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Delayed control action controllable systems in Banach space *)

by

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Control systems defined in Banach space with delayed control action in their dynamics are considered. The operator acting on the state is only assumed to be the infinitesimal generator of a strongly continuous semigroup. Necessary and/or sufficient conditions for approximate controllability only in terms of the operators appearing in the dynamics are sought and established. They generalise both previous results of Banks-Jacobs-Latina obtained for finite dimensional systems as well as previous results of the author obtained for in finite dimensional systems but no delays. The conditions are illustrated throughout by examples of physical interest. Also, a result lf Fattorini, reducing the unbounded to the bounded operator case, is extended to the present case of delays.

1. Introduction

Consider the control system $\mathscr{S}_h(A, B_0, B_1)$

$$\dot{x}(t)=Ax(t)+B_0\, u(t)+B_1\, u(t-h)$$

with time lag in the control action, subject to the following assumptions valid throughout the paper: X (state space) and U (control space) are separable Banach spaces; A is (a closed, linear operator, with domain $D(A)$ dense in X) and the infinitesimal generator of a strongly continuous semigroup of bounded operator $S(t)$, $t \geqslant 0$ [2, 3, 8, 11]; B_0 and $B_1 \in \mathscr{B}(U, X)$, the Banach space of bounded linear operators $U \to X$; $x(\cdot)$ (state) and $u(\cdot)$ (control) are X-valued and U-valued functions, respectively; h is a positive constant.

When $B_1=0$, the approximate controllability problem in finite time of $\mathscr{S}_n(A, B_0, B_1)$ — which we then denote by $\mathscr{S}(A, B_0)$ — has been treated in [5, 14]; see also ([4] and [13]). In [5] necessary and sufficient conditions for (in our terminology) approximate controllability were derived, when X is a Hilbert space,

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A is a self adjoint (or normal with some further properties) and B_0 is of finite dimensional range. The tool used was the ordered representation of a Hilbert space. In [14] a different viewpoint was taken, aiming at extending to arbitrary Banach spaces the classical rank condition when $U=R^m$ and $X=R^n$; the obtained extensions were then illustrated by deriving nice necessary and sufficient conditions, e.g. when X is a Hilbert space and A is normal with compact resolvent, in the form of a sequence of rank conditions, using the set of eigenvectors of A. As far as approximate controllability in finite time is concerned, that the case when A is bounded on X is all what is needed to investigate was proved in [4] Prop. 2.3 (see Sec. 3 below for a precise statement) and exploited in [13], where a general analysis was given; in particular a test was given together with an example involving a first order differential operator. Such a reduction to the bounded operator case was also used in [14] Sec. 4 to derive an alternative proof of theorem 4.3′. Also, the problem of lack of exact controlla bility in finite time for $\mathscr{S}(A, B_0)$ when B_0 is compact is treated in [13] Sec. 3.3 and [15].

On the other hand, when $U=R^m$ and $X=R^n$ a through investigation of three concepts of controllability for $\mathscr{S}_h(A, B_0, B_1)$, all of physical interest, was presented in [1] Sec. 3.

We can now state the content of the present paper. We first generalise, in Section 3 below the finite dimensional theory of [1] Sec. 3 for $\mathscr{S}_h(A, B_0, B_1)$ defined on arbitrary Banach spaces and with the operator A acting on the state generally unbounded. Alternatively, Section 3 below can be viewed as an extension of the infinite dimensional results for $\mathscr{S}(A, B_0)$ in [14] to the system $\mathscr{S}_h(A, B_0, B_1)$, where also a delayed control appears in the dynamics.

In Section 4, Fattorini's result — reducing the approximate controllability of $\mathscr{S}(A, B_0)$ from the unbounded to the bounded operator case — is extended to $\mathscr{S}_h(A, B_0, B_1)$.

Finally, the appendix gives *direct* proofs of implications arising in the previously described reduction.

The results are illustrated throughout by nontrivial examples of physical interest.

2. Preliminaries and definitions

For reasons explained above, we shall besically adopt the same notation and terminology used in [1] Sec. 3 in the case $U=R^m$ and $X=R^n$. Also, for simplicity of notation, only one delayed control term will be considered in the dynamics: however, it will be clear that the procedure works also for several delayed control terms, and the corresponding results will be obvious. So, consistently with [1], we denote by $\mathscr{S}_h(A, B_0, B_1)$ the system

$$\dot{x}(t)=Ax(t)+B_0\, u(t)+B_1\, u(t-h)$$

subject to the assumptions stated in the introduction. The system

$$\dot{x}=Ax(t)+Bw(t)$$

is denoted by $\mathcal{S}(A, B)$, where A is as above and $B\in\mathcal{B}(U, X)$.

Without loss of generality, we take the initial time $t_0=0$. Then, if the control $u(t)$ is a sufficiently smooth U-valued function on $[-h, t_1]$ (e.g. C^1), the solution of the Cauchy problem associated with $\mathcal{S}_h(A, B_0, B_1)$ is $(0\leqslant t\leqslant t_1)$ ([11] p. 486):

$$\begin{aligned} x(t, x_0, u)&=S(t)\,x_0+\int_0^t S(t-s)\,[B_0\,u(s)+B_1\,u(s-h)]\,ds=\\ &=S(t)\,x_0+\int_0^t S(t-s)\,B_0\,u(s)\,ds+\int_{-h}^{t-h} S(t-h-s)\,B_1\,u(s)\,ds \end{aligned}\tag{1}$$

for $x_0\in D(A)$. However, the expression (1) is well defined as soon as $u(t)$ is Bochner integrable on $[-h, t_1]$, and, moreover for all $x_0\in X$. For problems of approximate controllability, the imposition of smoothness on $u(t)$ is not restrictive; however, it is restrictive for general optimal control problems, in which case one can either resort to the existence and uniqueness theory of the differential equation as in [17] or, alternatively, *assume* that the control process is modelled directly by the integral version (1). We shall precisely follow this second route; an admissable control $u(t)$ on $[-h, t_1]$ is, then, following tradition, an L_∞-function with values in U. (One can restrict to C^1 control functions without changing the conditions of approximate controllability below (Sec. 3), since C^1 functions are dense in the class of L_∞-functions).

We write explicitely the special case

$$\dot{x}(t)=Ax(t)+b_0\,u(t)+b_1\,u(t-h)$$

with $b_0, b_1\in X$. Such system will be denoted by $\mathcal{S}_h(A, b_0, b_1)$. Again, restriction to just one nondelayed control term and one delayed control term is only for simplicity of notation.

Definition 1. Given a control constraint set $\Omega\subset U$, the symbol $\mathcal{S}_h^1(A, B_0, B_1)$ denotes the system $\mathcal{S}_h(A, B_0, B_1)$ with constraint $u(t)\in\Omega$, $t\in[-h, t_1]$.

Definition 2. Given a control constraint set $\Omega\subset U$ and an L_∞-function v_0 in $[-h, 0]$ with values on U, the symbol $\mathcal{S}_h^2(A, B_0, B_1)$ denotes the system $\mathcal{S}_h(A, B_0, B_1)$ with constraints:

$$u(t)\in\Omega;\ t\in[0, t_1] \text{ and } u_{t_0}=v_0,\ t_0=0$$

where $u_t(s)\equiv u(t+s)$, $s\in[-h, 0]$.

Definition 3. Given a control constraint set $\Omega\subset U$ and two L_∞-functions v_0 and v_1 on $[-h, 0]$ with values in U, the symbol $\mathcal{S}_h^3(A, B_0, B_1)$ denotes the system $\mathcal{S}_h(A, B_0, B_1)$ with constraints

$$u(t)\in\Omega,\ t\in[0, t_1-h] \text{ and } u_{t_0}=v_0,\ u_{t_1}=v_1,\ t_0=0.$$

Examples of physical problems and situations where systems of the type $\mathscr{S}^i$ occur are given in [1] and so we dispense with further comment.

Definition 4. The system $\mathscr{S}^i_h(A, B_0, B_1)$, $i=1, 2, 3$, is aproximately controllable on $[0, t_1]$ in case, given $x_0 \in X$, the totality of the solution points $x(t_1, x_0, u)$ of (2.1) corresponding to admissible controls $u(t)$ on $[-h, t_1]$, is dense in X.

It is no loss of generality to take $x_0=0$ and this will be done henceforth with no further mention.

The following consequence of the Hahn-Banach theorem will be used throughout the paper.

Proposition 1 ([8] p. 31). Let X be a normed linear space and E an arbitrary set in X. Then $\bar{sp}\{E\}=X$ if and only if the zero functional is the only bounded linear functional that vanishes on E.

3. Approximate controllability of $\mathscr{S}_h(A, B_0, B_1)$

To make the paper self-contained, we first collect recent results — to which we shall refer in the sequel — established in the case when no delay is present in the dynamics. So let $\mathscr{S}(A, B)$ be the non-delayed system defined in Section 2. We now associate to the system $\mathscr{S}(A, B)$ the system $\mathscr{S}(R(\lambda_0, A), B)$

$$\dot{x}=R(\lambda_0, A)\,x+Bu$$

defined on the same spaces X and U; $R(\cdot, A)$ is the (bounded!) resolvent operator of A and λ_0 is an arbitrary fixed point in $\rho_0(A)$, the connected component of the resolvent set $\rho(A)$ of A, that contains the half-plane $\operatorname{Re}\lambda > w_0=\lim_{t\to\infty}\|S(t)\|/t<\infty$ ([3] p. 618—619).

In particular cases of physical interest when: (i) either U is finite dimensional, say of dimension m, and hence is (isometrically isomorphic to) R^m; or (ii) B is an operator with finite dimensional range, say of dimension m, $\mathscr{S}(A, B)$ can be written more conveniently as $\mathscr{S}(A, (b_1, \ldots, b_m))$

$$\dot{x}=Ax+\sum_{i=1}^{m} b_i\,u_i$$

with b_i vectors in X and $u=[u_1, \ldots, u_m]$, u_i scalar.

Fattorini showed [4] that if $K_T(\cdot)$ denotes the set of attainability from the origin (with no control constraints $\Omega=U$) of the system $(\cdot)$, the following holds:

$$Cl \bigcup_{0<T<\infty} K_T(L)=Cl \bigcup_{0<T<\infty} K_T(L_{\lambda_0}), \tag{2}$$

(Cl=closure), where $L=\mathscr{S}(A, B)$: $L_{\lambda_0}=\mathscr{S}(R(\lambda_0, A), B)$ Fattorini's results reduces the problem of approximate controllability in finite time from the unbounded to the bounded operator case.

The case when A is bounded. The above reduction was exploited in [13], where it was shown that, when A is bounded on X (and so the semigroup $S(t)$ is a uniformly continuous, analytic group given by

$$S(t)=\exp(At)=\sum_{n=0}^{\infty} A^n t^n/n!, \quad -\infty<t<\infty$$

with convergence in the uniform topology) approximate controllability of $\mathcal{S}(A, B)$ is equivalent to $(\Omega=U)$

$$\overline{sp}\{A^n BU, n=0, 1, \ldots\}=X \tag{3}$$

and hence is independent on the time interval length. In the case of the system $\mathcal{S}(A, (b_1, \ldots, b_m))$, (3) becomes

$$\overline{sp}\{A^n b_i, i=1, \ldots, m; \; n=0, 1, \ldots\}=X. \tag{3'}$$

Hence, when A is bounded, we have

$$Cl \bigcup_{0<T<\infty} K_t=Cl\, K_T, \; T \text{ arbitrary and } 0<T<\infty=\overline{sp}\{A^n BU, n=0, 1, \ldots\}.$$

General case. When the original operator A is only assumed to be an infinitesimal generator, a corollary of (2) combined with (3) reads as follows $\mathcal{S}(A,B)$ (resp. $\mathcal{S}(A, (b_1, \ldots, b_m))$ is approximately controllable in finite time and only if $(\Omega=U)$

$$\overline{sp}\{R^n(\lambda_0, A)\, BU, \; n=0, 1, \ldots\}=X \tag{3''}$$

$$(\text{resp. } \overline{sp}\{R^n(\lambda_0, A)\, b_i, \; i=1, \ldots, m; \; n=0, 1, \ldots\}=X). \tag{3'''}$$

Despite the fact that the characterization (3) fully solves the problem (see example 3.2.7 in [13] and Sec. 4 in [14]) an analysis of possible generalization(s) of (3) for $\mathcal{S}(A, B)$ (resp. (3′) for $\mathcal{S}(A, (b_1, \ldots, b_m))$ *directly* in terms of the original unbounded operator A was explored in [14] and is reported below.

First, let $D_\infty(A)=\bigcap_{n=1}^{\infty} D(A^n)$. $D_\infty(A)$ is still a dense subspace of X ([2] p. 12). Define $U_\infty=\{u \in U: Bu \in D_\infty(A)\}$, i.e. U_∞ is the largest subspace such that $BU_\infty \subset D_\infty(A)$. U_∞ is non empty, but need not be dense in U. In some of the subsequent results, we shall assume that the subspace BU_∞ is dense in the subspace BU (resp. the vectors $b_i \in D_\infty(A)$). Since $D_\infty(A)$ is dense in X, this will always hold, maybe after a slight perturbation of the operator B (resp. the vectors b_i). Then, with $\Omega=U$, we have:

(i) Appropriate versions of (3) and (3′), namely

$$\overline{sp}\{A^n BU_\infty, \; n=0, 1, \ldots\}=X \tag{4}$$

$$\overline{sp}\{A^n b_i, \; i=1, \ldots, m; \; n=0, 1, \ldots\}=X, \; b_i \in D_\infty(A), \tag{4'}$$

are still sufficient for approximate controllability on [0, T] for $\mathcal{S}(A, B)$ and $\mathcal{S}(A, (b_1, \ldots, b_m))$ respectively; however they case to be necessary (see examples in [14]).

(ii) Conversely, if the semigroup $S(t)$ generated by A is analytic, $t>0$, then approximate controllability on $[0, T]$ (which in this case is the same as approximate controllability in finite time[1]) implies, for $\mathscr{S}(A, B)$ and $\mathscr{S}(A, (b_1, ..., b_m))$ respectively:

$$\overline{sp}\{A^n S(\bar{t}) BU, \; n=0, 1, ...\}=X \tag{5}$$

$$\overline{sp}\{A^n S(\bar{t}) b_i; \; i=1, ..., m; \; n=0, 1, ...\}=X, \tag{5'}$$

where $\bar{t}$ is an arbitrary *positive* time.

Another version of the necessary condition for approximate controllability on $[0, T]$ is given, under analyticity, by[2])

$$\overline{sp}\{S(\bar{t}) A^n BU_\infty, \; n=0, 1, ...\}=X, \tag{5''}$$

when the subspace BU_∞ is dense in the subspace BU; and

$$\overline{sp}\{S(\bar{t}) A^n b_i; \; i=1, ..., m; \; n=0, 1, ...\}=X, \tag{5'''}$$

when $b_i \in D_\infty(A)$.

For instance, if A is selfadjoint with compact resolvent, then (4′) and (5‴) are equivalent to each other and also to (5′) when $b_i \in D_\infty(A)$ [14].

We can now state the sought for extension. Define the subspace U^j_∞ for B_j, $j=0, 1$, in the same way as U_∞ was defined for B before.

The following results generalise those in Sec. 3 of [1] from finite infinite dimensional spaces, and also the above conditions (4) and (5) for undelayed infinite dimensional systems to the case when an additional delayed control is added in the dynamics. We first give necessary conditions for the approximate controllability of $\mathscr{S}^i_h(A, B_0, B_1)$, $i=1, 2, 3$, and sufficient conditions only for $i=1, 2$. The case $i=3$ will be treated in more detail at the end of the present section.

THEOREM 1. Let the semigroup $S(t)$ generated by A be analytic, $t>0$. Then:

(i) A necessary condition that $\mathscr{S}^i_h(A, B_0, B_1)$, $i=1, 2, 3$, be approximately controllable on any $[0, t_1]$, $t_1>h$, is that

$$\overline{sp}\{A^n S(\bar{t}) B_0 U, \; A^n S(\bar{t}) B_1 U, \; n=0, 1, ...\}=X, \tag{6}$$

where $\bar{t}$ is an arbitrary *positive* time.

If the subspaces $B_j U^j_\infty$ are dense in the subspaces $B_j U$, an alternate necessary condition is given by

$$\overline{sp}\{S(\bar{t}) A^n B_0 U^0_\infty, \; S(\bar{t}) A^n B_1 U^1_\infty, \; n=0, 1, ...\}=X. \tag{6'}$$

[1]) That is $Cl \bigcup_{0<T<\infty} K_T(L)=X$.

[2]) The difference between the two groups of formulae: (5), (5′) on the one hand and (5″) and (5‴) on the other hand, is that, under assumptions in fact weaker that analyticity, the operator $A^n S(t)$ is *bounded* on X, for $t>0$ ([2] p. 15) and hence can be applied to the whole subspace BU or *any* b_i. For $y \in D_\infty(A)$, we however have $S(t) A^n x = A^n S(t) y$, $t \geqslant 0$ ([2] p. 11).

(ii) Conversely, let $\Omega=U$. A sufficient condition that $\mathcal{S}_h^i(A, B_0, B_1)$, $i=1, 2$, be approximately controllable on any $[0, t_1]$ $t_1>h$, is that

$$\overline{sp}\{A^n B_0 U_\infty^0, A^n B_1 U_\infty^1;\ n=0, 1, \ldots\}=X \tag{7}$$

or, more generally, that, for any $\bar{t}$, $\bar{t}\geqslant 0$:

$$\overline{sp}\{S(\bar{t}) A^n B_0 U_\infty^0, S(\bar{t}) A^n B_1 U_\infty^1;\ n=0, 1, \ldots\}=X. \tag{7'}$$

Proof.

(i) $\mathcal{S}_h^i(A, B_0, B_1)$ approximately controllable implies $\mathcal{S}(A,(B_0, B_1))$ approximately controllable; this, in turn, under the present assumptions, implies (6) and (6'), by applying (5) and (5'') with B replaced by (B_0, B_1) und U replaced by $U\times U$.

(ii) It suffices to give a proof for $\mathcal{S}^2$. Let $y_1=\int_{-h}^{0} S(t_1-s-h) B_1 v_0(s)\, ds$. Suppose, by contradiction, that the totality of points $\{x(t_1, 0, u)-y_1\}$, when u runs over all admissible controls on $[-h, t_1$ is not dense in X. Then, by Proposition 1 there is $0\neq\bar{x}^*\in X^*$ such that $(t_1>h)$

$$\int_0^{t_1} \bar{x}^*\big(S(t_1-s) B_0 u(s)\big)\, ds+\int_0^{t_1-h} \bar{x}^*\big(S(t_1-h-s) B_1 u(s)\big)\, ds=0 \tag{8}$$

for all u admissible.

This easily implies

$$\bar{x}^*\big(S(t_1-s) B_0 U\big)\equiv 0,\ t_1-h\leqslant s\leqslant t_1.$$

Otherwise, in fact, if $\bar{x}^*\big(S(t_1-\bar{t}) B_0 \bar{u}\big)\neq 0$ for some $\bar{t}$ in $[t_1-h, t_1]$ and $\bar{u}\in U$, define a control $u(t)$ on $[0, t_1]$ to be identically zero except near $\bar{t}$ and this leads to a contradiction of (8). The analyticity of $S(\cdot)$ then implies

$$\bar{x}^*\big(S(t_1-s) B_0 U\big)\equiv 0,\ 0\leqslant s\leqslant t_1.$$

In particular

$$\bar{x}^*\big(S(t_1-s) B_0 U_\infty^0\big)\equiv 0,\ 0\leqslant s\leqslant t_1.$$

Plugging (9) into (8), one gets

$$\int_0^{t_1-h} \bar{x}^*\big(S(t_1-h-s) B_1 u(s)\big)\, ds=0$$

for all u admissible and hence, as before

$$\bar{x}^*\big(S(t_1-h-s) B_1 U\big)\equiv 0,\ 0\leqslant s\leqslant t_1-h. \tag{10}$$

In particular

$$\bar{x}^*\big(S(t_1-h-s) B_1 U_\infty^1\big)\equiv 0,\ \ 0\leqslant s\leqslant t_1-h. \tag{10'}$$

Now recall that when $y\in D_\infty(A)$, $S(t)y$ is infinitely many times differentiable and the following holds ([2] p. 11)

$$d^n S(t)\, y/dt^n=S(t) A^n y,\ t\geqslant 0,\ n=0, 1, \ldots$$

Consequently
$$d^n x^* (S(t) y)/dt^n = x^* (S(t) A^n y) \tag{11}$$
for $x^n \in X^*$.

Differentiate successively in a both (9′) and (10′), using (11), and set $s=t_1$ and $s=t_1-h$, respectively, at each stage. One then gets $\bar{x}^* (A^n B_0 U^0_\infty)=0$ and $\bar{x}^* (A^n B_1 U^1_\infty)=0$, $n=0, 1, \ldots$ This, in view of Proposition 1, contradicts (7) since $\bar{x}^*$ is nonzero. The slight modification to prove (7′) is obvious. Q.E.D.

Corollary 1. Let the semigroup $S(t)$ generated by A be analytic, $t>0$. Then:

(i) A necessary condition that $\mathcal{S}^i_h (A, b_0, b_1)$, $i=1, 2, 3$, be approximately controllable on $[0, t_1]$, $t_1>h$, is that
$$\overline{sp}\{A^n S(\bar{t})\, b_j,\ j=0, 1;\ n=0, 1, \ldots\}=X, \tag{12}$$
where $\bar{t}$ is an arbitrary *positive* time.

If $b_j \in D_\infty (A)$, an alternate necessary condition is given by
$$\overline{sp}\{S(\bar{t})\, A^n b_j,\ j=0, 1, \ldots\}=X. \tag{12′}$$

(ii) Conversely, let $\Omega=U$. A sufficient condition that $\mathcal{S}^i_h (A, B_0, B_1)$, $i=1, 2$, be approximately controllable in every $[0, t_1]$ $t_1>h$, is that
$$\overline{sp}\{A^n b_j,\ j=0, 1;\ n=0, 1, \ldots\}=X, \tag{13}$$
or, more generally, that for any $\bar{t}$, $\bar{t}\leqslant 0$:
$$\overline{sp}\{S(\bar{t})\, A^n b_j,\ j=0, 1;\ n=0, 1, \ldots\}=X.$$

If, in particular, A is bounded on X, then $S(t)=\exp(At)$ is automatically analytic for *all* $-\infty<t<\infty$ (group), so in this case (and *only* in this case[3])) we can take $\bar{t}=0$; moreover $D_\infty (A)=X$, $B_j U^j_\infty = B_j U$ in this case. Hence:

Corollary 2. Let A be bounded on X.

(i) A necessary condition that $\mathcal{S}^i_h (A, B_0, B_1)$, $i=1, 2, 3$, be approximately controllable on $[0, t_1]$, $t_1>h$, is that
$$\overline{sp}\{A^n B_j U,\ j=0, 1;\ n=0, 1, \ldots\}=X. \tag{14}$$

(ii) Conversely, let $\Omega=U$; a sufficient condition that $\mathcal{S}^i_h (A, B_0, B_1)$, $i=1, 2$, be approximately controllable on every $[0, t_1]$, $t_1>h$, is that (14) holds

Corollary 3. Let A be bounded on X.

(i) A necessary condition that $\mathcal{S}^i_h (A, b_0, b_1)$, $i=1, 2, 3$, be approximately controllable on $[0, t_1]$, $t_1>h$, is that
$$\overline{sp}\{A^n b_j,\ j=0, 1;\ n=0, 1, \ldots\}=X. \tag{15}$$

(ii) Conversely, let $\Omega=U$; a sufficient condition that $\mathcal{S}^i_h (A, b_0, b_1)$, $i=1, 2$, be approximately controllable on every $[0, t_1]$, $t_1>h$, is that (15) holds.

[3]) Analytic groups generated by unbounded operators cannot exist ([8] p. 278, 477).

REMARK 1. If Ω is a proper subset of U, then Theorem 1 (ii), and its Corroleries 1 (ii), 2 (ii), 3 (ii) are no longer sufficient, even in the finite dimensional case $X=R^n$, $U=R^m$. See examples 7.3 and 7.4 in [1].

REMARK 2. When the semigroup is analytic for $t>0$, the necessary conditions for approximate controllability on $[0, t_1]$, $t_1>h$, delayed system $\mathcal{S}_h^i(A, B_0, B_1)$, $i=1, 2, 3$, as well as the sufficient conditions for $i=1, 2$ and $\Omega=U$ are the same as the corresponding conditions for the system $\mathcal{S}(A, (B_0, B_1))$ with two nondelayed scalar controls; in fact, (4), (5) and (5″) applied to the operator $B=(B_0, B_1)$: $U\times U\to X$, become precisely (7), (6) and (6′) respectively. So, for A bounded on X, the characterization (14) for approximate controllability that of $\mathcal{S}_h^i(A, B_0, B_1)$ for $i=1, 2$, and $\Omega=U$ conincides with that of $\mathcal{S}(A, (B_0, B_1))$ on any $[0, T]$.

REMARK 3. Also, if on neglects the time delay and so sets $h=0$ in the equation defining $\mathcal{S}_h(A, B_0, B_1)$, one gets the system $\mathcal{S}(A, (B_0+B_1))$, whose approximate controllability on $[0, T]$, in the analytic case, implies (resp., if A is bounded, is equivalent to)

$$\overline{sp}\{S(t)\, A^n(B_0+B_1)\, U_\infty^{0,1},\ n=0, 1, \ldots\}=X$$

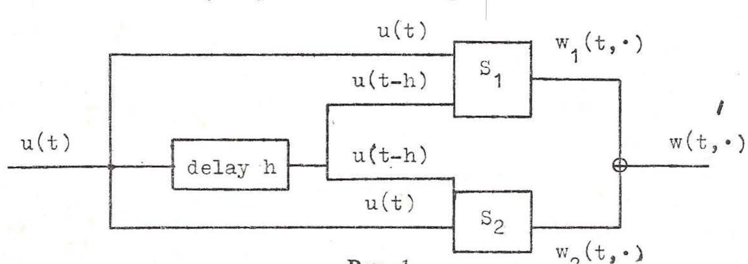
with $U_\infty^{0,1}=U_\infty^0\cap U^1$ (resp. with $U_\infty^{0,1}=U$). This relation, coupled with the obvious one

$$\overline{sp}\{S(t)\, A^n(B_0+B_1)\, U_\infty^{0,1},\ n=0, 1, \ldots\}\subset sp\,\{S(t)\, A^n B_0\, U_\infty^0, A^n B_1\, U_\infty^1,\ n=0, 1\},$$

proves that approximate controllability on $[0, T]$ of $\mathcal{S}(A, (B_0+B_1))$ in the analytic case implies that of $\mathcal{S}_h^i(A, B_0, B_1)$, $i=1, 2$, for $\Omega=U$ and $T>h$. The converse is of course false (e.g. $B_1=-B_0$). Before tackling the controllability problem for the system $\mathcal{S}_h^3(A, B_0, B_1)$, we wish to illustrate the above results concerning the systems $\mathcal{S}_h^1$ and $\mathcal{S}_h^2$ with examples of physical significance.

Examples

Bounded operator case. We start with one example involving a bounded operator on a somewhat unusual Hilbert space, whose choice may appear artificial at first. However, it will be appearent that such a choice provides the appropriate setting for studying a composite system, consisting of subsystems in parallel connection. In our example, two subsystems modelled by integro-differential equations of Volterra type are connected in parallel and driven, each, by a scalar control $u(t)$ and its retardation $u(t-h)$. See block diagram below.



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Before presenting explicitly the dynamical system, we need to introduce a special Hilbert space.

Let $L_2^2[0, 1]$ be the Hilbert space [6] consisting of all (real) 2-dimensional vector functions $f(\xi)=\{f_1(\xi), f_2(\xi)\}$, $0\leqslant\xi\leqslant 1$, with measurable coordinates, such that

$$\|f\|^2=\int_0^1 (|f_1(\xi)|^2+|f_2(\xi)|^2)\, d\xi<\infty.$$

The scalar product $\langle,\rangle$ is defined on $L_2^2[0, 1]$ by

$$\langle f, g\rangle=\int_0^1 (f_1(\xi)\, g_1(\xi)+f_2(\xi)\, g_2(\xi))\, d\xi=(f_1, g_1)+(f_2, g_2)$$

with (,) inner product of $L_2[0, 1]$.

Consider now the composite system S:

$$S_1: \frac{\partial w_1(t, \xi)}{\partial t} \int_0^\xi w_1(t, s)\, ds+b_0^1(\xi)\, u(t)+b_1^1(\xi)\, u(t-h)$$

$$S_2: \frac{\partial w_2(t, \xi)}{\partial t}=-\int_0^\xi w_2(t, s)\, ds+b_0^2(\xi)\, u'\, t)+b_1^2(\xi)\, u(t-h)$$

consisting of the subsystems S_1 and S_2 operating in parallel, that is, driven by the same input (the pair $u(t)$ and its retardation $u(t-h)$) and with output $w(t, \xi)= =w_1(t, \xi)+w_2(t, \xi)$. See above block diagram. Let $b_j^i(\cdot)$ be $L_2[0, 1]$-functions and choose the state space X to be $L_2^2[0, 1]$. Let $x(t)$ be a vector in X, given by $x(t)=\{w_1(t, \cdot), w_2(t, \cdot)\}$.

One then checks that

$$\dot{x}(t)=\left\{\frac{\partial w_1(t, \cdot)}{\partial t}, \frac{\partial w_2(t, \cdot)}{\partial t}\right\}$$

[8] ($\dot{x}$ is the derivative in the norm of X!).

Next, consider the following Volterra operator A [6] defined on $L_2^2[0, 1]$ by

$$(Af)(\xi)=\left\{\int_0^\xi f_1(s)\, ds, -\int_0^\xi f_2(s)\, ds, 0\leqslant\xi\leqslant 1\right.,$$

that we write more concisely as

$$(Af)(\xi)=\{(Vf_1)(\xi), -(Vf_2)(\xi)\}$$

where V is the Volterra operator on $L_2[0, 1]$ defined by

$$(Vg)(\xi)=\int_0^\xi g(s)\, ds.$$

According to the definition of A, we have

$$Ax(t)=\left\{\int_0^\xi w_1(t, s)\, ds, -\int_0^\xi w_2(t, s)\, ds\right\}.$$

Then the system S can be written in the abstract form an $\mathcal{S}_h(A, b_0, b_1)$ on $L_2^2\,[0, 1]$:

$$\dot{x} = Ax(t) + b_0\, u(t) + b_1\, u(t-h)$$

where $b_j = \{b_j^1, b_j^2\}$, $j=0, 1$ and $b_j^i = b_j^i(\cdot) \in L_2\,[0, 1]$.

One easily realises that the definitions of approximate controllability on $[0, T]$ for the system $\mathcal{S}_h^i(A, b_0, b_1)$ above mean that it is possible to drive *simultaneously* the subsystems S_1 and S_2 of S (by means of a common admissible control $u(t)$ and its retardation $u(t-h)$ applied to both of them) from each initial pair of states $w_1(0, \xi)$ for S_1 and $w_2(0, \xi)$ for S_2 to a pair of final states $w_1(T, \xi)$ for S_1 and $w_2(T, \xi)$ for S_2, each of them arbitrarily close (in the $L_2\,[0, 1]$-norm) to a prefixed vector in $L_2\,[0, 1]$.

Claim. If

$$\overline{sp}\,\{A^n f,\ n=0, 1, \ldots\} = L_2^2\,[0, 1],$$

then, necessarity

$$\overline{sp}\,\{V^n f_1, n=0, 1, \ldots\} = \overline{sp}\,\{V^n f_2, n=0, 1, \ldots\} = L_2\,[0, 1].$$

In fact, if say

$$\overline{sp}\,\{V^n f_1, n=0, 1, \ldots\} \subsetneq L_2\,[0, 1]$$

and hence by Proposition 1

$$(y_1, V^n f_1) = 0,\quad n=0, 1, \ldots$$

for some $y_1 \neq 0$ in $L_2\,[0, 1]$, then from

$$A^n f = \{V^n f_1, (-1)^{n+1}\, V^n f_2\},\quad n=0, 1, \ldots$$

it follows that

$$\langle y, A^n f\rangle = (y_1, V^n f_1) + (-1)^{n+1}\,(y_2, V^n f_2) = 0,\quad n=0, 1, \ldots$$

with $y_2 = 0$ and $y = \{y_1, y_2\} \neq 0$. But this contradicts the assumption, again by Proposition 1. Q.E.D.

Consequently:

(i) For the vector $b_0 = \{b_0^1, b_0^2\}$ with

$$b_0^1(\xi) = \begin{bmatrix} 0, & 0 \leqslant \xi < \frac{1}{2} \\ \frac{1}{2}, & \frac{1}{2} \leqslant \xi \leqslant 1 \end{bmatrix} \qquad b_0^2(\xi) = \begin{bmatrix} 1, & 0 \leqslant \xi < \frac{1}{2} \\ \frac{1}{2}, & \frac{1}{2} \leqslant \xi < 1 \end{bmatrix}$$

the following holds

$$\overline{sp}\,\{V^n b_0^1, n=0, 1, \ldots\} \subsetneq L_2\,[0, 1]$$

([10], aldo [9] and [6] and hence, by the above claim,

$$\overline{sp}\,\{A^n b_0, n=0, 1, \ldots\} \subsetneq L_2\,[0, 1]. \tag{16}$$

(ii) Similarly, for the vector $b_1 = \{b_1^1, b_1^2\}$ with

$$b_1^1(\xi) = \begin{bmatrix} 1, & 0 \leqslant \xi < \frac{1}{2} \\ \frac{1}{2}, & \frac{1}{2} \leqslant \xi \leqslant 1 \end{bmatrix} \qquad b_1^2(\xi) = \begin{bmatrix} 0, & 0 \leqslant \xi < \frac{1}{2} \\ \frac{1}{2}, & \frac{1}{2} \leqslant \xi \leqslant 1 \end{bmatrix}$$

we have

$$\overline{sp}\,\{V^n b_1^2,\ n=0,1,\dots\}\subsetneq L_2\,[0,1]$$

([10]; also [9] and [6]) and hence, by the above claim

$$\overline{sp}\,\{A^n b_1,\ n=0,1,\dots\}\subsetneq L_2^2\,[0,1]. \tag{17}$$

(iii) Yet for the sum of the two vectors

$$b_0+b_1=\{b_0^1+b_1^1,\ b_0^2+b_1^2\}$$

for which we have

$$b_0^1(\xi)+b_1^1(\xi)\equiv 1 \text{ and } b_0^2(\xi)+b_1^2(\xi)\equiv 1,\ 0\leqslant\xi\leqslant 1$$

the following holds

$$\overline{sp}\,\{A^n(b_0+b_1),\ n=0,1,\dots\}=L_2^2\,[0,1] \tag{18}$$

([6] p. 351). Since

$$\overline{sp}\,\{A^n(b_0+b_1),\ n=0,1,\dots\}\subset\overline{sp}\,A^n b_1, j=0,1;\ n=0,1,\dots\}$$

we conclude that

$$\overline{sp}\,\{A^n b_j,\ j=0,1;\ n=0,1,\dots\}=L_2^2\,[0,1]. \tag{19}$$

We can now gather our results for the example in question:

(i) According to Corollary 3, (19) says that the above system $\mathscr{S}_h(A,b_0,b_1)$, with one delayed control, when interpreted either as $\mathscr{S}_h^1(A,b_0,b_1)$ or $\mathscr{S}_h^2(A,b_0,b_1)$ is approximately controllable on $[0,T]$, $T>h$, with $b_0(\xi)$ and $b_1(\xi)$ defined as before and $\Omega=U$.

According to Remark 2 and (i) the system $\mathscr{S}(A,(b_0,b_1))$ with two non-delayed scalar controls is also approximately controllable on $[0,T]$, (any $T>0$).

(iii) Finally, of the following three one-scalar control systems: $\mathscr{S}(A,b_0)$, $\mathscr{S}(A,b_1)$ and $\mathscr{S}(A,(b_0+b_1))$, the third is approximately controllable (cf. (18)), while the first two are not, (cf. (16) and (17)).

(iv) Also, (ii) and (iii) together imply that the system $\mathscr{S}(A,(b_0,b_1))$ with two scalar controls is non trivially reduced to the system $\mathscr{S}(A,(b_0+b_1))$ with one scalar control, without loosing its approximate controollability.

Unbounded operator case. To find approximately controllable systems $\mathscr{S}_h(A,b_0,b_1)$ on $[0,t_1]$, $t_1>h$, we again resort to Remark 2 and to our previous results for nondelayed system $\mathscr{S}(A,(b_0,b_1))$ with, say, two scalar (nondelayed) controls ([14] Sec. 4.1).

Let X be a Hilbert space and A be a selfadjoint operator: $X\supset D(A)\to X$ with compact resolvent and spectrum bounded above (so that A is an infinitessimal generator). Then, with $b_j\in D_\infty(A)$, $j=1,2$, the necessary condition (12′) for approximate controllability and the sufficient condition (13) are equivalent ([14], Sec. 4.1). Hence, the delayed system $\mathscr{S}_h^i(A,b_0,b_1)$ is approximately controllable on $[0,t_1]$, $t_1>h$, with $i=1,2$ and $\Omega=U$, if and only if the same holds for the nondelayed system $\mathscr{S}(A, (b_0, b_1))$ in any $[0, t_1]$. This is the case, in turn, if and only if ([14], Sec. 4.1)

$$\operatorname{rank}\begin{bmatrix}(b_0, x_{k1}) & (b_1, x_{k1})\\ \vdots & \\ (b_0, x_{kr_k}) & (b_1, x_{kr_k})\end{bmatrix} = r_k, \quad k=1, 2, \dots$$

where r_k is the multiplicity of the eigenvalue λ_k with associated eigenvectors $x_{k1}, \dots, x_{kr_k}$. Since, in our particular example, the above ranks are all $\leqslant 2$, A must then have eigenvalues with multiplicity not greater than two. This is the case, e.g. for the heat equation on a disk with zero boundary conditions, Sturm-Liouville operator etc. ([14] Sec. 4.5).

We now turn to the system $\mathscr{S}_h^3(A, B_0, B_1)$ whose response at time $t_1 > h$ can be easily checked to be

$$x(t_1, x_0, u) = S(t_1)\, x_0 + \int_0^{t_1} S(t_1 - t)\, B_0\, u(t)\, dt + \int_0^{t_1} + S(t_1 - t)\, B_1\, u(t-h)\, dt =$$

$$= S(t_1)\, x_0 + \int_0^{t_1 - h} S(t_1 - t)\, B_0\, u(t)\, dt + \int_0^{t_1 - h} + S(t_1 - t - h)\, B_1\, u(t) dt + y_1 + y_0,$$

where

$$y_1 = \int_{-h}^{0} S(-s)\, B_0\, v_1(s)\, ds \quad \text{and} \quad y_0 = \int_{-h}^{0} S(t_1 - s - h)\, B_1\, v_0(s)\, ds.$$

Now the totality of response points $\{x(t_1, x_0, u)\}$ are dense in X, when u runs over all $L_\infty - [0, t_1 - h]$ — controls, if and only if the translations $\{x(t_1, x_0, u) - S(t_1)\, x_0 - y_1 - y_0\}$ are dense in X.

By Proposition 1, this happens just in case, with $\Omega = U$

$$\begin{cases} \displaystyle\int_0^{t_1 - h} x^* [S(t_1 - t)\, B_0\, u(t) + S(t_1 - t - h)\, B_1\, u(t)]\, dt = 0 \\ \text{for all } L_\infty - [0, t_1 - h] \text{ — controls and all } x^* \in X^* \\ \Rightarrow x^* = 0 \end{cases} \tag{21}$$

or, equivalently, just in case

$$\begin{cases} x^* (S(t_1 - t)\, B_0\, U + S(t_1 - t - h)\, B_1\, U) \equiv 0, \quad 0 \leqslant t \leqslant t_1 - h \\ \Rightarrow x^* = 0 \end{cases} \tag{22}$$

(21)$\Leftarrow$(22): obvious, arguing e.g. by contradiction with $0 \neq \bar{x}^* \in X^*$;

(21)$\Leftarrow$(22): if, by contradiction,

$$\bar{x}^* (S(t_1 - \bar{t})\, B_0\, \bar{u} + S(t_1 - \bar{t} - h)\, B_1\, \bar{u}) \neq 0$$

for some $0 \neq \bar{x}^* \in X^*$, $\bar{t}$ in $[0, t_1 - h]$ and $\bar{u}$ in U, then define a controller $u(t)$ to be identically zero on $[0, t_1 - h]$ except near $\bar{t}$, to contradict (21). We can now state the following:

THEOREM 2.

(i) The system $\mathcal{S}_h^3(A, B_0, B_1)$ is approximately controllable on $[0, t_1]$, $t_1 > h$, if (with $\Omega = U$) and only if (22) holds.

(ii) If A generates a (strongly continuous) group, then (22) becomes

$$\begin{cases} x^*(S(t_1-t)(B_0\,U + S(-h)\,B_1\,U)) \equiv 0, \quad 0 \leqslant t \leqslant t_1 - h \\ \Rightarrow x^* = 0. \end{cases} \tag{23}$$

(iii) If A is bounded on X, we obtain, for $t_1 > h$:

(a) Let $\mathcal{S}_h^3(A, B_0, B_1)$ be approximately controllable on $[0, t_1]$. Then

$$\overline{sp}\{A^n(B_0\,U + \exp(-Ah)\,B_1\,U),\ n = 0, 1, \ldots\} = X. \tag{24}$$

(b) Conversely, let $\Omega = U$ and let (24) hold. Then $\mathcal{S}_h^3(A, B_0, B_1)$ is approximately controllable on $[0, t_1]$.

Proof. The proof of (i) was given above, and then (ii) follows immediately. We now show (iii).

(a) Suppose (24) fails and so, by Proposition 1

$$\bar{x}^*\big(A^n(B_0\,U + \exp(-Ah)\,B_1\,U)\big) = 0, \quad n = 0, 1, \ldots$$

for some $0 \neq \bar{x}^* \in X^*$. This obviously implies

$$\bar{x}^*\big(\exp At(B_0\,U + \exp(-Ah)\,B_1\,U)\big) \equiv 0, \quad t \geqslant 0$$

which contradicts (ii).

(b) If $\mathcal{S}_h^3(A, B_0, B_1)$ is not approximately controllable on $[0, t_1]$, then, by (ii), we have

$$\bar{x}^*\big(\exp A(t_1 - t)(B_0\,U + \exp(-Ah)\,B_1\,U)\big) \equiv 0, \quad 0 \leqslant t \leqslant t_1 - h$$

for some $0 \neq \bar{x}^* \in X^*$. Actually by analyticity of $\exp At$, the above holds for all $-\infty < t < \infty$. Successive differentiations of the above odentity, setting $t = t_1$ at each stage, yields

$$\bar{x}^*\big(A^n(B_0\,U + \exp(-Ah)\,B_1\,U)\big) = 0, \quad n = 0, 1, \ldots$$

which, by Proposition 1, contradicts. (22). Q.E.D.

The case involving the system $\mathcal{S}_h^3(A, b_0, b_1)$ with scalar controls is singled out in the next.

COROLLARY 4.

(i) The system $\mathcal{S}_h^3(A, b_0, b_1)$ is approximately controllable on $[0, t_1]$, $t_1 > h$, if (with $\Omega = U$) and only if

$$\begin{cases} x^*(S(t_1-t)\,b_0 + S(t_1-t-h)\,b_1) \equiv 0, \quad 0 \leqslant t \leqslant t_1 - h \\ \Rightarrow x^* = 0. \end{cases} \tag{22'}$$

(ii) If A generates a (strongly continuous) group then (22′) becomes

$$\begin{cases} x^*(S(t_1-t)\,(b_0 + S(-h)\,b_1) \equiv 0, 0 \leqslant t \leqslant t_1 - h \\ \Rightarrow x^* = 0. \end{cases} \tag{23'}$$

(iii) If A is bounded on X, we obtain, for $t_1>h$:

(a) Let $\mathcal{S}_h^3(A, b_0, b_1)$ be approximately controllable on $[0, t_1]$. Then

$$\overline{sp}\{A^n(b_0+\exp(-Ah)\,b_1),\ n=0, 1, \ldots\}=X. \tag{24'}$$

(b) Conversely, let $\Omega=U$ and let (24′) hold. Then $\mathcal{S}_h^3(A, b_0, b_1)$ is approximately controllable on $[0, t_1]$.

Remark 4. The dependence on the lag h in the condition (24), (24′) is not illusory, even in the finite dimensional case: $X=R^n$, $U=R^m$ ([1], Remark 3.4.) Also, since

$$\overline{sp}\,\{A^n\,b_0+\exp(-Ah)\,A^n\,b_1\}\subset\overline{sp}\,\{A^n\,b_0, \exp(-Ah)\,A^n\,b_1\}$$

it follows that (24′) implies

$$\overline{sp}\,\{A^n\,b_0, \exp(-Ah)\,A^n\,b_1\}=X,$$

which, in turn, implies (15), since $\exp -Ah$ is a $1-1$ onto operator on X. Conversely, (15) does not imply (24′) even for $U=R^m$ and $X=R^n$ ([1], example 3.2). Hence the condition (24′) of Corollary 4 is stronger than the condition (15) of Corollary 3.

4. Reduction of approximate controllability from the unbounded to the bounded case in presence of delays in the control action

In the following, Fattorini's result — reported in (2) — reducing the study of the approximate controllability for $\mathcal{S}(A, B_0)$ from the unbounded to the bounded operator case, is generalized to the system $\mathcal{S}_h^i(A, B_0, B_1)$, $i=1, 2$, when A generates an analytic semigroup.

The notation in the next theorem will be simplified as follows: set

$$L_h^i\equiv\mathcal{S}_h^i(A, B_0, B_1) \text{ and } L_h^i(\lambda)\equiv\mathcal{S}_h^i(R(\lambda, A), B_0, B_1)$$

for $i=1, 2$. Also recall that $K_t(\cdot)$ denotes the set of attainability from the origin (with $\Omega=U$) of the system $(\cdot)$ at time t; also $\rho_0(\lambda)$ is the connected component of $\rho(A)$ defined at the beginning of Section 3.

Theorem 3. Let the semigroup $S(t)$ generated by A be analytic, $t>0$, and let $\Omega=U$. Then

$$Cl\bigcup_{h<t<\infty}K_t(L_h^i)=Cl\bigcup_{h<t<\infty}K_t(L_h^i(\lambda))=$$

$$=Cl\,K_T(L_h^i(\lambda))=\overline{sp}\,\{R^n(\lambda, A)\,B_j\,U,\ j=0, 1;\ n=0, 1, \ldots\},$$

where $i=1, 2$; λ is an arbitrary point in $\rho_0(A)$ and T is an arbitrary time $>h$.

Hence, in particular, L_h^i is approximately controllable on $[0, T]$, $T>h$, if and only if the same holds for $L_h^i(\lambda)$; and this happens just in case

$$\overline{sp}\,\{R^n(\lambda, A)\,B_j\,U,\ j=1,2;\ n=0,1,\ldots\}=X.$$

Proof. By the Hahn-Banach theorem, we only need to show that all $x^*\in X^*$ annihilating $\bigcup_{h<t<\infty} K_t(L_h^i)$ also annihilate $\bigcup_{h<t<\infty} K_t(L_h^i(\lambda))$, and conversely. Let

$$\bar{x}^*\Big(\bigcup_{h<t<\infty} K_t(L_h^i)\Big)=0 \text{ for } \bar{x}^*\in X^*. \tag{25}$$

Then, as in the sufficient part of the proof of Theorem 1, we have that, because of the analyticity assumption, (9) and (10) hold. Actually, this same assumption allows us to extend the argument to the entire sonnegative real axis and hence to write

$$\bar{x}^*\big(S(t)\,B_0\,U\big)\equiv 0 \text{ and } \bar{x}^*\big(S(t)\,B_1\,U\big)\equiv 0,\ 0\leqslant t. \tag{26}$$

Now the argument as in [4] can be adopted. From

$$\bar{x}^*\big(R^n(\lambda_0, A)\,B_j\,U\big)=\frac{1}{(n-1)!}\int_0^\infty t^{n-1}\exp(-\lambda_0 t)\,\bar{x}^*\big(S(t)\,B_j\,U\big)\,dt \tag{27}$$

with $j=1,2$, $\operatorname{Re}\lambda_0>w_0$, $n=1,2,\ldots$ [3], using (26) we get

$$\bar{x}^*\big(R^n(\lambda_0, A)\,B_j\,U\big)=0,\ n=0,1,\ldots,\ \operatorname{Re}\lambda_0>w_0. \tag{28}$$

(The case $n=0$ stems from (26) for $t=0$.

Using

$$R(\lambda, A = \sum_{n=0}^{\infty}(\lambda_0-\lambda)^n\,R^{n+1}(\lambda_0, A),\ \lambda \text{ close to } \lambda_0 \tag{29}$$

$$\left(\frac{d^n\,R(\lambda, A)}{d\lambda^n}=(-1)^n\,n!\,R^{n+1}(\lambda, A),\ n=0,1,\ldots\right)$$

[12] yields

$$\bar{x}^*\big(R(\lambda, A)\,B_j\,U\big)\equiv 0,\ j=1,2 \tag{30}$$

for all λ close to λ_0, and, by analytic continuation, for all $\lambda\in\rho_0(A)$. It then follows by differentiation in λ that

$$\bar{x}^*\big(R^n(\lambda, A)\,B_j\,U\big)\ 0,\ n=0,1,\ldots,j=1,2$$

for all $\lambda\in\rho(A)$ and hence that

$$\bar{x}^*\big(\exp R(\lambda, A)\,t\big)\,B_j\,U\big)\equiv 0,\ 0\leqslant t<\infty \text{ (also } -\infty<t<\infty) \tag{31}$$

i.e.

$$\bar{x}^*\Big(\bigcup_{h<t<\infty} K_t(L_h^i(\lambda))=0 \tag{32}$$

for all $\lambda\in\rho_0(A)$ and one way is done.

Conversely, let (32) hold for some $\lambda=\lambda_0\in\rho_0(A)$. Then (32) implies (for such $\lambda=\lambda_0$) (31) in as much the same way as (25) implies (26).

Differentiating (31) in t and setting $t=0$ at each stage, yields (28). Using (29) again implies (30) for all $\lambda \in \rho_0(A)$. Hence, from (27), applied for $n=1$ and Re $\lambda > w_0$, we get

$$\int_0^\infty \exp(-\lambda t)\, \bar{x}^*(S(t) B_j U)\, dt \equiv 0$$

for all λ with Re $\lambda > w_0$.

Therefore by uniqueness of the Laplace transform [3] we finally get (26) (for every t, not merely a.e.) and hence (25). Q.E.D.

REMARK 5. The above argument does not carry over for the system $\mathscr{S}_h^3(A, B_0, B_1)$ with A unbounded. In such case, in fact, (25) implies the top line of (22) for $x^* = \bar{x}^*$ (and not (26)!), which we now rewrite as

$$\bar{x}^*\big(S(t) B_0 U + S(t-h) B_1 U\big) \equiv 0, \quad h \leqslant t \leqslant t_1, \tag{33}$$

and, if we assume analyticity for $S(t)$, $t>0$, we can extend the validity of (33) to all $t \geqslant h$. We cannot however extend the validity of (33) for all $t \geqslant 0$, when A is unbounded, since this would imply evaluation of $S(\cdot)$ for negative argument, in fact on $[-h, 0]$. But $S(t)$ cannot be an analytic group, if A is unbounded [8] which is precisely the case of intereset for Theorem 3.

The validity of (33) for all $t \geqslant 0$ would be required to conclude in analogous way as to the step from (26) to (28).

For completeness we mention that the above argument does carry over, when A is bounded on X and so $S(t) = \exp At$ is an analytic group. In this case approximate controllability of $\mathscr{S}_h^3(A, B_0, B_1)$ on $[0, t_1]$, $t_1 > h$, and $\Omega = U$ is equivalent to $(\lambda \in \rho_0(A))$

$$\overline{sp}\,\{R^n(\lambda, A)\big(B_0 + \exp(-Ah) B_1\big) U,\ n=0, 1, \ldots\} = X \tag{34}$$

which, in turn, in view of Theorem 2, is equivalent to (24). The equivalence between (24) and (34) can easily be established directly; for instance in the classical case, $X = R^n$ and $U = R^m$, it can be easily proved within the linear algebra framework. What such equivalence says, in this case, is that the two $n \times (m \cdot n)$ matrices

$$C = [B, AB, \ldots, A^{n-1} B] \text{ and } [B, R(\lambda, A) B, \ldots, R^{n-1}(\lambda, A) B]$$

have the same rank n, where $R(\lambda, A)$ is the $n \times n$ nonsingular matrix $[\lambda I - A]^{-1}$, for λ not an eigenvalue of A. If A itself is nonsingular, this assertion with $\lambda = 0$ follows simply by multiplying the matrix C on the left by the nonsingular matrix $(-A^{-1})^{n-1}$; here one uses the standard fact that multiplication of a matrix on the left (or on the right) by a nonsingular matrix preserves its rank [5]. The case when A is singular is nahdled similarly, using the above, coupled with the observation that the rank of C and the rank of

$$[B\,(\lambda I - A) B, \ldots, (\lambda I - A)^{1-n} B]$$

are the same.

APPENDIX

From the meaning, in terms of controllability properties, of the conditions (3) throught (5″), one plainly concludes:

(i) When A is bounded on X, then (3) (resp. (3′)) and (3″) (resp. 3‴) are equivalent.

(ii) When A is an unbounded infinitesimal generator, however, then:

(a) (4) implies (3″)

(resp. (4′) implies (3‴)];

(b) conversely, in the analytic case:

(3″) implies (5) (as well as (5″) if BU_∞ in dense in BU)

[resp. (3‴) implies (5′) (as well as (5‴) if $b_i \in D_\infty(A)$)].

Notice that if $\bar{t}$ could be taken zero in (5″) or (5‴), then they would reduce to (4) and (4′) respectively: here the point is, however, that $\bar{t}$ can be taken zero, *only* when A is bounded, since an analytic *group* cannot be generated by an unbounded infinitesimal generator ([8] p. 278 also p. 477). We think it is appropriate then to insert *direct* proofs of the implications in (i) and (ii) above, based on functional analysis techniques: this will enable one to realize why the necessary and sufficient condition (3) for A bounded, splits, when A is unbounded, into two conditions: the sufficient condition (4) and, when the semigroup is analytic for $t>0$, the necessary conditions (4) and (5″). The reason will be that the operational calculus for bounded operators fails to have a full counterpart when A is simply closed. For simplicity of notation, we shall consider only the conditions of (a) and (b) in (i) marked with prime (corresponding to a finite number of scalar controls). The case (i) will be a special case.

Proof. (4′)⇒(3‴).

By Proposition 1 we must show that

$$\begin{cases} x^*(A^n b_i)=0, \ i=1, \ldots, m; \ n=0, 1, \ldots, x^* \in X^*, \ b_i \in D_\infty(A) \\ \Rightarrow x^*=0 \end{cases}$$

implies

$$\begin{cases} x^*\big(R^n(\lambda_0, A)\, b_i\big)=0, \ i=1, \ldots, m; \ n=0, 1, \ldots, x^* \in X^* \\ \Rightarrow x^*=0. \end{cases}$$

By contradiction, let

$$\bar{x}^*\big(R^n(\lambda_0, A)\, b_i\big)=0, \ n=0, 1, \ldots$$

for some $0 \neq \bar{x}^* \in X^*$. By analyticity of $R(\lambda, A)$ [12], $R(\lambda, A) \sum_{n=0}^{\infty} (\lambda_0-\lambda)^n R^{n+1}(\lambda_0, A)$ λ close to λ_0, if follows that $\bar{x}^*\big(R(\lambda, A)\, b_i\big)\equiv 0$ for all λ close to λ_0 and, by analytic continuation, for all λ in $\rho_0(A)$. From[4])

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[4]) When A is bounded on X, we can use the operation calculus formula ([12] p. 289) $A^n = (\frac{1}{2}\pi i) \int_\Gamma \lambda^n R(\lambda, A)\, d\lambda$, $n=0, 1, \ldots$, (Γ is the boundary of any bounded Cauchy domain, say a circle enclosing the spectrum of A) to quickly get $\bar{x}^*(A^n b_i)=0$, which is a contradiction. A counterpart of this formula (with convergence in the strong topology) does not hold in full generality when A is just closed. For special cases, e.g. when A is selfadjoint, see [3] p. 1196 Theorem 6.

$$R(\lambda, A)\, b_i = \int_0^\infty \exp(-\lambda t)\, S(t)\, b_i\, dt, \qquad \operatorname{Re} \lambda > w_0 \tag{A.1}$$

([3] p. 627) it follows, by applying $x^* \in X^*$ on both sides and using the uniqueness of the Laplace transform ([3] p. 626) that

$$\bar{x}^* \big(S(t)\, b_i\big) \equiv 0, \quad t \geqslant 0, \quad i = 1, \ldots, m, \tag{A.2}$$

Next observe that, since $b_i \in D_\infty(A)$, the following holds [2]

$$\frac{d^n S(t)\, b_i}{dt^n} = S(t)\, A^n\, b_i, \quad i = 1, \ldots, m; \quad n = 0, 1, \ldots$$

and hence

$$\frac{d^n \bar{x}^* \big(S(t)\, b_i\big)}{dt^n} = \bar{x}^* \big(S(t)\, A^n\, b_i\big), \quad i = 1, \ldots, m, \quad n = 0, 1, \ldots \tag{A.3}$$

Successive differentiations of (A.2), together with (A.3) imply, taking $t=0$ after each step $\big(S(0) = I\big)$

$$\bar{x}^* (A^n\, b_i) = 0, \quad i = 1, \ldots, m; \quad n = 0, 1, 2, \ldots,$$

which is a contradiction, since $\bar{x}^* \neq 0$. Q.E.D.

Conversely, in the analytic case:

$$(3''') \Rightarrow (5') \text{ [and } (5''') \text{ if } b_i \in D_\infty(A)].$$

By Proposition 1 we must show that

$$\begin{cases} x^* \big(R^n(\lambda_0, A)\, b_i = 0, \quad i = 1, \ldots, m, n = 0, 1, \ldots, x^* \in X^* \\ \Rightarrow x^* = 0 \quad \lambda_0 \in \rho_0 \quad A) \end{cases}$$

implies

$$\begin{cases} x^* \big(A^n\, S(\bar{t})\, b_i\big) = 0, \quad i = 1, 2, \ldots, m, n = 0, 1, \ldots, x^* \in X^* \\ \Rightarrow x^* = 0 \quad \bar{t} = \text{arbitrary positive time}, \end{cases}$$

as well

$$\begin{cases} x^* \big(S(\bar{t})\, A^n\, b_i\big) = 0, \quad i = 1, \ldots, m; \quad n = 0, 1, \ldots, x^* \in X^*, \quad b_i \in D_\infty(A) \\ \Rightarrow x^* = 0 \quad \bar{t} = \text{arbitrary positive time}. \end{cases}$$

Let by contradiction $\bar{x}^* \big(S(\bar{t})\, A^n\, b_i\big) = 0$, as well as $\bar{x}^* \big(A^n\, S(\bar{t})\, b_i\big) = 0$, for $0 \neq \bar{x}^* \in X^*$. These, coupled with the analyticity of $S(t)$, $t > 0$, formula (A.3) as well as

$$\frac{d^n \bar{x}^* \big(S(t)\, b_i\big)}{dt^n} = \bar{x}^* \big(A^n\, S(t)\, b_i\big) \quad \textit{any } b_i \in X$$

([2] p. 11) imply $\bar{x}^* \big(S(t)\, b_i\big) = 0$ for all t in a neighbourhood of $\bar{t}$, hence for all $t \geqslant 0$. Consequently, making use of (A.1), one gets $\bar{x}^* \big(R(\lambda, A)\, b_i\big) = 0$ for all λ with $\operatorname{Re} \lambda > w_0$ and, by analytic continuation of $R(\cdot, A)$, for all $\lambda \in \rho_0(A)$. Using

$$\frac{d^n R(\lambda, A)}{d\lambda^n} = (-1)^n\, n!\, R^{n+1}(\lambda, A), \quad n = 0, 1, \ldots$$

([12] p. 257) and hence

$$\frac{d^n \bar{x}^* (R(\lambda, A))}{d\lambda^n} = (-1)^n n! \, \bar{x}^* (R^{n+1}(\lambda, A)),$$

we conclude that: $\bar{x}^* (R^n(\lambda, A) b_i) = 0$, $n = 0, 1, \ldots$, for all $\lambda \in \rho_0(A)$, in particular for $\lambda = \lambda_0$ and this is a contradiction, since $\bar{x}^* \neq 0$. (More quickly, from $\bar{x}^* (S(t) b_i) \equiv 0$, $t \geqslant 0$, use the known formula

$$R^n(\lambda, A) BU = \int_0^\infty t^{n-1} \exp(-\lambda t) \exp At \, BU \, dt \, ((n-1)!)^{-1}$$

([3] p. 623), obtained from the above formulae for $R(\lambda, A)$ and $d^n R(\lambda, A)/d\lambda^n$).

Examples

We finish, by illustrating the above results with non trivial examples.

An example of claim a. Let $X = L_2 [0, 1]$ and let A be the simplest integral operator

$$(Af)(\xi) = \int_0^\xi f(s) \, ds, f(\cdot) \in X.$$

A is a Volterra operator, i.e. compact and whose spectrum is just the origin [6]. Moreover, $\|A\| = 2/\pi < 1$ ([7] p. 300) and so we can take $\lambda_0 = 1$, to verify the above claim *a*. We have ([12] p. 291):

$$[(I-A)^{-1} g](\xi) = g(\xi) + \int_0^\xi \exp(\xi - s) \, g(s) \, ds.$$

If b is a vector of X (written as a function $b(\xi)$), it is not at all obvious from the above definitions of A and $R(1, A) = (1-A)^{-1}$ that

$\overline{sp}\{A^n b, n = 0, 1, \ldots\} = X$ if and only if

$\overline{sp}\{R^n(1, A) b, n = 0, 1, \ldots\} = X$ as claim a dictates.

We now wish to show this fact *directly*.

The above expression for $R(1, A)$ reads $R(1, A) = 1 + V$, where V is the Volterra operator defined by

$$(Vf)(\xi) = \int_0^\xi \exp(\xi - s) \, f(s) \, ds, f(\cdot) \in X.$$

It then follows that

(i) $R^n(1, A)$ is a particular linear combination of $I, V, V^2, \ldots, V^n$; $n = 1, 2, \ldots$; conversely.

(ii) V^n is a particular linear combination of

$$I, R(1, A), R^2(1, A), \ldots, R^n(1, A); \; n = 1, 2, \ldots$$

So, for the vector b in X, (i) and (ii) imply

$$\overline{sp}\{R^n(1, A)\, b, n=0, 1, \ldots\}=X \text{ if and only if}$$

$$\overline{sp}\{V^n b, n=0, 1, \ldots\}=X.$$

In order to verify claim a, it remains to show therefore that

$$\overline{sp}\{A^n b, n=0, 1, \ldots\}=X \text{ if and only if}$$

$$\overline{sp}\{V^n b, n=0, 1, \ldots\}=X.$$

This follows as an immediate consequence of the unicellularity of both A and V ([6] p. 38, [10] for A and [9] for V), since the kernel $k_v(\xi, s)=\exp(\xi-s)$ is C^2 and $k_v(s, s)\neq 0$: actually [9] refers to the adjoint operator V^*:

$$(V^* f)(\xi)=\int_{\xi}^{1} k_v(\xi, s) f(s)\, ds;$$

but the unicellularity of V^* implies that of V, and conversely ([6] p. 36); also, the invariant subspaces, called reducing manifolds in [9], of V are the orthogonal subspaces of the invariant subspaces of V^* ([6] p. 36). Moreover, the only closed invariant subspaces both A and V are the subspaces $L_2[a, 1]$ for all $a\in[0, 1]$ ([6] p. 38, [10], [9]); differently stated, the conditions

$$\overline{sp}\{A^n b, n=0, 1, \ldots\}=X \text{ and}$$

$$\overline{sp}\{V^n b, n=0, 1, \ldots\}=X$$

are both equivalent to the condition: for each $\delta>0$, the set

$$\{\xi: b(\xi)\neq 0 \text{ a.e. on } [0, \delta]$$

has non-zero (Lebesgue) measure.

So we have verified claim a *directly*, in our particular example.

An example of claim b. Let X be a Hilbert space and let A be selfadjoint with compact resolvent. A generates a strongly continuous semigroup if and only if its spectrum is bounded above; also the semigroup in this case is seladjoint and analytic for $t>0$ (more generally, the semigroup of a selfadjoint operator with spectrum bounded above is seladjoint and analytic for $t>0$ ([8] p. 588—589). The following equivalences involving the conditions (4'), (5'''), (3''') were proved, in the present case, in ([14] Sec. 4):

(4')⇔(5''')⇔(3''')⇔rank $C_j=r_j$, $j=1, 2, \ldots$ with $b_i\in D(A)$ and

$$C_j=\begin{vmatrix} (b_1, x_{j1}), & \ldots, & (b_m, x_{j1}) \\ (b_1, x_{j2}), & \ldots, & (b_m, x_{j2}) \\ \vdots & & \vdots \\ (b_1, x_{jr_j}), & \ldots, & (b_m, x_{jr_j}) \end{vmatrix}.$$

Here r_j is the finite multiplicity of an eigenvalue of A with associated eigenvectors $x_{j1}, \ldots, x_{jr_j}$. (Also, the spectrum of A consists *precisely* of these eugenvalues).

The equivalence (4')⇔(3''') appears directly from the proof of Theorem 4.3' based on the explicit expression of the resolvent ([14] Sec. 4.2) and relies on the following formulas

$$A^n b_i = \sum_{j=1}^{\infty} \lambda_j^n \sum_{k=1}^{r_j} (b_i, x_{jk}) x_{jk}$$

$$R^n(\lambda, A) b_i = \sum_{j=1}^{\infty} \frac{1}{(\lambda - \lambda_j)^n} \sum_{k=1}^{r_j} (b_i, x_{jk}).$$

More specific examples are given in [14] Sec. 5.

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System sterowalny w przestrzeni Banacha z opóźnionym działaniem sterowania

Rozważono układy zdefiniowane w przestrzeni Banacha z opóźnionym działaniem sterowania. Założono, że operator oddziałujący na stan jest jedynie elementarnym generatorem silnie ciągłej półgrupy, warunki konieczne i/lub wystarczające są wyszukiwane i wyznaczane jedynie w funkcji operatorów zjawiających się w dynamice. Są one uogólnieniem wyników otrzymanych poprzednio przez Banksa, Jacobsa i Latinę dla układów nieskończenie wymiarowych bez opóźnień. Warunki ilustrowane są przykładami fizycznymi. Wynik Fattoriniego, redukujący przypadek operatora nieograniczonego do ograniczonego, wykorzystano również do problemu z opóźnieniem.

Управляемая система в банаховом пространстве с запаздыванием управления

В статье рассмотрены системы, определяемые в банаховом пространстве, с запаздывающим действием управления. Предполагается, что оператор воздействующий на состояние является лишь элементарным генератором сильно непрерывной полугруппы. Необходимые и/или достаточные условия находятся и определяются только в фукции операторов, появляющихся в динамике. Они являются обобщением результатов полученных ранее Бэнксом-Якобсом-Лэйтином для случая бесконечномерных систем без запаздываний. Условия иллюстрируются примерами из физики. Результат Фатторини, редуцирующий случай неограниченного оператора в ограниченный, используется также в задаче с запаздыванием.