable (strongly continuously, uniformly) with respect to λ and twice differentiable with respect to t, then the derivative $\bar{\xi}_1(t)$ exists and satisfies equation (22) where $A_{ij}(\lambda)$ are substituted by $A_{ij}(\mathrm{t}, \lambda)$.

THEOREM 3.3. The solution $x_1(t, \lambda)$ is twice differentiable with respect to λ for $\lambda \in (0, \varepsilon]$, and its second-order derivative converges continuously to the one-side derivative at $\lambda = 0^+$, defined as follows

$$\dot{\bar{\xi}}_1^2 = \mathfrak{A}\bar{\xi}_1^2 - A_{12} A_{22}^{-2} A_{21} \dot{\bar{\xi}}_1 \tag{23}$$

$$\bar{\xi}_1^2(0) = \bar{\xi}_1(0)$$

when $\lambda \to 0^+$.

The proof is similar to that of Theorem 3.2.

4. Basic theorems — the optimization problem

Consider now the optimization problem, formulated in the introduction. Setting $\psi_2 = \frac{1}{\lambda}\tilde{\psi}_2$ (where $\tilde{\psi}_2$ denotes the constante variable corresponding to x_2) we obtain the canonical equations for problem (1a, b), (3) in a form:

$$\dot{x}_1 = A_{11} x_1 + A_{12} x_2 + B_1 R^{-1}(B_1^* \psi_1 + B_2^* \psi_2), \tag{24a}$$

$$\lambda\dot{x}_2 = A_{21} x_1 + A_{22} x_2 + B_2 R^{-1}(B_1^* \psi_1 + B_2^* \psi_2), \tag{24b}$$

$$\dot{\psi}_1 = -A_{11}^* \psi_1 - A_{21}^* \psi_2 + Qx_1, \tag{25a}$$

$$\lambda\psi_2 = -A_{12}^* \psi_1 - A_{22}^* \psi_2, \tag{25b}$$

with boundary conditions

$$x_1(0) = x_{10},\ x_2(0) = x_{20},\ \psi_2(t_1) = -Fx(t_1),\ \psi_2(t_1) = 0 \tag{26}$$

and the optimal control

$$\hat{u}(t, \lambda) = R^{-1}\left(B_1^* \psi_1(t, \lambda) + B_2^* \psi_2(t, \lambda)\right). \tag{27}$$

THEOREM 4.1. If the assumptions of Theorem 3.1 hold, then for every

$$t \in [0, t_1] \qquad \lim_{\lambda \to 0^+} \hat{x}_1(t, \lambda) = \hat{\bar{x}}_1(t), \tag{28a}$$

$$t \in (0, t_1] \qquad \lim_{\lambda \to 0} \hat{u}(t, \lambda) = \hat{\bar{u}}(t). \tag{28b}$$

The proof follows immediately from [2].

THEOREM 2.4. The optimal solution $\hat{x}_1(t, \lambda)$, $\hat{\psi}_1(t, \lambda)$, $\hat{u}(t, \lambda)$ has one-side derivatives $\bar{\xi}_1(t)$, $\bar{\eta}_1(t)$, $\bar{\sigma}(t)$ with respect to λ at $\lambda = 0^+$, which are determined by the equations:

$$\dot{\bar{\xi}}_1 = \mathfrak{A}\bar{\xi}_1 + \mathfrak{B}\bar{\sigma} - A_{12} A_{21}^{-2} A_{21} \hat{\bar{x}}_1, \tag{29a}$$

$$\dot{\bar{\eta}}_1 = -\mathfrak{A}^* \bar{\eta}_1 + Q\bar{\xi}_1 - A_{21}^* A_{22}^{*-2} A_{12}^* \hat{\bar{\psi}}_1, \tag{29b}$$

where

$$\bar{\xi}_1(0) = A_{12} A_{22}^{-1}\left(x_{20} + A_{22}^{-1} A_{21} x_{10} + A_{22}^{-1} B_2 \hat{u}(0)\right), \tag{30a}$$

$$\bar{\eta}_1(t_1) = -F\bar{\xi}_1(t_1) + A_{21}^* A_{22}^{*-2} A_{12}^* F\bar{x}_1(t_1) \tag{30b}$$

and

$$\bar{\sigma}(t)=R^{-1}\mathfrak{B}^*\bar{\eta}_1+R^{-1}B_2^*A_{22}^{-2}A_{21}\hat{\bar{\psi}}_1. \quad (31)$$

The proof is given in the Appendix 1.

COROLLARY 4.1. The derivatives $\xi_1(t, \lambda), \sigma(t, \lambda)$ of the optimal $\hat{x}_1(t, \lambda), \hat{u}(t, \lambda)$, determined for $\lambda>0$, tend continuously to the one-side derivatives $\bar{\xi}_1(t), \bar{\sigma}(t)$ when $\lambda\to 0^+$.

NOTE 4.1. If the operators A_{ij}, B_i depend on t, and if some additional assumptions are fulfilled — see Note 3.3., then the derivatives $\bar{\xi}_1, \bar{\sigma}$ can be defined in an analogous way.

NOTE 4.2 If can be easily proven that the optimal solution has second-order one-side derivatives with respect to λ at $\lambda=0^+$ — see Theorem 3.3.

In terms of the sensitivity analysis according to the foregoing theorem, there exist basic sensitivity functions $\bar{\xi}_1, \bar{\sigma}$ which can be obtained by solving the two-point boundary value problem (29a, b). The most suitable way to compute these functions is to introduce the substitution

$$\bar{\eta}_1(t)=K(t)\,\bar{\xi}_1(t)+L(t), \quad (32)$$

where K will satisfy the Riccati equation (7) and L is determined by the equation

$$\dot{L}=-(K\varphi+\mathfrak{A}^*)\,L-K\delta_1+\delta_2, \quad (33)$$

$$L(t_1)=+A_{21}^*\,A_{22}^{*-2}\,A_{12}^*\,F\bar{x}_1(t_1),$$

where $\delta_1=\mathfrak{B}R^{-1}B_2^*A_{22}^{*-2}A_{12}^*\hat{\bar{\psi}}_1-A_{12}A_{22}^{-2}A_{21}\hat{\bar{x}}_1$, $\delta_2=-A_{21}^*A_{22}^{*-2}A_{12}^*\hat{\bar{\psi}}_1$.

Thus, the basic control sensitivity function is defined as

$$\bar{\sigma}=R^{-1}\mathfrak{B}^*K\bar{\xi}_1+R^{-1}\mathfrak{B}^*L \quad (34)$$

and this relation expresses exactly the linearised closed-loop optimal control law-see [12].

5. Sensitivity analysis of several optimal control structure

The theorems presented in Sections 3 and 4 provide a basis for the λ-sensitivity analysis. We consider now several well-known optimal control structures — see [12]. In order to determine the second-order sensitivity approximation we apply the relativity principle, i.e. assume that the model is singularly perturbed and the real process is described by the low-order state equation (2a, b). If is easy to show that the conditions of Lemmas 2.1, 2.2 hold for the considered structures. Thus, the sensitivity analysis consists in determining the structural sensitivity function (which will be denoted by X_1^i, U^i) and approximating the sensitivity measure according to (14).

$(i=0)$ — *open loop structure*

The control is applied to the system in the same way as it is determined, hence $U^0=\bar{\sigma}$. The structural state sensitivity function X_1^0 does not depend on λ in the model explicitely, and can be computed from the equation

$$\dot{X}_1^0=\mathfrak{A}X_1^0+\mathfrak{B}\bar{\sigma}, \qquad X_1^0(0)=0 \tag{35}$$

$(i=1)$ — *closed-loop structure*

It is assumed that the optimal control law is synthesized on the basis of a perturbed model, and its linear approximation is given by (34). Hence

$$\dot{X}_1^1=\mathfrak{A}X_1^1+\mathfrak{B}U^1, \qquad X_1^1(0)=0, \tag{36}$$

where

$$U^1=R^{-1}\mathfrak{B}^*(KX_1^1+L),$$

where K and L are given from (7), (33).

Note that the closed-loop controller is independent of the process initial conditions. Hence we formulate one important property of the closed-loop structure — *the second-order sensitivity approximation does not depend on the degeneracy of the model.* This is not true for the open-loop structure.

If the process is absolutely controlable along the trajectory $X_1(t)$ — see [11], and if we can measure the current state $x_1(t)$ of the process exactly, then the optimal trajectory of the model $\hat{x}_1$ can be strictly realized in the process. Then we may say that the *optimal trajectory tracking structure* is applicable-see [11], [12]. In Appendix 2 the conditions of absolute controlability of a singular perturbed process (1a, b) are discussed.

The λ-sensitivity analysis of the optimal trajectory tracking structure is very complicated and will not be presented here (some remarks are given in Appendix 2).

Under some additional assumptions — see [12], *the open- and closed-loop optimizing feedback structures* can be applied. If can be easily proven that, in this case, these structures have the same sensitivity as the classical open-(closed-)-loop structures respectively.

Because the process is linear, the second-order term of the sensitivity approximation has the form

$$S_{\lambda\lambda}^i=X_1^{i*}(t_1)FX_1^i(t_1)+\int_0^1(X_1^{i*}QX_1^i+U^{i*}RU^i)\,dt \tag{37}$$

and can be easily computed.

6. An example

Consider the following simple example which can be interpreted as a model of destributed heat exchange. The state equation has the form

$$\dot{x}_1=-x_1+2\int_0^1 x_2(t,z)\,dz+u, \qquad x_1(0)=1, \tag{37a}$$

$$\lambda \frac{\partial x_2(t,z)}{\partial t} = -x_2(t,z) + z x_1(t),\ x_2(0,z) = \frac{3}{2} z^2, \tag{37b}$$

and the assumed performance functional

$$J(x_1, u) = 0.5\left(x_1^2(1) + \int_0^1 \left(b x_1^2(t) + \frac{1}{b} u^2(t)\right) dt\right). \tag{38}$$

where $x_1(t) \in R^1$, $x_2(t, \cdot) \in L^2_{[0,1]}$. Setting $\lambda = 0$ we obtain the low-order model

$$\dot{x}_1 = u, \qquad x_1(0) = 1. \tag{39}$$

Hence the optimal solution

$$\hat{\bar{x}}_1 = e^{-bt}, \qquad \hat{\bar{u}} = -be^{-bt},$$

and $K(t) = -1$. The basic sensitivity functions satisfy the equation

$$\left|\begin{array}{ll} \dot{\bar{\xi}}_1 = b\bar{\eta}_1 + be^{-bt}, & \bar{\xi}_1(0) = 0, \\ \dot{\bar{\eta}}_1 = b\bar{\xi}_1 - be^{-bt}, & \bar{\eta}_1(1) = -\bar{\xi}_1(1) + e^{-b}. \end{array}\right|$$

Hence $L = e^{b(t-2)}$.

In order to determine the structural sensitivity functions we solve the equations
— for the open-loop structure

$$\dot{X}_1^0 = -b\bar{\eta}_1, \qquad X_1^0(0) = 0;$$

— for the closed-loop structure

$$\dot{X}_1^1 = U^1, \qquad X_1^1(0) = 0,$$

where: $U^1 = -b\,(X_1^1 - e^{b(t-2)})$.

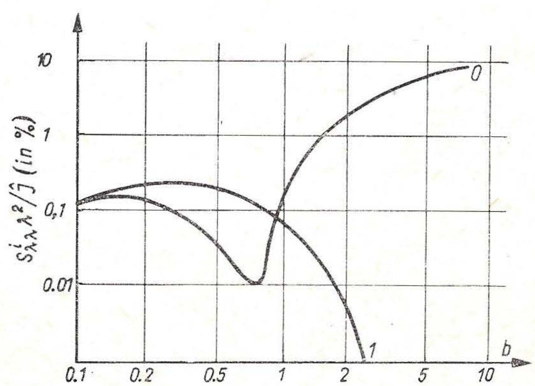

Fig. 1

In Fig. 1 the relative performance losses approximations for $\lambda = 0.1$ are shown. If $b > 1$ then the performance losses for the closed loop structure are close to zero. Moreover, the feedbach optimal controller can be very easily constructed.

7. Conclusions

The paper describes an application of the general theory of sensitivity to the singular perturbed optimal control system, described by differential equations in Banach space. The preliminary notions of the sensitivity analysis and the λ-sensitivity problem have been presented. Basic results similar to the theorems proved before have been obtained. Namely, if the, reduced state equation, is stable, then the λ-sensitivity problem is well-defined and a computational method can be applied.

The reduced model is usually much more relevant and practical for numerical optimization and the synthesis of various optimal control structures. The performance losses due to employing the low-order model can be rather easily estimated. The computational effort of the method is comparable to the solving of the reduced optimization problem. Moreover, the sensitivity analysis allows the comparison of different optimal control structures.

By giving a simple ilustrative example the advantage of the proposed method is shown.

It is expected that in future research more general results for nonlinear systems will be obtained.

APPENDIX 1. THE PROOFS OF THEOREMS 3.3 AND 4.2

Equation (1b) can be expressed in an integral form:

$$x_2(t, \lambda) = T_2(t, \lambda)\, x_{20} + \frac{1}{\lambda} \int_0^t T_2(t-\tau, \lambda)\, A_{21}\, x_1(\tau, \lambda)\, d\tau . \tag{1.1}$$

Applying the Green formula we obtain

$$x_2(t, \lambda) = T_2(t, \lambda)\,(x_{20} + A_{22}^{-1} A_{21} x_{10}) - A_{22}^{-1} A_{21}\, x_1(t, \lambda) + \\ + \int_0^t T_2(t-\tau, \lambda)\, A_{22}^{-1} A_{21} \big(A_{11}\, x_1(\tau, \lambda) + A_{12}\, x_2(\tau, \lambda)\big)\, d\tau . \tag{1.2}$$

Let us denote $\Delta x_i(t) = \big(x_i(t, \lambda) - \bar{x}_i(t)\big)/\lambda$. (Since (1.2) we have

$$\Delta x_2(t, \lambda) = \frac{1}{\lambda} T_2(t, \lambda)\,(x_{20} + A_{22}^{-1} A_{21} x_{10}) - A_{22}^{-1} A_{21}\, \Delta x_1(t, \lambda) + \\ + \frac{1}{\lambda} \int_0^t T_2(t-\tau, \lambda)\, A_{22}^{-1} A_{21}\, \dot{x}_1(\tau, \lambda)\, d\tau . \tag{1.3}$$

We now transform the integral part of the above equation

$$\frac{1}{\lambda} \int_0^t T_2(t-\tau, \lambda)\, A_{22}^{-1} A_{21}\, \dot{x}_1(\tau, \lambda)\, d\tau = \\ = T_2(t, \lambda)\, A_{22}^{-1} A_{21} (A_{11} x_{10} + A_{12} x_{20}) - A_{22}^{-2} A_{21}\, \dot{x}_1(t, \lambda) + \\ - \int_0^t T_2(t-\tau, \lambda)\, A_{22}^{-2} A_{21} \big(A_{11}\, \dot{x}_1(\tau, \lambda) + A_{12}\, \dot{x}_2(\tau, \lambda)\big)\, d\tau . \tag{1.4}$$

Hence, by the uniform boundness of $\dot{x}_i(t, \lambda)$ — see theorem 3.2, we obtain

$$\lim_{\lambda\to 0+} \frac{1}{\lambda} \int_0^t T_2(t-\tau, \lambda)\, A_{22}^{-1}\, A_{21}\, \dot{x}_1(\tau, \lambda) = A_{22}^{-2}\, A_{21}\, \dot{\bar{x}}_1(t)$$

and

$$\Delta \dot{x}_1(t, \lambda) = \mathfrak{A} \Delta x_1(t, \lambda) = A_{12}\, A_{22}^{-2}\, A_{21}\, \dot{\bar{x}}_1(t) + $$
$$+ \frac{1}{\lambda} A_{12}\, T_2(t, \lambda)\, (x_{20} + A_{21}^{-1}\, A_{21}\, x_{10}) + 0(\lambda), \tag{1.5}$$
$$\Delta x_1(0) = 0$$

Let us consider the following integral

$$I(t, \lambda) = \frac{1}{\lambda} \int_0^t \bar{T}(t-\tau)\, A_{12}\, T_2(\tau, \lambda)\, d\tau = -A_{12}\, A_{22}^{-1}\, T_2(t, \lambda) +$$
$$+ \bar{T}(t)\, A_{12}\, A_{22}^{-1} - \int_0^t \bar{T}(t-\tau)\, \mathfrak{A} A_{12}\, A_{22}^{-1}\, T_2(\tau, \lambda)\, d\tau$$

Hence

$$\lim_{\lambda\to 0+} I(t, \lambda) = \bar{T}(t)\, A_{12}\, A_{22}^{-1} \tag{1.6}$$

and

$$\Delta x_1(t, \lambda) = \bar{T}(t)\, A_{12}\, A_{22}^{-1}\, (x_{20} + A_{22}^{-1}\, A_{21}\, x_{10}) +$$
$$- \int_0^t \bar{T}(t-\tau)\, A_{12}\, A_{22}^{-2}\, A_{21}\, \dot{\bar{x}}_1(\tau)\, d\tau + 0(\lambda). \tag{1.7}$$

If follows that

$$\lim_{\lambda\to 0+} \Delta x_1(t, \lambda) = \bar{\xi}_1(t) \tag{1.8}$$

where $\bar{\xi}_1(t)$ is determined in (19a, b). The proof is complete.

If the operators A_{ij} depend on t — see Note 3.1, the relation (1.2) will have the following form:

$$x_2(t, \lambda) = T_2(t, 0, \lambda)\, (x_{20} + A_{22}^{-1}(0)\, A_{21}(0)\, x_{20}) — A_{22}^{-1}(t)\, A_{21}(t)\, x_1(t, \lambda) +$$
$$+ \int_0^t T_2(t, \tau, \lambda) \left(A_{22}^{-1}(\tau)\, A_{21}(\tau)\, \dot{x}_1(\tau, \lambda) + \frac{d(A_{22}^{-1}(\tau)\, A_{21}(\tau))}{d\tau}\, x_1(\tau, \lambda) \right) d\tau \tag{1.9}$$

and the relation (1.4) can also be easily modified.

In the proof of the result, given in the Note 3.2, the relation (1.3) will be expressed as

$$\Delta x_2(t, \lambda) = \frac{1}{\lambda} T_2(t, \lambda)\, (x_{20} + A_{22}^{-1}(\lambda)\, A_{21}(\lambda)\, x_{10}) +$$
$$- \frac{1}{\lambda} (A_{22}^{-1}(\lambda)\, A_{21}(\lambda) - A_{22}^{-1}(0)\, A_{22}(0))\, \bar{x}(t) + A_{22}^{-1}(\lambda)\, A_{21}(\lambda)\, x_1(t, \lambda) +$$
$$+ \int_0^t T(t-\tau, \lambda)\, A_{22}^{-1}(\lambda)\, A_{21}(\lambda)\, \dot{x}_1(\tau, \lambda)\, d\tau. \tag{1.10}$$

The proof of theorem 4.2 is similar. From equation (25b) we have

$$\psi_2(t,\lambda)=-\frac{1}{\lambda}\int_0^t \exp\left(-A_{22}^*\frac{t-\tau}{\lambda}\right)A_{21}^*\psi_1(\tau,\lambda)\,d\tau. \tag{1.11}$$

Denoting $\Delta\psi_1(t,\lambda)$, $\Delta\psi_2(t,\lambda)$ as before

$$\Delta\psi_2(t,\lambda)=\frac{1}{\lambda}\exp\left(-A_{22}^*\frac{t_1-t}{\lambda}\right)A_{22}^{*-1}A_{12}^*\psi_1(t,\lambda)-$$

$$-A_{22}^{*-1}A_{12}^*\Delta\psi_1(t,\lambda)-\frac{1}{\lambda}\int_{t_1}^t \exp\left(-A_{22}^*\frac{t-\tau}{\lambda}\right)A_{22}^{*-1}A_{12}^*\dot{\psi}_1(\tau,\lambda)\,d\tau \tag{1.12}$$

or, in an another form,

$$\Delta\psi_2(t,\lambda)=-\frac{1}{\lambda}\exp\left(A_{22}^*\frac{t_1-t}{\lambda}\right)A_{22}^{*-1}A_{12}^*Fx_1(t,\lambda)+$$

$$-A_{22}^{*-1}A_{12}^*\Delta\psi_1(t,\lambda)+A_{22}^{*-2}A_{12}^*\dot{\bar{\psi}}_1(t)+0(\lambda). \tag{1.13}$$

Since (24b) we have

$$\Delta x_2(t,\lambda)=-A_{22}^{-1}A_{21}\Delta x_1(t,\lambda)-A_{22}^{-2}A_{21}\dot{\bar{x}}_1(t)+$$

$$-A_{22}^{-1}B_2R_1(B_1^*\Delta\psi_1(t,\lambda)+B_2^*\Delta\psi_2(t,\lambda)+$$

$$+\frac{1}{\lambda}\exp\left(A_{22}\frac{t}{\lambda}\right)(x_{20}+A_{22}^{-1}A_{21}x_{10}+A_{22}^{-1}B_2\hat{u}(0))+0(\lambda) \tag{1.14}$$

where $\hat{u}(0)=\hat{u}(t=0,\lambda)$ obviously depends on x_{10}, x_{20}. Hence

$$\Delta\dot{x}_1=\mathfrak{A}\Delta x_1(t,\lambda)-A_{12}A_{22}^{-2}A_{21}\dot{\bar{x}}_1-A_{12}A_{22}^{-1}B_2R^{-1}(B_1^*\Delta\psi_1+B_2^*\Delta\psi_2)+$$

$$+B_1R^{-1}(B_1^*\Delta\psi_1+B_2^*\Delta\psi_2)+\frac{1}{\lambda}A_{12}\exp\left(A_{22}\frac{t}{\lambda}\right)(x_{20}+A_{22}^{-1}A_{21}x_{10}+$$

$$+A_{22}^{-1}B_2\hat{u}(0))+0(\lambda), \tag{1.15}$$

$$\Delta\dot{\psi}_1(t,\lambda)=-\mathfrak{A}^*\Delta\psi_1(t,\lambda)+Q\Delta x_1(t,\lambda)-A_{21}^*A_{22}^{*-2}\dot{\bar{\psi}}_1(t)+$$

$$+\frac{1}{\lambda}A_{21}^*\exp\left(+A_{22}^*\frac{t_1-t}{\lambda}\right)A_{22}^{*-1}A_{12}^*F\bar{x}_1(t_1)+0(\lambda) \tag{1.16}$$

with the boundary conditions

$$\Delta x_1(0,\lambda)=0,\quad \Delta\psi_1(t_1,\lambda)=-F\Delta x_1(t_1,\lambda).$$

We can easily prove that

$$\lim_{\lambda\to 0+}\frac{1}{\lambda}\int_{t_1}^t \exp(-\mathfrak{A}^*(t-\tau))A_{21}^*\exp\left(A_{22}^*\frac{t_1-t}{\lambda}\right)d\tau=$$

$$=\exp(-\mathfrak{A}^*(t-t_1))A_{21}^*A_{22}^{*-1}.$$

Because of (1.6) and because of the continuouity of the solution of this two-point boundary value.problem, we obtain

$$\lim_{\lambda \to 0^+} \Delta x_1(t, \lambda) = \xi_1(t), \lim_{\lambda \to 0^+} \Delta \psi_1(t, \lambda) = \bar{\eta}_1(t)$$

which completes the proof of theorem 4.2.

APPENDIX 2. CONTROLABILITY IN THE OPTIMAL TRAJECTORY TRACKING STRUCTURE

By definition, the singular perturbed process (1a, b) is absolute controlable along the trajectory $\hat{x}_1$ of there exists a control u^2 such that for every $\lambda \in [0, \lambda_1]$ the corresponding real state fulfilles the state equation for the model. Therefore, we choose the control u^2 which satisfies the equations

$$\dot{\hat{x}}_1 = A_{11} \hat{x}_1 + A_{12} x_2 + B_1 u^2, \quad \hat{x}_1(0) = x_{10}, \tag{2.1a}$$

$$\lambda \dot{x}_2 = A_{21} \hat{x}_1 + A_{22} x_2 + B_2 u^2, \quad x_2(0) = x_{20}, \tag{2.1b}$$

for $\lambda \in [0, \lambda_1]$.

Let there exists B_1^{-1}. After some transformations we obtain the following integral equation for the control u^2:

$$\lambda u^2(t) = |\lambda f(t) - \lambda B_1^{-1} A_{12} \exp\left(A_{22} \frac{t}{\lambda}\right) x_{20} +$$

$$-B_1^{-1} A_{12} \int_0^t \exp\left(A_{22} \frac{t-\tau}{\lambda}\right) \hat{x}_1(\tau)\, d\tau - B_1 A_{12} \int_0^t \exp\left(A_{22} \frac{t-\tau}{\lambda}\right) u^2(\tau)\, d\tau, \tag{2.2}$$

where

$$f(t) = -B_1^{-1} (\varphi K - A_{12} A_{22}^{-1} A_{21}) \hat{x}_1(t) \tag{2.3}$$

It can be shown that the solution $u^2(t, \lambda)$ of this equation exists, and if A_{22} is negative definite $u^2(t, \lambda)$ tends continuously to the optimal control $u(t)$, when $\lambda \to 0^+$. In order to prove it, the Gronwall inequality and the Green formula can be applied.

The differentiability of $u^2(t, \lambda)$ (the structural control sensitivity function) with respect to λ is analysed similar to the proof of theorem 3.2. Additional assumptions for the initial conditions x_{10}, x_{20} will be necessary. Nevertheless, the sensitivity analysis can be performed without conceptual difficulties.

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Analiza wrażliwości nieskończenie wymiarowych liniowych układów sterowania optymalnego przy zmianach rzędu układu

Przedmiotem artykułu jest rzeczywista analiza wrażliwości w różnych strukturach pewnych zakłócanych układów sterowania optymalnego. Podano podstawowe sformułowanie problemu wrażliwości. Omówiono własności rozwiązań liniowych równań różniczkowych w przestrzeni Banacha z pochodną przy małym parametrze. Otrzymane wyniki wykorzystano w analizie λ-wrażliwości. Omawiana metoda polega na aproksymacji miary wrażliwości jej pierwszą i drugą pochodną. Przykład jest ilustracją zastosowania tej metody.

Анализ чувствительности бесконечномерных линейных систем оптимального управления при изменениях величины порядка системы

В статье рассматривается вопрос реального анализа чувствительности при разных структурах некоторых систем оптимального управления, при воздействии помех. Дана основная формулировка проблемы чувствительности. Рассмотрены свойства решений линейных дифференциальных уравнений в банаховом пространстве с производной при малом параметре. Полученные результаты используются при анализе — чувствительности. Рассмотренный метод состоит в аппроксимации меры чувствительности посредством ее первой и второй производной. Дан пример, иллюстрирующий применение этого метода.

Control
and Cybernetics
VOL. 3 (1974) No. 3/4

Application of numerical methods for multipole description of the electrical field of heart

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The paper deals with the studies on multipole components of the cardioelectric field using the multi-electrode network leads constructed following the principle of platonian polyhedra. Numerical methods were applied to develop the potential of electric field into a multipole series. Recent studies reveal that the cardioelectric field has complex structure and attempts to describe it using the dipole approximation in an oversimplification.

1. Introduction

For simplicity sake, electrocardiography assumes the electric field of the heart to have dipole nature [2].

Multipole [3] and multidipole [1] assays reveal complex structure of the electric field of the heart [13, 14].

Our studies on multipole components of the electric field of the heart using the multi-electrode network leads constructed following the principle of platonian polyhedra [10] reveal that the cardioelectric field has a complex structure and attempts to describe it using dipole approximation is an oversimplification [5, 8].

In our approach to multipole description of the electric field of the heart we are making use of the definitions accepted in the physical theory of the multipole fields [9, 15].

This theory enables description of any system of electrical charges of the heart [9].

Studies on multipole components of the electric field of the heart were carried out qualitatively and quantitatively. The simple selection rules resulting from the theory of representations of groups [4, 11] were used for qualitative studies [6, 7]. Numerical methods were applied for quantitative studies.

2. Multipole description of the electric field

Let $\sigma(x')$ denote charge density located inside of a sphere S with radius R'; $x=(x_1', x_2', x_3')$ — coordinates of a point located inside of S; r, ϑ, φ — spherical coordinates of a point $x=(x_1, x_2, x_3)$; $Y_{lm}(\vartheta, \varphi)$ — spherical functions, $l=0, 1, 2, \ldots$ $\ldots, m=-l, -(l-1), -(l-2), \ldots, -2, -1, 0, 1, 2, \ldots, l-1, l$; $q_{lm}=\int \bar{Y}_{lm}(\vartheta', \varphi')\sigma(x')$ $r'^l\, d^3 x'$ — m-th component of 2^l-pole moment

$$Y_{l,-m}(\vartheta, \varphi)=(-1)^m \bar{Y}_{lm}(\vartheta, \varphi). \tag{1}$$

Then the potential $\Phi(x)$ at a point $x=(x_1, x_2, x_3)$, $(|x|>R')$ may be written in the form [9]

$$\Phi(x)=\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{4}{2l+1}q_{lm}\frac{Y_{lm}(\vartheta, \varphi)}{r^{l+1}}=\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\Phi_{lm}(x). \tag{2}$$

According to the usual terminology, the terms

$$\Phi_{lm}(x)=\frac{4}{2l+1}q_{lm}\frac{Y_{lm}(\vartheta, \varphi)}{r^{l+1}}, \quad l=0, 1, 2, \ldots, \tag{3}$$

represent the 2^l-pole contribution.

In particular, for $l=0, 1, 2$ we obtain the following contributions:

(i) $l=0$, $2^0=1$ — pole (monopole) moment

$$\Phi_{00}(x)=4\pi\, q_{00}\frac{1}{r}Y_{00}(\vartheta, \varphi)=2\sqrt{\pi}\, q_{00}\frac{1}{r}. \tag{4}$$

(ii) $l=1$, $2^1=2$ — pole (dipole) moment

$$\Phi_{1m}(x)=\frac{4\pi}{3}q_{1m}\frac{1}{r^2}Y_{1m}(\vartheta, \varphi). \tag{5}$$

(iii) $l=2$, $2^2=4$ — pole (quadrupole) moment

$$\Phi_{2m}(x)=\frac{4\pi}{5}q_{2m}\frac{1}{r^3}Y_{2m}(\vartheta, \varphi). \tag{6}$$

It is easy to check that if we restrict ourselves to the simplest systems of electric charges treated traditionally as monopole, dipole, quadrupole etc., then the respective potential coincide with the above defined multipole potentials of the degres $l=0, 1, 2, \ldots$, etc., respectively [9].

3. Approximation of electric field potential function Program POTENTIAL (see Appendix)

Application

The program POTENTIAL is designed to approximate the $F=F(\rho, \vartheta, \varphi)$ function (ρ, ϑ, φ being the spherical coordinates of a point) with given values $F_k= =F(P_k)$ upon a finite discrete set of points:

$$P_k=(r, \vartheta_k, \varphi_k), \quad k=1, 2, \ldots, n, \tag{7}$$

lying upon a sphere having the radix r and the centre in the coordinate system zero point, by means of a function

$$G_{st}(\rho, \vartheta, \varphi)=\sum_{l=s}^{t} \rho^{-l-1} w_l \sum_{m=-l}^{l} q_{lm} Y_{lm}(\vartheta, \varphi), \rho \geqslant r, \tag{8}$$

where

$$w_l=\frac{4\pi}{2l+1}$$

$$Y_{lm}(\vartheta, \varphi)=\left(\frac{2l+1}{4\pi} \frac{(l-m)!}{(l+m)!}\right)^{1/2} p_l^m(\cos \vartheta) e^{-im\varphi},$$

$p_l^m(x)$ is associated legendre function of the first Kind, while the coefficients q_{lm}, $l=s, s+1, \ldots, t$; $m=0, 1, \ldots, l$; $q_{l,-m}=(-1)^{m-} \overline{q_{lm}}$; are determined from a condition the expression

$$R(\{q_{lm}\}) \overset{\mathrm{df}}{=} \sum_{k=1}^{n} [F_k - G_{st}(P_k)]^2 \tag{9}$$

to achieve the least value.

The program has been written in the language ALGOL 1204 for ODRA 1204 computer.

Method used

Let U_{lm} and V_{lm} denote real and omaginary part of q_{lm} ($q_{l0}=U_{l0}$ is real).

As (1) then the formula (8) may transformed in the following manner

$$G_{st}(\rho, \vartheta, \varphi)=\sum_{l=s}^{t} w_l \rho^{-l-1} H_l(\vartheta, \varphi),$$

where

$$H_l(\vartheta, \varphi)=U_{l0} Q_{l0}(\cos \vartheta)+2 \sum_{m=1}^{l} (U_{lm} \cos m\varphi - V_{lm} \sin m\varphi) Q_{lm}(\cos \vartheta),$$

$$Q_{lm}(x)=\left(\frac{2l+1}{4\pi} \frac{(l-m)!}{(l+m)!}\right)^{1/2} p_l^m(x).$$

The necessary condition of obtaining minimum value by the function (9) leads to the set $N \overset{\mathrm{df}}{=} (t+1)^2 - s^2$ of linear equations with N unknowns:

$$U_{lm}, l=s, s+1, \ldots, t; m=0, 1, \ldots, l;$$

$$V_{lm}, l=s, s+1, \ldots, t; m=1, 2, \ldots, l.$$

This set is solved by elimination method with partial pivoting as described in many manuals of numerical methods.

Data

n — number of points (7)
r — radius of a sphere

ϑ_k, φ_k — spherical coordinates of the points (7)
s, t — numbers appearing in the formula (8)
D — number of the first cell of the drum area allocated for the use during the program run
S — string, comment dealing with further following data
T — number of detailed data (see note below)
F_k — values of function F in the points (7).

Data should be perforated on the tape in the following order:

$$\left.\begin{array}{l} n \quad r \\ \vartheta_1, \varphi_1, \vartheta_2, \varphi_2 \ldots \vartheta_n, \varphi_n \\ s \quad t \\ D \\ S \end{array}\right\} \tag{10}$$

$$T\, F_1\, F_2 \ldots F_n \tag{11}$$

999

Note. In the practice it frequently occurs that the problem of approximation is solved for an established network of nodes (7) and for many systems $\{F_k\}$. The data tape should then contain data (10) and data (11) pertaining to the first system $\{F_k\}$, second, etc. The number 999, as an accesory datum is a conventional end of data sentinel.

Results

q_{lm} — coefficients appearing in (8). The results are tabulated in $t-s+1$ lines. Moreover, a table with a heading

k pot mes pot calc,

is printed which combines the values F_k and $G_{st}(P_k)$. We gixe also the mean square error $M=R(\{q_{lm}\})^{1/2}$.

Run time

The program run time depends mainly on s, t and n. In the test runs the following times have been obtained:

n	s	t	Run time in seconds	
			for the first set F_k	for the second and next in-turn sets
30	2	2	8	2
	0	2	18	3
	0	3	61	5
	0	4	180	7
	2	4	152	6
60	2	2	14	2
	0	2	30	3
	0	3	105	5
	0	4	317	8
	2	4	276	7

Correctness check

The program was checked among others for $n=30$ and $s=t=2$. The results were obtained for which relative mean square error equalled to about 3%.

4. Conclusions

The use of numerical methods enables to develop practically the potential of cardioelectric field into a multipole series. This made it possible to determine experimentally the dominating role of miltipols components of cardioelectric field, in particular of those of rank six [16]. The multipole description gives access to new information on the heart being not revealed by dipole interpretation of electrocardiograms.

At the current status of development of the quantitative studies on the multipole components of the electric field of the heart the conclusions are of approximate character.

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APPENDIX

Program POTENTIAL for multipole description of the electrical field of the heart

```
begin
  comment Program POTENTIAL;
  integer ab1,ab2,al,al1,al2,al3,be,drumpl,drumpl1,i,i1,j,j1,k,l,l1,l2,l3,l1m,l2m,m,n,p,p1,p2,p4,t,t1,t2;
  real cabk, ctk, c0,c1,c2,clmk,fk,gj,gl,Qik,Qlm,r,s,sabk,slmk,s0,s1,s2,s3,Vk;
  read(n,r);
  begin
    array ct,f,V[1:n];
    real procedure Q(l,m,t);
      value t;
      integer l,m;
      real t;
      begin
        integer lm;
        real st,tt,tt1,c;
        real procedure P(l,m);
          integer l,m;
          if if abs(t)=1.0 then m>0 else false
            then P:=.0
            else
            begin
              integer i,lm;
              switch wl:=l0,l1,l2,l3;
              go to wl[if l≤2 then l+1 else 4];
l0:           P:=1.0;
              go to endP;
l1:           P:=if m=0 then t else —st;
              go to endP;
l2:           P:=if m=0 then .5×(3×tt—1) else if m=1 then —3×t×st else 3×tt1;
              go to endP;
l3:           P:=if m=0∨m=1 then ((l+l—1)×t×P(l—1,m)—(l+m—1)×P(l—2,m))/(l—m)
                else —2×(m—1)×t/st×P(l,m—1)—(l—m+2)×(l+m—1)×P(l,m—2);
endP: end p;
        tt:=t×t;
        tt1:=1.0—tt;
        st:=sqrt(tt1);
        c:=1.0;
        lm:=l+m;
        for i:=l—m+1 step 1 until lm do
          c:=c×i;
        Q:=sqrt(.0795774715×(l+l+1)/c)×P(l,m)
```

```
  end Q;
for k:=1 step 1 until n do
  begin
  read(s,ctk);
    ct[k]:=cos(ctk);
    f[k]:=s
  end k;
read(t,p);
t1:=t—1;
t2:=t×t;
p1:=p+1;
p2:=.5×p1×(p+2)—t2;
p4:=p1×p1—t2;
p1:=p1—.5×(t2+t);
format('12⊔⊔');
line(4);
if t=p
  then print('l⊔=',p)
  else print('lmin⊔=',t,'lmax=',p);
begin
  integer array sub[1:p4];
  array a[1:p4,1:p4],cQ,c1Q,sQ,s1Q[1:n],g[t:p],Q0[t:p,1:n],rh,w[1:p4];
      drumpl:=drumplace:=ininteger;
      for j:=1 step 1 until p4 do
        sub[j]:=j;
      for l:=t step 1 until p do
        begin
          g[l]:=12.5663706143/(r↑(l+1)×(l+l+1));
          for m:=1 step 1 until l do
            begin
              for k:=1 step 1 until n do
                begin
                  fk:=m×f[k];
                  Qlm:=Q(l,m,ct[k]);
                  sQ[k]:=sin(fk)×Qlm;
                  cQ[k]:=cos(fk)×Qlm
                end k;
              todrum(n,sQ[1])
              todrum(n,cQ[1]);
            end m;
          for k:=1 step 1 until n do
            Q0[l,k]:=Q(l,0,ct[k])
        end l;
      for i1:=t step 1 until p do
        begin
          i:=i1—t1;
          for j:=t step 1 until p do
            begin
```

```
                s:=0;
                for k:=1 step 1 until n do
                  s:=s+Q0[i1,k]×Q0[j,k];
                a[i,j—t1]:=g[j]×s
              end j;
            drumplace:=drumpl;
            for l:=t step 1 until p do
              begin
                l3:=.5×l×(l—1);
                l2:=p2+l3;
                l1:=p1+l3;
                gl:=2×g[l];
                for m:=1 step 1 until l do
                  begin
                    s:=s1:=0;
                    fromdrum(n,sQ[1]);
                    fromdrum(n,cQ[1]);
                    for k:=1 step 1 until n do
                      begin
                        Qik:=Q0[i1,k];
                        s:=s+cQ[k]×Qik;
                        s1:=s1+sQ[k]×Qik
                      end k;
                    a[i,l1+m]:=gl×s;
                    a[i,l2+m]:=—gl×s1
                  end m
              end l
          end i;
        drumplace:=drumpl;
        for al:=t step 1 until p do
          begin
            al3:=.5×al×(al—1);
            al1:=p1+al3;
            al2:=p2+al3;
            for be:=1 step 1 until al do
              begin
                ab1:=al1+be;
                ab2:=al2+be;
                fromdrum(n,sQ[1]);
                fromdrum(n,cQ[1]);
                drumpl1:=drumplace;
                for j1:=t step 1 until p do
                  begin
                    j:=j1—t1;
                    s:=s1:=0;
                    for k:=1 step 1 until n do
                      begin
                        Qik:=Q0[j1,k];
                        s:=s+cQ[k]×Qik;
```

```
                s1:=s1+sQ[k]×Qik
              end k;
            gj:=g[j1];
            a[ab1,j]:=gj×s;
            a[ab2,j]:=gj×s1
          end j;
        drumplace:=drumpl;
        for l:=t step 1 until p do
          begin
            l3:=.5×l×(l—1);
            l1:=p1+l3;
            l2:=p2+l3;
            gl:=2×g[l];
            for m:=1 step 1 until l do
              begin
                s:=s1:=s2:=s3:=0;
                fromdrum(n,s1Q[1]);
                fromdrum(n,c1Q[1]);
                for k:=1 step 1 until n do
                  begin
                    clmk:=c1Q[k];
                    slmk:=s1Q[k];
                    cabk:=cQ[k];
                    sabk:=sQ[k];
                    s:=s+clmk×cabk;
                    s1:=s1+slmk×cabk;
                    s2:=s2+clmk×sabk;
                    s3:=s3+slmk×sabk
                  end k;
                l1m:=l1+m;
                l2m:=l2+m;
                a[ab1,l1m]:=gl×s;
                a[ab1,l2m]:=—gl×s1;
                a[ab2,l1m]:=gl×s2;
                a[ab2,l2m]:=—gl×s3
              end m
          end l;
        drumplace:=drumpl1
      end be
    end al;
for i:=1 step 1 until p4 do
  begin
    s:=0;
    for j:=i step 1 until p4 do
      begin
        s1:=abs(a[i,sub[j]]);
        if s1>s
          then
```

```
                    begin
                      s:=s1;
                      k:=j
                    end s1>gt s
                end j;
              if s=.0
                then
                begin
                  print('?macierz␣osobliwa');
                  go to ENDP
                end s:=.0;
              l1:=sub[k];
              sub[k]:=sub[i];
              sub[i]:=l1;
              s:=a[i,l1];
              i1:=i+1;
              for k:=i1 step 1 until p4 do
                begin
                  l2:=sub[k];
                  s1:=a[i,l2]:=a[i,l2]/s;
                  for j:=i1 step 1 until p4 do
                    a[j,l2]:=a[j,l2]—a[j,l1]×s1
                end k
            end i;
        end first;
        begin
        integer array title[1:100];
          instring(title[1]);
          line(10);
          outstring(title[1])
        end;
NEWDATA:
      i:=ininteger;
      if i=999
        then go to ENDP;
      read(V);
      format('␣TIME␣:␣1234');
      line(10);
      print(i);
      for i1:=t step 1 until p do
        begin
          s:=.0;
          for k:=1 step 1 until n do
        s:=s+Q0[i1,k]×V[k];
          rh[i1—t1]:=s
        end i1;
      drumplace:=drumpl;
      for al:=t step 1 until p do
```

```
        begin
          al3:=.5×al×(al—1);
          al1:=p1+al3;
          al2:=p2+al3;
          for be:=1 step 1 until al do
            begin
              s1:=s2:=.0;
              fromdrum(n,sQ[1]);
              fromdrum(n,cQ[1]);
              for k:=1 step 1 until n do
                begin
                  Vk:=V[k];
                  s1:=s1+Vk×cQ[k];
                  s2:=s2+Vk×sQ[k]
                end k;
              rh[al1+be]:=s1;
              rh[al2+be]:=s2
            end be
        end al;
      for i:=1 step 1 until p4 do
        begin
          l1:=sub[i];
          s:=rh[i]:=rh[i]/a[i,l1];
          for j:=i+1 step 1 until p4 do
            rh[j]:=rh[j]—a[j,l1]×s
        end i;
      for i:=p4 step —1 until 1 do
        begin
          s:=rh[i];
          for j:=i+1 step 1 until p4 do
            begin
              k:=sub[j];
              s:=s—a[i,k]×w[k]
            end j;
          w[sub[i]]:=s
        end i;
      line(4);
      print('
l         m Re q[l,m] Im q[l,m]
      ');
      for l:=t step 1 until p do
        begin
          format('?12⊔⊔12⊔⊔—1234.12345');
          print(l,0,w[l—t+1]);
          format('12⊔⊔—1234.12345⊔⊔—1234.12345');
          llm:=.5×l×(l—1);
          for m:=1 step 1 until l do
            begin
              llm:=llm+1;
```

```
            print('? ',m,w[p1+l1m],w[p2+l1m]);
          end m;
      end l;
    format('?123⊔⊔—123.12⊔⊔—123.12');
    print('


⊔⊔k⊔⊔pot⊔mes⊔⊔⊔pot⊔calc
    ');
    drumplace:=drumpl;
    for k:=1 step 1 until n do
      s1Q[k]:=.0;
        for l:=t step 1 until p do
      begin
        l1m:=.5×l×(l—1);
        for k:=1 step 1 until n do
          c1Q[k]:=.0;
        for m:=1 step 1 until l do
    begin
            l1m:=l1m+1;
            fromdrum(n,sQ[1]);
            fromdrum(n,cQ[1]);
            s1:=w[p1+l1m];
            s2:=w[p2+l1m];
            for k:=1 step 1 until n do
              c1Q[k]:=c1Q[k]+cQ[k]×s1—sQ[k]×s2
          end m;
        gl:=g[l];
        s:=w[l—t+1];
        for k:=1 step 1 until n do
          s1Q[k]:=s1Q[k]+gl×(s×Q0[l,k]+2×c1Q[k])
      end l;
    s:=.0;
    for k:=1 step 1 until n do
      begin
        s1:=s1Q[k];
        Vk:=V[k];
        print(k,Vk,s1);
        s1:=Vk—s1;
        s:=s+s1×s1
      end k;
    format('??mean⊔error⊔=⊔⊔123.12');
    print(sqrt(s/n));
    go to NEWDATA
  end;
ENDP:
  end
end
```

COMMENTS

The program has been formulated generally, and it is possible to widen its application for multipole description of the electric field. This program has been tested on the Odra-1204 computer. The input values are those of potentials measured over the sphere. The application of the above described program to the study on the cardioelectric field became possible due to potential measurements using the resistor network which reduced the measurement to the conditions of a sphere.

Practically, there were input the values measured at the points of the sphere corresponding with the vertices of dodecahedron, icosahedron, icosadodecahedron and 62-hedron being various modifications of the output of network lead system based on the principle of Platonian dual polyhedra [10]. Out of these modifications, the icosadodecahedron output point set can be obtained by means of numerical equivalent of the diamentoid network as proposed by Paszkowski [12].

Zastosowanie metod numerycznych do badania składników multipolowych pola elektrycznego serca

Omówiono badania składników multipolowych pola elektrycznego serca z użyciem wieloelektrodowych odprowadzeń sieciowych skonstruowanych zgodnie z zasadą "figur platońskich". Do rozwinięcia potencjału pola elektrycznego w szereg multipolowy użyto metod numerycznych. Wykonane badania wykazują, że pole elektryczne serca ma strukturę złożoną i próby opisania go za pomocą aproksymacji dipolowej są nadmiernym uproszczeniem.

Применение численных методов для многополюсного описания электрического поля сердца

Статья касается исследований мультипольовых моментов электрического поля сердца при использовании многоэлектродных сетевых отводов, построенных согласно принципу платонового многогранника.

Для разложения потенциала электрического поля в многополюсных ряд используются численные методы. Последние исследования показали, что электрическое поле сердца иммет сложную структуры и попытка описать его с помощью дипольной аппроксимации является чрезмерным упрощением.

Study support by the Medical Section of the Polish Academy of Sciences.