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On minimum point cutsets of a point weighted communication graph

by

JUHANI NIEMINEN
Finnish Academy
Department of Technical Sciences
Helsinki, Finland

Let G be a point weighted communication graph. Using the Malgrange method for determining a point cutset of a graph as a base, a bivalent programming scheme is constructed for calculating the point outsets of minimum weight in G.

1. Introduction and definitions

A central problem in communication graph analysis is the determination of point cutsets of a graph. There are several ways for trying to find a point cutset of minimum weight in an undirected communication graph, but non of them seems to be appropriate to solving this problem as stated in the monograph of Frank and Frisch ([2], p. 176). In this short paper we shall translate the Malgrange method [4] for determining the point cutsets of a commlnication graph G into finding, maximum complete subgraphs of a derived graph G''. This translation offers immediately a base on constructing a linear bivalent programming scheme for calculating the point cutsets of minimum weight in a point weighted communication graph.

In this paper, a communication graph, briefly graph, will be finite, undirected and connected, and will contain no loops or multiple lines. If G is a graph, its set of points will be denoted by $V(G)$, and its set of lines will be denoted by $X(G)$. We assume that the reader is familiar with the central concepts in graph theory; the terminology of graph theory follows that of Harary in [3].

By a Boolean matrix we shall mean a $n \times h$ matrix $M=[m_{ij}]$ of zeros and ones. A submatrix B of M, in which the elements are ones, is called a complete submatrix of M. A submatrix of M is prime, if it is complete and not contained in any other complete submatrix of M. By a maximum prime submatrix B of a Boolean matrix M we shall mean a $u \times w$ matrix B for which the sum $u+w$ reaches its maximum in M.

A complete subgraph of G is called maximal, if it is not contained in any other, and a complete subgraph of greatest cardinality in G is called a maximum complete subgraph of G.

A point cutset S of a graph G is a minimal set of points whose removal from G separates the graph into two or more non-empty connected components.

2. A translation

Let $A=[a_{ij}]$ be the $p\times p$ adjacency matrix of a given graph G and $\bar{A}=[\bar{a}_{ij}]$ its Boolean complement. By a submatrix B of A, $\bar{A}$ or matrix derived from these we shall always mean a matrix determined by a pair (R_B, C_B) of rows and columns of A (or $\bar{A}$) such that b_{ij} is an element of B only if there is a row r_i and a column c_j of A $(\bar{A})$, $r_i\in R_B$ and $c_j\in C_B$, and b_{ij} is an element of r_i and c_j in A $(\bar{A})$. As the results in [4] show, a prime $u\times w$ submatrix B of $\bar{A}$ determines a point cutset $S(B)$ in G if and only if any two points v_{ri}, v_{cj} represented by row r_i and column c_j of B in $\bar{A}$, are distinct. Furthermore, $S(B)=V(G)-\{v_{r1}, \dots, v_{ru}, v_{c1}, \dots, v_{cw}\}$. Now $v_{ri}=v_{cj}$ only if the corresponding element in the matrix $\bar{A}$ is one; this can be avoided by putting any diagonal element $\bar{a}_{ii}$ of $\bar{A}$ to a zero. The matrix thus obtained from $\bar{A}$ is denoted by $\bar{A}^*=[\bar{a}^*_{ij}]$.

Let G be a given graph and A its adjacency matrix. We associate with $\bar{A}^*$ a graph $G''=(V(G''), X(G''))$ defined as follows: $X(G'')=V_r\cup V_c$, where the points of the sets $V_r=\{v_{r1}, \dots, v_{rp}\}$ and $V_c=\{v_{c1}, \dots, v_{cp}\}$ correspond to the rows and columns of $\bar{A}^*$, respectively. A line $(v_{ri}, v_{cj})\in X(G'')$ only if $\bar{a}^*_{ij}=1$. Further, for any row i of $\bar{A}^*$ the points $v_{cj_1}, v_{cj_2}, \dots, v_{cj_s}$ which correspond to the ones in this row, form a complete subgraph in G''; analagously, if the points $v_{ri_1}, \dots, v_{ri_t}$ correspond to the ones in the row j of $\bar{A}$, they determine a complete subgraph of G''. There are no other lines in G''. The following lemma reduces the determination of point cutsets of G to that of a class of maximal complete subgraphs of G''.

LEMMA 1. Let G be a given graph and $\bar{A}^*$ a Boolean matrix associated with G. For any prime submatrix B of $\bar{A}^*$ there is a maximal complete subgraph $G\dot{B}$ in the graph G'' such that $V(G''_B)\cap V_r\neq\emptyset\neq V(G''_B)\cap V_c$, and conversely. Moreover, any maximum prime submatrix B of $\bar{A}^*$ determines a maximum complete subgraph among the graphs G''_B with $V(G''_B)\cap V_r\neq\emptyset=V(G''_B)\cap V_c$, and conversely.

Proof. Let B be a prime $u\times w$ submatric of $\bar{A}^*$. Then any row $i(B)$ of B in $\bar{A}^*$ contains a one at least with the elements $\bar{a}^*_{i(B)j(B)_1}, \bar{a}^*_{i(B)j(B)_2}, \dots, \bar{a}^*_{i(B)j(B)_w}$, where the indices $j(B)_1, \dots, j(B)_w$ correspond to the columns of B in $\bar{A}^*$. An analogous fact holds for any column $j(B)$ of B in $\bar{A}^*$. From the definition of the graph G'' it follows now that the points $v_{ri(B)_1}, \dots, v_{ri(B)_u}, v_{cj(B)_1}, \dots, v_{cj(B)_w}$ induce a complete subgraph G''_B in G''. If a point, say v_{ro}, can be added to G''_B in G''. If a point, say v_{ro}, can be added to G''_B such that $G''_B\cup\{v_{ro}\}$ would be a complete subgraph of G'', then $\bar{a}^*_{oj(B)_s}=1$ for any value of s, $s=1, \dots, w$. But then the elements $\bar{a}^*_{oj(B)_s}$ can be added to B in $\bar{A}^*$, and B would not be prime, which is a contradiction. The proof is similar for a point v_{co}. Hence, G_B'' is a maximal subgraph of G''. Hence, G_B'' is a maximal subgraph of G''. Trivially, $V(G_B'') \cap V_r \neq \emptyset = V(G_B'') \cap V_c$.

The validity of the remaining assertions is obvious according to the proof above, and hence we omit the detailed proofs.

The methods for enumerating maximal complete subgraphs of a graph [1], can now be applied to generating point cutsets of a given graph G. The main difficulties are in selecting away those subgraphs G^o for which $V(G^o) \cap V_r = \emptyset$ or $V(G^o) \cap V_c = \emptyset$. In addition, if S divides G into k connected, non-empty components, there are $2k$ disjoint maximal complete subgraphs G_B'' which determine S.

3. A bivalent programming scheme

If G is a graph and $\bar{A}^*$ a matrix associated with G, we denote by $\bar{G}^*$ the graph with $\bar{A}^*$ as its adjacency matrix. As well known, the concept of maximal complete subgraph of a graph G coincides with that of maximal independent set of $\bar{G}^*$. As noted e.g. in [5], a maximal independent set I of G can be found by a linear process in which the complement $\bar{I}$, the minimal point cover of G, is determined. Thus the reduction of the determination of a point cutset to that of a maximal independent set of $\overline{G''^*}$ allows us to construct a linear bivalent programming scheme for finding the point cutsets of minimum weight in a point weighted graph G.

Let $K = [k_{ij}]$ be the incidence matrix (point-line matrix) of the graph $\overline{G''^*}$, when the original graph G is point weighted. We shall call a graph G point weighted only if the weight w_i of point v_i in G is real and $w_i > 0$, i.e. there are no points of zero weights in G. A bivalent variable z_i corresponds to the point v_i of $\overline{G''^*}$, $i = 1, \ldots, 2p$. We assume that the points $v_1, \ldots, v_p$ of $\overline{G''^*}$ correspond to the rows of $\bar{A}^*$ and $v_{p+1}, \ldots, v_{2p}$ to the columns. Hence, if in G a point t_s corresponding to the row and column with label s, $s = 1, \ldots, p$, has a weight w_s^o, then v_s and v_{p+s} have in $\overline{G''^*}$ the wegihts w_s and w_{s+p}, respectively, and $w_s^o = w_s = w_{s+p}$.

LEMMA 2. Let $(z_1', \ldots, z_{2p}')$ be a solution of the following bivalent programming scheme with an object function: minimize

$$f(z_1, \ldots, z_{2p}) = w_1 z_1 + w_2 z_2 + \ldots + w_{2p} z_{2p}, \tag{1}$$

and with constraints

$$\sum_{i=1}^{i=2p} k_{ij} z_i \geqslant 1, \; j = 1, \ldots, q \tag{2}$$

and

$$\sum_{s=1}^{s=p} z_s \leqslant p-1, \quad \sum_{s=1}^{s=p} z_{p+s} \leqslant p-1, \tag{3}$$

where q denotes the number of lines in $\overline{G''^*}$. If $(z_1', \ldots, z_{2p}')$ is an absolutely minimizing point of the function (1) with subject to the conditions in (2) and (3), then the points $\{g_1, \ldots, g_m\}$ corresponding to zeros in $(z_1', \ldots, z_{2p}')$ determine a point cutset S_{min} of minimum weight in G, $S_{min}=V(G)-\{g_1, ..., g_m\}$, and any S_{min} is determined by an absolutely minimizing point of the bivalent programming scheme of (1), (2) and (3).

Proof. Let $T=\{v_{1(T)}, ..., v_{n(T)}\}$ be a set of point of minimum weight in a point weighted graph G whose removal separates G into at least two non-empty connected components. As the weight of any point is greater than zero, T must be a point cutset of G.

Let $(z'_1, ..., z'_{2p})$ be an absolutely minimizing point of (1) which satisfies also the conditions in (2) and (3). As shown in [5], since $(z'_1, ..., z'_{2p})$ satisfies the constraints in (2), the points of the ones in $(z'_1, ..., z'_{2p})$ form a point cover of $\overline{G''^*}$, and hence the points of $\overline{G''^*}$ corresponding to the zeros in $(z'_1, ..., z'_{2p})$ determine an independent set of $\overline{G''^*}$. According to (3), this independent set has points both from V_r and from V_c. This independent set is also maximal, as in other cases by chaning a one in $(z'_1, ..., z'_{2p})$ to a zero, we would further have a solution satisfying (3) and (2) with lower value of $f(z_1, ..., z_{2p})$ than before; this is a contradiction. So, $V(G)-\{g_1, ..., g_m\}$ is a point cutset of G according to Lemma 1 and the proof above.

The construction principle of G'' was such that a maximal complete subgraph G^o of G'' cannot have two points such that they correspond to a single point of the original graph G. The same fact holds also for any maximum independent set of $\overline{G''^*}$, i.e. any two zesros of $(z'_1, ..., z'_{2p})$ correspond two distinct points of the original graph G. But then any point of G is represented by a one in $(z'_1, ..., z'_{2p})$, and, in particular, if a point t_d of G belongs to the set $V(G)-\{g_1, ..., g_m\}$, then the variables z_d and z_{p+d} are ones.

Let us denote by indices $d_1, ..., d_n$, $p+d_1, ..., p+d_n$ the variables corresponding to the points in the set $V(G)-\{g_1, ..., g_m\}$. Then $f(z'_1, ..., z'_{2p})=(\sum_{i=1}^{i=p} w_i)+w_{d_1}+$ $+w_{d_2}+...+w_{d_n}$, and hence the minimum of the function $f(z_1, ..., z_{2p})$ is equal to the minimum value of the sum of weights $w_{d_1}, ..., w_{d_n}$, the weights of the points in the cutset $V(G)-\{g_1, ..., g_m\}$.

The remaining assertions are obvious according to Lemma 1 and the proof above.

Clearly the programming scheme can be used for determining the connectivity of a graph G, but for this object there are several rapid labelling algorithms as reported in [2].

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O optymalnych przekrojach grafu ważonego z wagami przypisanymi węzłom

Omówiono ważony graf komunikacyjny G z wagami przypisanymi węzłom. W celu wyznaczenia przekrojów grafu G o minimalnej wadze konstruuje się schemat programowania dwuwartościowego zgodnie z metodą Malgrange'a wyznaczania przekrojów grafu.

Об оптимальных сечениях взвешенного графа с весами приписанными узлам

Рассмотрен взвешенный транспортный граф G с весами приписанными узлам. С целью определения сечений графа G с минимальным весом разрабатывается схема двоичного программирования согласно методу Мальгранжа для определения сечений графа.