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Approximation methods for optimal control problems *)

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Projection-iteration methods for the approximate solution of convex minimum problems in infinite dimensional spaces, recently developed by R. Kluge, are presented, and applications to the problem of computation of optimal controls in systems described by ordinary or partial differential equations are indicated.

1. Approximation methods for convex minimum problems in Hilbert spaces

Let H be a real Hilbert space with inner product $(\cdot, \cdot)$ and norm $\|\cdot\|$, let C be a non-empty subset of H, and let $f: H \rightarrow R^1$ be a finite real-valued functional defined on the whole space H. We are interested in computing a solution of the following

Problem 1. Find a $u_0 \in C$ such that

$$f(u_0) = \min_{u \in C} f(u).$$

It is well known that under the quite general assumptions

(C_0) C is weakly compact,

(f_0) f is weakly lower semicontinuous,

the existence of at least one solution of Problem 1 can be proved, but more restricting assumptions are needed in order to formulate efficient calculation procedures.

The theorems listed below apply to problems where

(C_1) C is a closed convex set,

(f_1) f is strongly convex,

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and

(f_2) f', the gradient of f, exists.

Here a functional f is said to be strongly convex if there exists a real number $c>0$ such that the inequality

$$f\left(\frac{u_1+u_2}{2}\right) \leqq \frac{1}{2}[f(u_1)+f(u_2)]-c\,\|u_1-u_2\|^2$$

is fulfilled for all $u_1, u_2 \in H$. The gradient of f is said to exist if there is a linear operator $f': H \to H$ with the property

$$\lim_{t\to 0}\left[\frac{f(u+th)-f(u)}{t}-(f'(u), h)\right]=0$$

for all $u, h \in H$.

Under the suppositions (C_1), (f_1) the existence of a unique solution u_0 for Problem 1 is assured (cf. [9]). Supposition (f_2) allows to use gradient methods in H for its calculation. Some related computation methods working in finite dimensional spaces have been developed recently by R. Kluge [1—3] in the more general framework of nonlinear variational inequalities.

Let $\{H_i\}$, $i=1, 2, \ldots$, be a sequence of subspaces of H with the property

$$H_1 \subset H_2 \subset \ldots \subset H,$$

let us use the notations

$$C_i = C \cap H_i, \quad i=1, 2, \ldots,$$

and let us assume $C_1 \neq \emptyset$. A point $u \in H$ is said to be (C_i)-approximable if there exists a sequence $\{w_i\}$, $w_i \in C_i$, $i=1, 2, \ldots$, with the property

$$\lim_{i\to\infty} w_i = u.$$

Then the following theorems are valid.

THEOREM 1. (Kluge [1], Satz 15.1). Let the assumptions (C_1), (f_1), (f_2),

(f_3) f' is Lipschitz continuous with Lipschitz constant L (i.e. $|f'(u_1)-f'(u_2)| \leqq L\,\|u_1-u_2\|$ for all $u_1, u_2 \in H$),

and

$$u_0 \text{ is } (C_i)\text{-approximable} \tag{*}$$

be fulfilled. Then the sequence $\{v_i\}$ generated by the projection-iteration procedure

$$v_i = P(C_i)\,P_i\,(I-af')\,v_{i-1}, \; v_0 \in H,$$

with

$$0 < a < 2c/L^2$$

strongly converges to the unique solution u_0 of Problem 1, and there is an error estimation of the form

$$\|u_0-v_i\| \leqq M_1\,\|u_0-w_i\|^{1/2} + M_2 \sum_{j=1}^{i-1} (k_a)^{i-j}\,\|u_0-w_j\|^{1/4} + (k_a)^i\,\|u_0-v_0\|$$

with $w_i \in C_i$ and

$$k_a = \sqrt{1 - 2ac + a^2 L^2}.$$

Here $P(C_i)$ is the projection operator onto the convex set C_i, it is defined by

$$\|u - P(C_i)\, u\|^2 = \min_{v \in C_i} \|u - v\|^2;$$

$P_i = P(H_i)$ is the projection operator onto the linear subspace H_i. In general, the operators $P(C_i)$, $i = 1, 2, \dots$, are difficult to realize. In the following theorems, projections are to be performed only onto linear sets.

THEOREM 2. (Kluge [1], Satz 15.4). Let the assumptions of Theorem 1 be fulfilled. Let there be given another real functional $h: H \to R^1$ with the properties

(h_1) h is convex,
(h_2) h' exists,
(h_3) h' is Lipschitz continuous with Lipschitz constant $\tilde{L}$, and

$$(C_2) \quad C = \{u \in H \mid h(u) = \min_{v \in H} h(v)\}.$$

Then the sequence $\{v_i\}$ generated by the projection-iteration procedure

$$v_i = (P_i\, W_i)^{n_i}\, v_{i-1}, \quad v_0 \in H,$$

with

$$W_i = I - b_i\, [\varepsilon_i\, af' + (1 - \varepsilon_i)\, h']$$

and

$$0 < \varepsilon_i < 1, \quad \lim_{i \to \infty} \varepsilon_i = 0,$$

$$0 < a < 2c/L^2,$$

$$0 < b_i < 2c_i/L_i^2,$$

$$c_i = \varepsilon_i (1 - k_a),$$

$$L_i = \varepsilon_i (1 + k_a) + (1 - \varepsilon_i)\, \tilde{L},$$

$$k_a = \sqrt{1 - 2ac + a^2 L^2},$$

$$n_i \geq \frac{d}{1 - k_i}, \quad d > 0,$$

$$k_i = \sqrt{1 - 2b_i\, c_i + b_i^2\, L_i^2}$$

strongly converges to the unique solution of Problem 1.

THEOREM 3 (Kluge [2], Korollar 1). Let the assumptions (f_1), f_2), (C_1), and

int C is not empty and (C_i)-approximable (**)

be fulfilled. Let there be given a real functional $g: H \to R^1$ with the properties

(g_1) g is convex,
(g_2) g' exists,

and

(C_3) $C=\{u\in H \mid g(u)\leq 0\}$.

Then the sequence $\{v_i\}$ generated by the projection-iteration procedure

$$v_{i+1}=\begin{cases} P_i\left(v_i-\dfrac{a_i f'(v_i)}{\|P_i f'(v_i)\|}\right) & \text{for } v_i\in C_i, \\ P_i\left(v_i-\dfrac{a_i g'(v_i)}{\|P_i g'(v_i)\|}\right) & \text{for } v_i\notin C_i, \end{cases} \qquad v_1\in H_1,$$

with

$$a_i>0, \quad \lim_{i\to\infty} a_i=0, \quad \sum_{i=1}^{\infty} a_i=\infty$$

strongly converges to the unique solution of Problem 1.

Remarks

1. For the proofs of Theorems 1—3 (see [1, 2]).

2. The procedures given by Theorems 2, 3 are totally linearized: At the i-th step of iteration only the solution of a system of linear algebraic equations of order $\dim H_i$ has to be computed.

3. In Theorem 3, Lipschitz continuity of the gradient operators f', g' is not needed. In Theorems 1, 2, suppositions (f_3), (h_3) can be weakneed, it is sufficient to assume some type of local Lipschitz continuity (cf. Kluge and Bruckner [6]).

4. Some results concerning the approximability conditions (*), (**) are given in [1], [3]. If $\overline{\bigcup_i H_i}=H$ and $C_1\neq\emptyset$ then every point $u\in\operatorname{int} C$ is (C_i)-approximable. If a set C is given by (C_3) then every point u with $g(u)<0$ belongs to int C.

2. Application to optimal control problems

Let H_0, H_1 be two Hilbert spaces, and let T be a positive number. Let

$$F\colon H_1\times H_0\times[0,T]\to R^1,$$

$$G\colon H_1\to R^1$$

be two real functionals and

$$A(t)\colon H_1\to H_1,$$

$$B(t)\colon H_0\to H_1$$

($0\leq t\leq T$) be two families of bounded linear operators. Consider the functional

$$f(u(\cdot))=\int_0^T F(x(t),u(t),t)\,dt+G(x(T)) \tag{1}$$

with side condition

$$\frac{dx(t)}{dt} = A(t)\,x(t) + B(t)\,u(t),\quad 0 \leq t \leq T,\quad x(0) = c \in H_1, \tag{2}$$

where the integral in (1) has to be taken as a Bochner integral. (1) has to be considered as a functional in

$$H = L_2\,(0, T; H_0),$$

where $L_2\,(0, T; H_0)$ is the space of square-integrable abstract functions $u(\cdot)$ defined on the interval $[0, T]$ with values $u(t) \in H_0$. We are concerned with

Problem 2. Let $C \subset H$ be a given set of admissible controls, find $u_0(\cdot) \in C$ and $x_0(\cdot) \in AC\,(0, T; H_1)$ such that (2) holds and

$$f(u_0(\cdot)) = \min_{u(\cdot) \in C} f(u(\cdot)).$$

If we take $H_1 = R^n$, $H_0 = R^r$ then Problem 2 reduces to the well-known problem of computing an optimal control for an object described by a finite system of ordinary differential equations. If we assume H_1 to be l_2 (the space of square-summable sequences of real numbers) we are dealing with objects described by a countable set of ordinary differential equations. And there are interesting problems also in the domain of partial differential equations: If H_1 is taken to be the Sobolev space $W_2^l(\Omega)$, $\Omega \subset R^n$, then, for instance, pseudoparabolic equations of the form

$$\frac{\partial}{\partial t}\left((-1)^l \Delta^l y\right) + \sum_{|\alpha| \leq l} D^\alpha A_\alpha(t, y, \ldots, D^l y, u) = 0$$

arising in problems of viscoelasticity and viscoplasticity may be represented by abstract differential equations of the type (2) (cf. [4], [5]).

We want to point out to what extent the computing procedures described by Theorems 1—3 apply to the solution of Problem 2. First we consider the assumptions concerning the functional f.

Assumption (f_1) is realized if in (1) the functional G is convex and the functional F is strongly convex in its first and second arguments, e.g.

$$G\left(\frac{x_1 + x_2}{2}\right) \leq \frac{1}{2}\,[G(x_1) + G(x_2)],$$

$$F\left(\frac{x_1 + x_2}{2}, \frac{u_1 + u_2}{2}, t\right) \leq \frac{1}{2}\,[F(x_1, u_1, t) + F(x_2, u_2, t)] - c\,(\|x_1 - x_2\| + \|u_1 - u_2\|)^2, \quad c > 0,$$

for all $x_1, x_2 \in H_1$, $u_1, u_2 \in H_0$, $t \in [0, T]$.

Assumptions (f_2), (f_3) concern the gradient of the functional f. Its existence is assured if the gradient of G (denoted by G_x) and the partial gradients of F with respect to its first and second arguments (denoted by F_x, F_u) exist, its value at time t is given by

$$f'(u(\cdot))\,(t) = F_u(x(t), u(t), t) - B^*(t)\,\psi(t), \tag{3}$$

where $B^*(t): H_1 \to H_0$ is the adjoint operator to $B(t)$, $\psi(\cdot)$ is the solution of the abstract linear differential equation

$$\frac{d\psi(t)}{dt} = -A^*(t)\,\psi(t) + F_x(x(t), u(t), t) \tag{4}$$

with boundary value

$$\psi(T) = -G_x(x(T)),$$

$A^*(t)$ is the adjoint operator to $A(t)$, and $x(\cdot)$ is the solution of system (2) corresponding to $u(\cdot)$. The gradient f' is Lipschitz continuous in H if the gradient $G_x(x)$ is Lipschitz continuous in H_1 and if the gradients $F_x(x, u, t)$, $F_u(x, u, t)$ are Lipschitz continuous with respect to x, u in $H_1 \times H_0 \times [0, T]$ (cf. [5]).

The basic assumption concerning the set of admissible control functions is (C_1). If C is given by pointwise restrictions,

$$C = \{u(\cdot) \in H \mid u(t) \in U(t) \text{ a.e. in } [0, T]\},$$

then for (C_1) to be valid it is sufficient to suppose the sets $U(t) \subset H_0$ to be closed and convex for all $t \in [0, T]$.

We will not look for conditions guaranteeing (C_1) in case of other types of restrictions. But we will give some additional remarks concerning the assumptions (C_2), (C_3) in finite dimensional problems. Let

$$H_0 = R^r, \quad H_1 = R^n,$$

and let the set C of admissible controls be given by pointwise restrictions

$$p_i(u(t), t) \leqslant 0 \text{ a.e. in } [0, T], \quad i = 1, \ldots, k_1, \tag{5}$$

and integral restrictions

$$\int_0^T q_i(u(t), t)\, dt \leqslant R_i, \quad i = k_1 + 1, \ldots, k_1 + k_2. \tag{6}$$

How to find the functionals h and g demanded for in Theorems 2, 3.

Let us first consider the quite simple case where only one restriction of type (6) is given ($k_1 = 0$, $k_2 = 1$). If we define a functional g to be given by

$$g(u(\cdot)) = \int_0^T q(u(t), t)\, dt - R,$$

then we have

$$C = \{u(\cdot) \in H \mid g(u(\cdot)) \leqslant 0\},$$

that is condition (C_3) of Theorem 3. Conditions (g_1), (g_2) are fulfilled if the function $q(u, t)$ is convex with respect to u and has continuous partial derivatives with respect to $u_1, \ldots, u_r$ for all $t \in [0, T]$, the gradient g' is given by

$$g'(u(\cdot))(t) = q_u(u(t), t),$$

it is Lipschitz continuous if $q_u(u, t)$ is bounded and Lipschitz continuous with respect to u on $R^r \times [0, T]$ (cf. [1]).

From the functional g, a functional h corresponding to the conditions of Theorem 2 can be derived in the following way: Let $e(r)$, $-\infty<r<+\infty$ be some real function which is identically equal to zero for $r\leqslant 0$ and strictly increasing, bounded and Lipschitz continuous for $r>0$ (in [1] the function

$$e(r)=\begin{cases} 0 & \text{for } r\leqslant 0, \\ \dfrac{(\sqrt{1+r}-1)^2}{1+r} & \text{for } r>0 \end{cases}$$

has been used), let

$$E(r)=\int_0^r e(s)\,ds,$$

and define h as

$$h(u(\cdot))=E(g(u(\cdot))). \tag{7}$$

From (C_3), the relation

$$C=\{u(\cdot)\in H \mid h(u(\cdot))=\min_{v\in H} h(v)\ (=0)\}$$

follows. Conditions (h_1), (h_2) are fulfilled if (g_1), (g_2) are valid, the gradient h' is given by

$$h'(u(\cdot))=e(g(u(\cdot)))\,g'(u(\cdot)), \tag{8}$$

it is Lipschitz continuous if g' is Lipschitz continuous and bounded for all $u(\cdot)\in H$ (cf. [1]).

Let us now consider the case where only one restriction of type (5) is given ($k_1=1$, $k_2=0$). We may transform it into a restriction of type (6) by introducing

$$q(u,t)=E(p(u,t)),$$

and the same procedure as before gives us a functional g which, at the same time, can be used as a functional h since $g\geqslant 0$ by definition,

$$h(u(\cdot))=g(u(\cdot))=\int_0^T E(p(u(t),t))\,dt.$$

Here, conditions (h_1), (h_2) are fulfilled if the function $p(u,t)$ is convex with respect to u and has continuous partial derivatives with respect to $u_1, \ldots, u_r$ for all $t\in[0,T]$, the gradient h' is given by

$$h'(u(\cdot))(t)=e(p(u(t),t))\,p_u(u(t),t),$$

it is Lipschitz continuous if $p_u(u,t)$ is bounded and Lipschitz continuous with respect to u on $R^r\times[0,T]$ (cf. [8]).

Considering the general case where k_1 restrictions of type (5) and k_2 restrictions of type (6) are given we may form the functionals

$$h_i(u(\cdot))=\begin{cases} \displaystyle\int_0^T E(p_i(u(t),t))\,dt, & i=1,\ldots,k_1, \\ E\left(\displaystyle\int_0^T q_i(u(t),t)\,dt-R_i\right), & i=k_1+1,\ldots,k_1+k_2, \end{cases}$$

and sum them up to get

$$h(u(\cdot))=\sum_{i=1}^{k_1+k_2} h_i(u(\cdot)).$$

Then again C can be characterized as

$$C=\{u(\cdot)\in H|h(u(\cdot))=\min_{v\in H} h(v)\},$$

the conditions (h_1), (h_2), (h_3) are fulfilled, and Theorem 2 can be applied. To apply Theorem 3 we may take $g(u(\cdot))=h(u(\cdot))$ (since we have $C=\{u(\cdot)\in H| |h(u(\cdot))\leqslant 0)$, but there may arise difficulties in confirming the additional condition int $C\neq\emptyset$ for there are no controls $u(\cdot))$ with $h(u(\cdot))<0$.

Remarks

1. It may be possible to find functionals h, g according to (C_2), (C_3) also for problems with control constraints including the state variables (cf. [8]). Let instead of (5), (6) the restrictions

$$p_i(x(t), u(t), t)\leqslant 0 \text{ a.e. in } [0, T],\ i=1, \ldots, k_1, \tag{9}$$

$$\int_0^T q_i(x(t), u(t), t)\, dt\leqslant R_i,\ i=k_1+1, \ldots, k_1+k_2, \tag{10}$$

be given, where $x(\cdot)$ is the solution of (2) corresponding to $u(\cdot)$. Then a restriction (10) can be represented by means of a functional

$$g(u(\cdot))=\int_0^T q(x(t), u(t), t)\, dt-R,$$

and conditions (g_1), (g_2) have to be assured by assumptions which are to some extent analogous to those concerning the functional f; especially, the gradient g' has to be calculated as

$$g'(u(\cdot))(t)=q_u(x(t), u(t), t)-B^*(t)\,\chi(t),$$

where $\chi(\cdot)$ is the solution of the differential equation

$$\frac{d\chi(t)}{dt}=-A^*(t)\,\chi(t)+q_x(x(t), u(t), t),\quad \chi(T)=0.$$

Again, a functional h is given by (7), (8). A restriction (9) can be represented by the functional

$$h(u(\cdot))=g(u(\cdot))=\int_0^T E(p(x(t), u(t), t)\, dt,$$

its gradient is given by

$$h'(u(\cdot))(t)=e(p(x(t), u(t), t)\, p_u(x(t), u(t), t)-B^*(t)\,\chi(t),$$

where $\chi(\cdot)$ is the solution of the differential equation

$$\frac{d\chi(t)}{dt} = -A^*(t)\,\chi(t) + e\big(p(x(t), u(t), t)\,p_x(x(t), u(t), t), \chi(T)=0$$

(see [8] for details).

2. Theorems 1—3 also apply to a (somewhat narrow) class of control problems with weakly nonlinear side conditions (2) (cf. [7]). Let us consider the finite dimensional case, $H_0 = R^r$, $H_1 = R^n$, and let instead of (2) a nonlinear differential equation

$$\frac{dx(t)}{dt} = \varphi(x(t), u(t), t)$$

($\varphi: R^n \times R^r \times [0, T] \to R^n$) be given. For this case, conditions guaranteeing the assumptions (f_1), (f_2), (f_3) have been given by Poljak [10]. The calculation of the gradient becomes somewhat more complicated, instead of (3), (4) we get the expression

$$f'(u(\cdot))(t) = F_u(x(t), u(t), t) - \varphi_u^*(x(t), u(t), t)\,\psi(t),$$

where $\psi(\cdot)$ is the solution of the differential equation

$$\frac{d\psi(t)}{dt} = -\varphi_x^*(x(t), u(t), t)\,\psi(t) + F_x(x(t), u(t), t),\ \psi(T) = -G_x(x(T)).$$

For calculation of the Lipschitz constant of f', see [7].

3. Problem 2 was formulated as a control problem with free right-hand side of the trajectory $x(\cdot)$. It is not difficult to generalize the computation procedures described above to situations where some terminal manifold $S(T)$ is given by using penalty methods.

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Metody aproksymacyjne problemów sterowania optymalnego

Przedstawiono metody iteracyjne rzutujące przybliżonego rozwiązania problemów wypukło-minimalnych w przestrzeniach nieskończenie wymiarowych, rozwinięte ostatnio przez R. Klugego, oraz wskazano na możliwość zastosowania ich w problemach obliczania sterowania optymalnego w układach opisanych zwyczajnymi lub cząstkowymi równaniami różniczkowymi.

Методы аппроксимации для задач оптимального управления

Представлены итерационно-проекционные методы приближенного решения выпукло-минимальных задач в бесконечномерных пространствах, развитие за последнее время в работах Р. Клюге. Показана возможность применения этих методов для задач вычисления оптимального управления в системах, описываемых дифференциальными уравнениями, обыкновенных и с частными производными.