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Basic relations in performance sensitivity analysis of optimal control systems *)

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The paper presents basic notions and relations in first — and second order performance sensitivity analysis of optimal control systems described by models in Banach spaces. Two basic formulations of the performance sensitivity analysis problem are discussed: the ideal sensitivity analysis based on one mathematical model, and the sensitivity analysis of a control system, based on two mathematical models and on the notion of a controller in a given structure. Basic lemmas related to the notion and properties of sensitivity measure are restated for the general case considered in the paper. The notion of second-order sensitivity operators, their existence, and the conditions of first order insensitivity are examined. Methods of computing second-order sensitivity operators in several control systems structures are given. These methods result in an effective way of comparing performance sensitivities of various optimal control structures and of choosing the best (the least sensitive) structure under uncertainty of parameters.

An example illustrates the application of these methods to an environment control problem, resulting in the choice of a hierarchical control system structure.

1. Introduction

The problem of sensitivity analysis of control systems has been formulated and investigated relatively early. The main result of the sensitivity analysis consist in showing that the output of a closed-loop control system is less sensitive to disturbances or under uncertainty of parameters than the output of an open-loop system.

The development of the optimal control theory motivated also an extensive research in the field of the analysis of performance changes due to disturbances or parameter uncertainty. This problem is called performance sensitivity analysis of optimal control systems. First particular problem that has been thoroughly investigated is the determination of optimal performance, state and control variations under known changes of parameter values — see e.g. [5]. This particular

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branch of sensitivity analysis is called here ideal adaptive sensitivity analysis; in the ideal adaptive situation there is no actuall necessity to distinguish between the model and the real process to be controlled. The ideal adaptive analysis stimulated many techniques of suboptimal control (nearoptimum design technique, optimaly sensitive structures etc. see [1], [5], [6]) where the differences between the model and the real process are taken into consideration but are compensated with help of measured or estimated parameter changes. The second, basic problem is the comparison of performance sensitivity of various optimal control system structures, under the assumption that parameter changes are not necessarity measured nor estimated during systems operation, i.e. under uncertainty of parameters. This situation is much more common in practical applications; it is natural to say that an optimal control system structure is better than other structures if its performance sensitivity is lower without the necessity of estimating parameter changes. In this case, both the model and the real process (or, strictly speaking, the actual model and a second model representing the parameter uncertainty) must be taken into consideration.

The investigation of the problem of performance sensitivity comparison started early [8], [11], but soon led to the so-called [11] Pagurek-Witsenhousen paradox: the first order performance sensitivity coefficients are identical for an open-loop and for a closed-loop optimal control system. This result has been since then generalized and reconfirmed in many other situations. Kreidler [1], [7], accepting the equal performance sensitivity coefficients for an open-loop and a closed-loop system, proved that a measure of trajectory sensitivity is lower in the closed-loop case. Kokotovic at al. [5] proved more: if the parameter changes can be measured or estimated, then sensitivity coefficients of arbitrary order can be made equal for an open-loop and a closed-loop structure by a Taylor-series compensation of parameter changes. This result, however, is closely related to the ideal adaptive situation, since the ideal adaptive structure based on the knowledge of parameter changes gives totally identical results for any open-loop and closed-loop structure.

Being reconfirmed in many ways, the Pagurek-Witsenhousen result had an effect to stopping further investigation of the basic question: is there any difference of performance sensitivity of various optimal control system structures, or, equivalently, *is there any reason for prefering one or another optimal control system structure under uncertainty of parameters*. Most of sensitivity investigation turned to other directions of research [1]. First in [12] a proof was given that the Pagurek-Witsenhousen result is an inherent and immediate by-product of the optimality of control and is valid for all possible optimal control system structures, hence is not meaningful; that at least second-order sensitivity coefficients must be considered in order to compare sensitivities of various structures under uncertainty of parameters, and that there exist rather large differences in performance sensitivity of various structures. However, this line of approach has not been followed further.

In this paper, second-order methods of performance sensitivity analysis are developed for a general class of models in Banach space (including ordinary differential equations, difference-differential equations, partial differential equations and many others) and for an arbitrary pre-specified optimal control system structure. An effective method of comparing sensitivities of various structures is presented; it is shown that a sensitivity comparison for several given structures requires no more additional computational effort than the determination of optimal control itself. It is possible thus to choose at the most effective control system structure from a given group of structures under uncertainty of aprameters.

Because of generality of the Banach space approach, many basic notions and relations introduced earlier for models described by ordinary differential equations are restated or generalized in the initial part of the paper. Therefore, the initial part of the paper presents a detailed formulation and basic results of the ideal adaptative sensitivity analysis. The performance sensitivity investigation under uncertainty of parameters, based on two models (the model and the real process) is called real sensitivity analysis; a detailed formulation and basic relations for this case are also given. These relations form a basis for the development of comparison methods for performance sensitivity of various optimal control system structures.

2. The optimal control and ideal sensitivity problem

Consider the following optimization problem. Let B_x, B_u, B_p, B_a be Banach spaces. Let $Q: B_x \times B_u \times B_a \to R^1$ be a "performance" functional. Let $P: B_x \times B_u \times B_a \to B_p$ be a "constraining" operator. The problem

$$\min_{(x, u) \in \Omega} Q(x, u, a) = \hat{Q}(a), \tag{1}$$

where Ω is the set of (x, u) such that

$$P(x, u, a) = \Theta \in B_p, \tag{2}$$

can be looked upon as an optimal control problem with the state x, control u and parameter a, provided the constraining relation (2) determines uniquely an x for any given u, a and thus can be interpreted as a state equation. Hence we shall often call P the process operator.

We assume that there exist a unique normal optimal solution of the problem (1) for each a in a given subset in B_a, without specifying the conditions of existence and normality[1]). Moreover, assume B_p, B_x be reflexive Banach spaces; let operator P be Frechet differentiable with respect to (x, u) and denote by $P_x: B_x \to B_p$ the Frechet derivative of P with respect to x; let there exist P_x^{-1}. Therefore, P_x is a topological isomorphism of the spaces B_x and B_p. Let the functional Q be convex in a neighbourhood of the optimal point $(\hat{x}, \hat{u})$ and Frechet differentiable with respect to (x, u). Then the corresponding Lagrange functional has the following normal form

$$L(\eta, x, u, a) = Q(x, u, a) + \langle \eta, P(x, u, a) \rangle \tag{3}$$

[1]) By a normal solution of an optimal control problem we understand a solution corresponding to a Lagrange functional in the normal form (3), whereas the general form is $L(\eta, x, u, a) = \eta_0 Q(x, u, a) + \langle \eta, P(x, u, a) \rangle$. Therefore, a problem is normal if $\eta_0 \neq 0$. For discussion of normality conditions see e.g. [13].

where $\eta \in B_p^*$ is a Lagrange multiplier and $\langle \cdot, \cdot \rangle$ is the duality between B_p^* and B_p. The necessary conditions of optimality [13] can be written as

$$L_u^* = Q_u^* (\hat{x}, \hat{u}, a) + P_u^* (\hat{x}, \hat{u}, a) \hat{\eta} = \Theta \in B_u^*, \tag{4a}$$

$$L_x^* = Q_x^* (\hat{x}, \hat{u}, a) + P_x^* (\hat{x}, \hat{u}, a) \hat{\eta} = \Theta \in B_x^*, \tag{4b}$$

$$L_\eta^* = \qquad P(x, u, a) \quad = \Theta \in B_p, \tag{4c}$$

where stars denote gradients of functionals, adjoint operators, or dual spaces, respectively. The equation (4a) corresponds to the gradient of the minimized functional reduced to the space of independent variables B_u, see [13]; (4b) can be interpreted as the basic adjoint equation; (4c) is the state equation.

According to the assumptions, the equations (4) determine the optimal solution $\hat{u} = \hat{U}(a)$, $\hat{x} = \hat{X}(a)$ and the functional $\hat{Q}(a) = Q(\hat{X}(a), \hat{U}(a), a)$. The functional $\hat{Q}(a)$ is the *basic optimal performance characteristics*, expressing the dependence of the performance on the parameters of the optimal control problem. Similarly, $\hat{U}(a)$ is the *basic optimal control* characteristics and $\hat{X}(a)$ is the *basic optimal state characteristics*.

Suppose P, Q, are Frechet differentiable with respect to a. For a given optimal $u = \hat{u}$ it is easy to estimate the first order variations of x and Q due to a change odd parameters δa by

$$\delta x = -(P_x^{-1} P_a)(\hat{x}, \hat{u}, a)\, \delta a \tag{5}$$

where $-P_x^{-1} P_a = X_a : B_a \to B_x$ is called the *state sensitivity operator*, and

$$\delta Q = \langle (Q_a^* - P_a P_x^{*-1} Q_x^*)(\hat{x}, \hat{u}, a), \delta a \rangle . \tag{6}$$

These variations represent the influence of the parameters on the state and performance if the optimal control is applied in a control system in the same way as it is determined, that is, in so called *open-loop structure*. If we could measure precisely the parameter change δa and use the knowledge to readjust the optimal control, the corresponding variations of state and performance should be, obviously, different; we shall call the situation the *ideal* (*adaptive*) *structure*. To compute the variations[2]), let us assume P, Q be twice Frechet differentiable in (x, u, a). The variations of optimal $(\hat{x}, \hat{u})$ due to a change of parameters δa result from a linearisation of (4a) ... (4c)

$$\hat{L}_{uu}\, \delta\hat{u} + \hat{L}_{ux}\, \delta\hat{x} + \hat{L}_{u\eta}\, \delta\hat{\eta} + \hat{L}_{ua}\, \delta a = \Theta , \tag{7a}$$

$$\hat{L}_{xu}\, \delta\hat{u} + \hat{L}_{xx}\, \delta\hat{x} + \hat{L}_{x\eta}\, \delta\hat{\eta} + \hat{L}_{xa}\, \delta a = \Theta , \tag{7b}$$

$$\hat{L}_{\eta u}\, \delta\hat{u} + \hat{L}_{\eta x}\, \delta x + \qquad \hat{L}_{\eta a}\, \delta a = \Theta . \tag{7c}$$

These equations are called the *basic variational equations*; $\hat{L}_{uu}$ denotes the operator of second-order derivatives of the functional L with respect to u, evaluated at $(\hat{\eta}, \hat{x}, \hat{u}, a)$, etc. Since $L_{\eta x} = P_x (x, u, a)$ is invertible, we get from (7b), (7c)

$$\delta\hat{x} = -\hat{L}_{\eta x}^{-1} (\hat{L}_{\eta u}\, \delta\hat{u} + \hat{L}_{\eta a}\, \delta a), \tag{8a}$$

$$\delta\hat{\eta} = -\hat{L}_{x\eta}^{-1} (\hat{L}_{xu} - \hat{L}_{xx} \hat{L}_{\eta x}^{-1} \hat{L}_{\eta u})\, \delta\hat{u} + (\hat{L}_{xa} - \hat{L}_{xx} \hat{L}_{\eta x}^{-1} \hat{L}_{\eta a})\, \delta a), \tag{8b}$$

[2]) See also [5] for a similar development for problems described by ordinary differential equations.

and from (7a)

$$\hat{A}\delta\hat{u}+\hat{B}\delta a=\Theta, \tag{9}$$

where

$$\hat{A}=\hat{L}_{u\eta}\,\hat{L}_{x\eta}^{-1}\,\hat{L}_{xx}\,\hat{L}_{\eta x}^{-1}\,\hat{L}_{\eta u}-\hat{L}_{u\eta}\,\hat{L}_{x\eta}^{-1}\,\hat{L}_{xu}\,\hat{L}_{ux}-\hat{L}_{\eta x}^{-1}\,\hat{L}_{\eta u}+\hat{L}_{uu}) \tag{10a}$$

$$\hat{B}=\hat{L}_{u\eta}\,\hat{L}_{x\eta}^{-1}\,\hat{L}_{xx}\,\hat{L}_{\eta x}^{-1}\,\hat{L}_{\eta a}-\hat{L}_{u\eta}\,\hat{L}_{x\eta}^{-1}\,\hat{L}_{xa}-\hat{L}_{ux}\,\hat{L}_{\eta x}^{-1}\,\hat{L}_{\eta a}+\hat{L}_{ua}. \tag{10b}$$

Since $\hat{A}$ is the hessian operator of the minimized functional, reduced to the space of control variables — see e.g. [13], it is usually a positive operator. Assume $\hat{A}^{-1}$ exists; then

$$\delta\hat{u}=-\hat{A}^{-1}\,\hat{B}\delta a \tag{11a}$$

$$\delta\hat{x}=\hat{L}_{\eta x}^{-1}\,(\hat{L}_{\eta u}\,\hat{A}^{-1}\,\hat{B}-\hat{L}_{\eta a})\,\delta a, \tag{11b}$$

$$\delta\hat{\eta}=\hat{L}_{x\eta}^{-1}\,((\hat{L}_{xu}-\hat{L}_{xx}\,\hat{L}_{\eta x}^{-1}\,\hat{L}_{\eta u})\,\hat{A}^{-1}\,\hat{B}-(\hat{L}_{xa}-\hat{L}_{xx}\,\hat{L}_{\eta x}^{-1}\,\hat{L}_{\eta a}))\,\delta a. \tag{11c}$$

The operators

$$\hat{U}_a=-\hat{A}^{-1}\,\hat{B};\ \hat{X}_a=\hat{L}_{x\eta}^{-1}\,(\hat{L}_{u\eta}\,\hat{A}^{-1}\,\hat{B}-\hat{L}_{\eta a}) \tag{12}$$

are the *optimal control sensitivity operator* and *the optimal state sensitivity operator* in the ideal structure, representing the influence of the parameters on the control and the state if they are optimally readjusted after each parameter change. However, in most of practical applications it is almost impossible to determine the operator (12) analytically, when determining the basic variations $\delta\hat{u}$, $\delta\hat{x}$, $\delta\hat{\eta}$ computationally, it is preferable not to use the relations (11a, b, c), but to apply a special procedure.

Usually, the inversion of $\hat{L}_{uu}$ is rather simple. We get then from (7a)

$$\delta\hat{u}=-\hat{L}_{uu}^{-1}\,(\hat{L}_{ux}\,\delta\hat{x}+\hat{L}_{u\eta}\,\delta\hat{\eta}+\hat{L}_{ua}\,\delta a). \tag{13}$$

Substituting $\delta\hat{u}$ in (7 b, c) we obtain

$$\begin{aligned}&\mathfrak{A}_1\,\delta\hat{x}+\mathfrak{A}_2\,\delta\hat{\eta}+\mathfrak{B}_1\,\delta a=\Theta,\\&\mathfrak{A}_3\,\delta\hat{x}+\mathfrak{A}_1^*\,\delta\hat{\eta}+\mathfrak{B}_2\,\delta a=\Theta,\end{aligned} \tag{14a}$$

where

$$\begin{aligned}&\mathfrak{A}_1=\hat{L}_{\eta x}-\hat{L}_{\eta u}\,\hat{L}_{uu}^{-1}\,\hat{L}_{ux};\ \mathfrak{A}_2=-\hat{L}_{\eta u}\,\hat{L}_{uu}^{-1}\,\hat{L}_{u\eta};\\&\mathfrak{B}_1=\hat{L}_{\eta a}-\hat{L}_{\eta u}\,\hat{L}_{uu}^{-1}\,\hat{L}_{ua};\\&\mathfrak{A}_3=\hat{L}_{xx}-\hat{L}_{xu}\,\hat{L}_{uu}^{-1}\,\hat{L}_{ux};\ \mathfrak{B}_2=L_{xa}+L_{xu}\,L_{uu}^{-1}\,L_{ua}.\end{aligned} \tag{14b}$$

The equations (14) are called *canonical variational equations*. Their solution is clearly expressed by the relations (11 b, c). Hence there are many linear operators K, M which would satisfy the relation

$$\delta\hat{\eta}=K\delta\hat{x}+M\delta a \tag{15}$$

but one way of choosing K, M is particularly useful. Setting (15) into (14a), premultiplying first of the equations by K^* and summing both the equations we get

$$(K^*\mathfrak{A}_2\,K+K^*\mathfrak{A}_1+\mathfrak{A}_1^*\,K+\mathfrak{A}_3)\,\delta\hat{x}+(K^*\mathfrak{A}_2\,M+\mathfrak{A}_1^*\,M+K\mathfrak{B}_1+\mathfrak{B}_2)\,\delta a=\Theta \tag{16}$$

and require that

$$K^* \mathfrak{A}_2 K + K^* \mathfrak{A}_1 + \mathfrak{A}_1^* K + \mathfrak{A}_3 = \Theta, \tag{17a}$$

$$(K^* \mathfrak{A}_2 + \mathfrak{A}_1^*) M + K^* \mathfrak{B}_1 + \mathfrak{B}_2 = \Theta. \tag{17b}$$

The equation (17a) is called Riccati operator equation. The operator K defined by the equation has many interesting properties which can be fully investigated only when we define — beside the space of state trajectories B_x — the space of momentary states $B_{x(t)}$, take into account the semigroup properties of the process operator P, etc — see [4], [13]. Without motivating more deeply the equations (17a, b) nor analysing their properties we shall only note two important facts:

(a) Solving the equations (17a, b) is very often the most effective computational way to determine $\delta\hat{x}$, $\delta\hat{\eta}$, $\delta\hat{u}$ — and thus to invert the hessian A.

(b) The basic control variation $\delta\hat{u}$, resulting from (13), (15), (17a, b):

$$\delta\hat{u} = -\hat{L}_{uu}^{-1} (\hat{L}_{ux} + \hat{L}_{u\eta} K) \delta\hat{x} - \hat{L}_{uu}^{-1} (\hat{L}_{ua} + \hat{L}_{u\eta} M) \delta a \tag{18}$$

is the linear approximation of the optimal control law in the *classical closed-loop structure*.

The property (a) is substantiated by large computational experience. The property (b) can be rigorously proven; since it was proven in many particular cases of process operators and a general proof for linear-quadratic case was given in [4], we omit here the proof.

Beside determing the basic variations $\delta\hat{u}$, $\delta\hat{x}$, $\delta\hat{\eta}$, it is also important to analyse the behaviour of the performance functional $\hat{Q}(a)$. Its variation has the form

$$\delta\hat{Q} = \langle Q_a^*, \delta a\rangle + \langle Q_x^*, \delta\hat{x}\rangle + \langle Q_u^*, \delta\hat{u}\rangle =$$
$$= \langle Q_a^* - \hat{L}_{a\eta} \hat{L}_{x\eta}^{-1} Q_x^* + \hat{B}^* \hat{A}^{-1} (\hat{L}_{u\eta} \hat{L}_{x\eta}^{-1} Q_x^* - Q_u^*), \delta a\rangle \tag{19}$$

But $\hat{L}_{u\eta} \hat{L}_{x\eta}^{-1} Q_x^* - Q_u^* = \Theta$ according to (4a), (4b). Hence

$$\delta\hat{Q} = \langle Q_a^* - P_a^* P_x^{*-1} Q_x^*, \delta a\rangle \tag{20}$$

and the ideal performance variation is the same as in the open-loop structure (6). Therefore, *the readjustment of optimal control related to parameter changes does not apparently result in any gain of performance.* Such a interpretation of this results is, however, superficial; the only true interpretation is that *the first-order approximation of optimal performance is not sufficient for the comparison of various optimal control structures.* Moreover, the investigation of $\hat{U}(a)$, $\hat{X}(a)$, $\hat{Q}(a)$ does not provide answers for several important problems related to applications of optimal control under uncertainty of parameters; therefore, this investigation shall be called the *ideal sensitivity analysis.*

3. Real sensitivity problem

To achieve practically relevant results, a much more complicated model of the sensitivity problem must be formulated (see Fig. 1). In real applications, the determination of an optimal control is based on a mathematical model (e.g.

$Q(x, u, a)=q$, $P(x, u, a)=\Theta$) of the problem, and the model obviously differs from the reality. Therefore, the optimal control is seldom applied in the same way as it is determined (in the open-loop structure), though the ideal measurement of

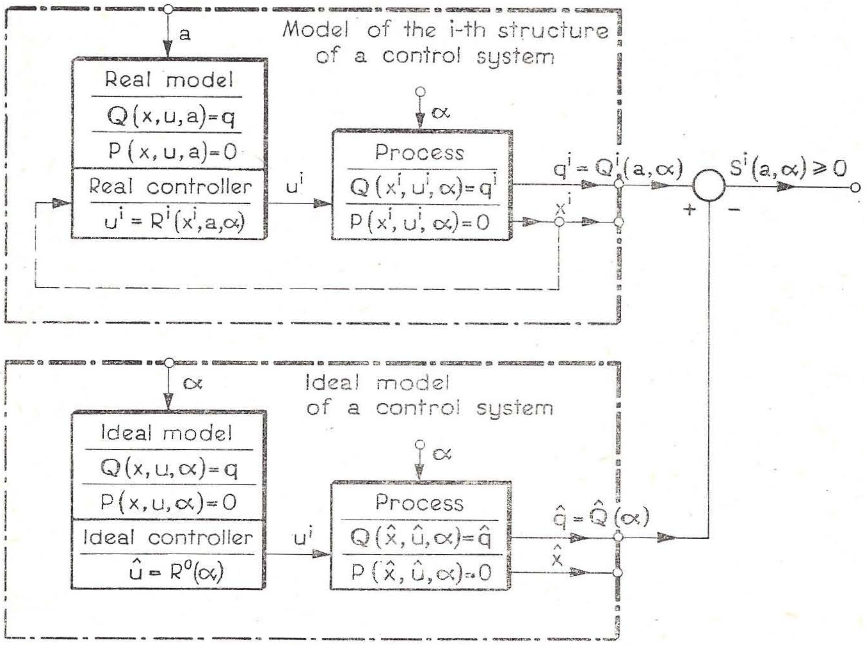

Fig. 1. Global sensitivity model

the parameters a (the ideal adaptive structure) is usually not possible. Usually, several output variables of the real process can be measured and utilized in order to improve the control which, being optimal for the model, is not optimal for the real process.

Mathematically speaking, we must first define what the "reality" is. The simplest assumption is that the "reality" can be represented by another model which differs from the original one only in the value of the parameters (e.g. $Q(x, u, \alpha)=q$, $P(x, u, \alpha)=\Theta$). In fact, such an assumption is quite general. For example, if $P(x, u, a)=\Theta$ corresponds to an ordinary differential equation, an appropriate choice of P and α can make $P(x, u, \alpha)=\Theta$ corresponding to a partial differential equation (although some boundary layer problems will arise in such a case).

Secondly, we must define what is to improve the control with the help of variables measured in the real process; this definition is much more difficult.

The optimal control $\hat{u}=\hat{U}(a)$ can be represented in various ways, as a function of various variables of the real process, i.e. of the second model. The way of representing the control

$$u^i=R^i(x, a, \alpha) \tag{21}$$

is denoted by the upper index i and called the *control* law or the *controller* in the i-th structure of the control system. By $i=0$ we denote the open-loop structure, where $R^0(x, a, \alpha) = \hat{U}(a)$. By $i=1$ we denote the classical closed-loop structure, where the current value of control depends on the current value of state but not on previous nor future state values — see [4]. The synthesis of the function R^1 is an extremely difficult task except in some fundamental cases, e.g. the linear-quadratic one. However, we can determine the derivatives R^1_x, R^1_a according to the relation (18).

The classical closed-loop structure is by far not the only one possible: for example, any linear combination (with coefficients summing up to one) of the open-loop and closed-loop optimal controllers results in an optimal controller. The number of possible structures is thus uncountable and limited only by the imagination of control-system designers.

The control law (21) results in the following relations:

$$P\left(x^i. R^i(x^i, a, \alpha), \alpha\right) = \Theta \in B_p, \tag{22}$$

$$Q\left(x^i, R^i(x^i, a, \alpha), \alpha\right) \stackrel{\text{df}}{=} Q^i(a, \alpha). \tag{23}$$

The relation (23) defines the functional $Q^i(a, \alpha)$, if the relation (22) determines x^i as a function of a, α. The functional $Q^i(a, \alpha)$ is related to $\hat{Q}(a)$, if the control law (21) represents the optimal control for $a=\alpha$. Hence we impose two axiomatic requirements on the nature of the relations (21), (22), (23).

Axiom of well-setting: For a problem of the real sensitivity analysis of a control system to be well set, it is necessary that the real process equation in the given system structure — the relation (22) — admits a unique solution $x^i = X^i(a, \alpha)$ for each α in a neighbourhood of a, and the performance functional — the relation (23) — is well defined in this neighbourhood.

Except in some degenerate cases, this axiom implays that the space B_p should be isomorphic to B_x and the operator $P_x + P_u R^i_x$ be invertible. Although quite natural to postulate, the well-setting axiom was often neglected in the sensitivity analysis of optimal control systems[3]).

Axiom of optimality: The control law (21) corresponds to an optimal control system, that is, $R^i(\hat{X}(a)\, a, a) = \hat{U}(a)$.

This axiom implies that $Q^i(\alpha, \alpha) = \hat{Q}(\alpha)$ and $Q^i(a, \alpha) \geqslant \hat{Q}(\alpha)$ since the control u^i can be only worse for the real process than the optimal $\hat{U}(\alpha)$, if the parameters of the real process are not known precisely and a parameter $a \neq \alpha$ is used to establish the control law. Hence the functional

$$S^i(a, \alpha) = Q^i(a, \alpha) - \hat{Q}(\alpha) \tag{24}$$

can be interpreted as the performance loss due to an imperfect knowledge of the process parameters α. The functional is called the *sensitivity measure* of the optimal control system in the i-th structure — see [12].

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[3]) The optimal control problem (1), (2) can have solutions even if the axiom is not satisfied. Consider, for example, an optimal control problem with a given final state, and incorporate the final conditions in the constraining relation $P(x, u, a) = \Theta$; let B_x be a Sobolev space and $B_p = R^n$. Suppose an optimal solution exists; but for $a \neq \alpha$ the relation (22) will not be generally satisfied. Hence we cannot perform the real sensitivity analysis, although the ideal sensitivity analysis problem is well-defined. This is probably the reason for omitting the well-setting axiom in some approaches; see also [1], [6], [12].

LEMMA 1. The sensitivity measure satisfies following relations

$$S^i(a, \alpha) \geqslant 0; \; S^i(a, \alpha) = 0 \text{ for all } a = \alpha. \tag{25}$$

If the sensitivity measure is differentiable, then

$$S^{i*}_a(a, \alpha) = S^{i*}_\alpha(a, \alpha) = \Theta \text{ for all } a = \alpha. \tag{26}$$

Lemma 1 is an immediate consequence of the optimality axiom; observe that the sensitivity measure is differentiable if the process equation, the control law and the performance functional are differentiable and the well-setting axiom is satisfied.

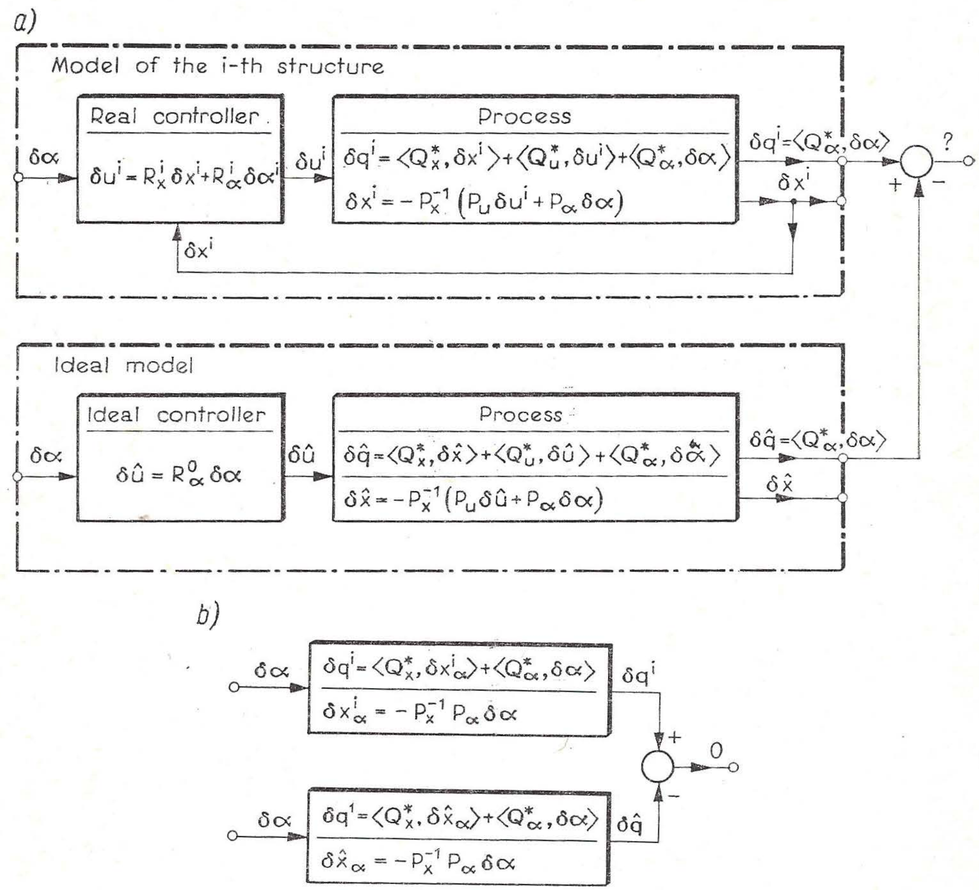

Fig. 2. First order local sensitivity model (change $\alpha_1 = a$ to $\alpha_2 = a + \delta\alpha$): a) full model, b) equivalent model under the assumption of optimality ($\langle Q^*_x, \delta x_u \rangle + \langle Q^*_u, \delta u \rangle = 0$ for $\delta x_u = -P^{-1}_x P_u \delta u$, all δu)

LEMMA 2. If the optimal performance characteristics $\hat{Q}(a)$ and the sensitivity measure $S^i(a, \alpha)$ are differentiable, then the first-order approximation of the change of the performance $Q^i(a, \infty)$ due to the process parameters change $\alpha = a + \delta\alpha$:

$$\delta Q^i = \langle Q^{i*}_\alpha, \delta\alpha \rangle = \langle Q^*_a - P^*_a P^{*-1}_a Q^*_x, \delta\alpha \rangle = \langle \hat{Q}_a, \delta\alpha \rangle = \delta\hat{Q} \tag{27}$$

is the same as the first-order approximation of the change of the optimal performance characteristics. Hence, one cannot distinguish various optimal control system structures by the first-order sensitivity analysis.

The proof of lemma is also immediate. Since $Q^i(a, \alpha) = \hat{Q}(\alpha) + S^i(a, \alpha)$, expanding both sides for $\alpha = a + \delta\alpha$ and taking into account (26) we obtain (27). An interpretation of the lemma 2 is given in Fig. 2.

LEMMA 3. If the sensitivity measure $S^i(a, \alpha)$ is twice Frechet differentiable with respect to a and α in an open set containing $\alpha = a$, and the second-order derivatives are continuous with respect to a and α, then

$$S^i_{\alpha\alpha} = S^i_{aa} = -S^i_{a\alpha} = -S^i_{\alpha a} \tag{28}$$

where the operators of second-order derivatives are evaluated at $a = \alpha$.

To prove the lemma, let us take $\alpha = a + \delta\alpha$ and expand $S^{i*}_\alpha(a, a+\delta\alpha)$: $S^{i*}_\alpha(a, a+\delta\alpha) = S^i_{\alpha\alpha}(a, a)\,\delta\alpha + o(\|\delta\alpha\|)$; $S^{i*}_\alpha(a, a, +\delta\alpha) = -S^i_{\alpha a}(a+\delta\alpha, a+\delta\alpha)\,\delta\alpha + o(\|\delta\alpha\|)$ since both $S^{i*}_\alpha(a, a) = S^{i*}_\alpha(a+\delta\alpha, a+\delta\alpha) = \Theta$. By the continuity of $S^i_{\alpha a}$ we get $S^i_{\alpha\alpha}(a, a) = -S^i_{\alpha a}(a, a)$. But then $S^i_{\alpha a}(a, a)$ must be self-adjoint and $S^i_{a\alpha}(a, a) = S^i_{\alpha a}(a, a)$. The same argument applied to $S^i_{a\alpha}(a, a+\delta\alpha)$ results in $S^i_{aa}(a, a) = -S^i_{a\alpha}(a, a)$.

The Lemma 3 can be also stated in form of a *local relativity principle* of sensitivity analysis: if the parameters of the model, a, and of the process, α, are sufficiently close, the sensitivity measure can be approximated by the difference of $\alpha - a$ only, no matter which of the parameters actually changes and which is kept constant:

$$S^i(a, \alpha) = 0.5\,\langle \alpha - a,\; S^i_{aa}(a, a)\,(\alpha - a)\rangle + o(\|\alpha - a\|^2). \tag{29}$$

An analogous statement is not true globally, since $S^i(a, \alpha)$ is not a function of $(\alpha - a)$.

The conditions of the second-order differentiability of the sensitivity measure result from the following lemma:

LEMMA 4. Suppose the process operator P and the performance functional Q are twice differentiable with respect to x, u. Suppose the control law operator R^i is differentiable with respect to x, a. Suppose:

(i) $\hat{Q}^*_u - \hat{P}^*_u \hat{P}^{*-1}_x \hat{Q}^*_x = \Theta$ (optimality),

(ii) $(\hat{P}_x + \hat{P}_u \hat{R}^i_x)^{-1}$ exists (well-setting).

Then the sensitivity measure $S^i(\alpha, a)$ is twice differentiable with respect to a and

$$S^i_{aa} = X^i_a \hat{L}_{xx} X^i_a + 2X^i_a \hat{L}_{xu} U^i_a + U^i_a \hat{L}_{uu} U^i_a = U^i_a \hat{A} U^i_a, \tag{30}$$

where $\hat{A}$ is specified by (10a) and

$$X^i_a = -(\hat{P}_x + \hat{P}_u \hat{R}^i_x)^{-1} \hat{P}_u \hat{R}^i_a;\; U^i_a = \hat{R}^i_x X^i_a + \hat{R}^i_a. \tag{31}$$

The operator X^i_a is called the structural state sensitivity operator, U^i_a — the structural control sensitivity operator. They determine the *structural state and* *control variations* $\delta u^i = U^i_a\,\delta a$, $\delta x^i = X^i_a\,\delta a$, which are the first-order approximations of the state and control changes due to a mistake δa in estimating the real process parameters α, $a = \alpha + \delta a$.

Observe that Lemma 4 gives only the conditions of differentiability of $S^i(a, \alpha)$ with respect to a; the conditions of differentiability with respect to α are similar. The situation where a is kept constant and α changes is actually of much larger practical importance; however, it is much more simple analytically and computationally to investigate the reverse situation, when a changes and α is kept constant — see Fig. 3. According to the Lemma 3 and the local relativity principle, both approaches are locally equivalent.

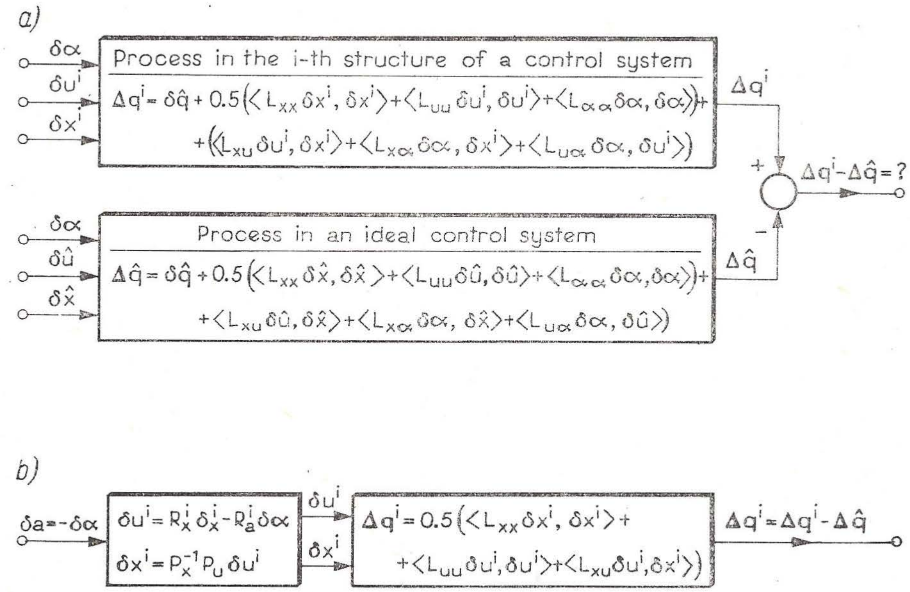

Fig. 3. Second order local sensitivity model: a) full model (change α from $\alpha_1 = a$ to $\alpha_2 = a + \delta x$); b) equivalent model based on the local relativity principle (change a from $a_1 = \alpha$ to $a_2 = \alpha + \delta a$, $\delta a = -\delta\alpha$)

To prove the Lemma 4, we have to determine x^i, u^i in the given structure by solving the equations

$$P(x^i, u^i, \alpha) = \Theta; \quad u^i = R^i(x^i, a, \alpha). \tag{32}$$

Under the assumptions of the lemma we can apply the implicit function theorem; setting $a = \alpha + \delta a$ we get

$$x^i - \hat{x} = \delta x^i + o(\|\delta a\|); \quad \delta x^i = -(\hat{P}_x + \hat{P}_u \hat{R}^i_x)^{-1} \hat{P}_u \hat{R}^i_a\, \delta a = X^i_a\, \delta a, \tag{33a}$$

$$u^i - \hat{u} = \delta u^i + o(\|\delta a\|),$$

$$\delta u^i = (I - \hat{R}^i_x (\hat{P}_x + \hat{P}_u \hat{R}^i_x)^{-1} \hat{P}_u)\, R^i_a\, \delta a = U^i_a\, \delta a, \tag{33b}$$

$$\delta x^i = -\hat{P}_x^{-1} \hat{P}_u\, \delta u^i, \tag{33c}$$

where $\hat{x}$, $\hat{u}$ are the optimal state and control satisfying $P(\hat{x}, \hat{u}, \alpha)=\Theta$ and $\hat{u}=R^i(\hat{x}, \alpha, \alpha)$. Now we have to expand $Q(x^i, u^i, \alpha)-Q(\hat{x}, \hat{u}, \alpha)=S^i(a, \alpha)$ in Taylor series with respect to $x^i-\hat{x}$, $u^i-\hat{u}$. In order to suppres the influence of the terms $o(\|\delta a\|)$ onto the second-order terms, it is convenient to use the identity $Q(x^i, u^i, \alpha)=Q(x^i, u^i, \alpha)+\langle\eta, P(x^i, u^i, \alpha)\rangle=L(\eta, x^i, u^i, \alpha)$. In fact, the first-order derivatives of the Lagrange functional are identically zero at $\hat{x}$, $\hat{u}$ — see (4a, b, c); hence we get

$$S^i(a, \alpha)=0.5\,(\langle X_a^i\,\delta a, L_{xx}\,X_a^i\,\delta a\rangle+2\,\langle X_a^i\,\delta a, L_{xu}\,U_a^i\,\delta a\rangle+ \\ +\langle U_a^i\,\delta a, L_{uu}\,U_a^i\,\delta a\rangle)+o(\|\delta a\|^2)=0.5\,\langle U_a^i\,\delta a, \hat{A}U_a^i\,\delta a\rangle+0(\|\delta \dot{a}\|^2) \tag{34}$$

thus establishing the relation (30) and the conclusions of the lemma.

As a summary of this paragraph, it is worth while to discuss the basic, conceptual difference between the first-order and the second-order sensitivity analysis of optimal control systems. The first-order sensitivity coefficients (the gradients $\hat{Q}_\alpha^*$, Q_α^{i*}) are equal for all possible structures and express the dependence of the

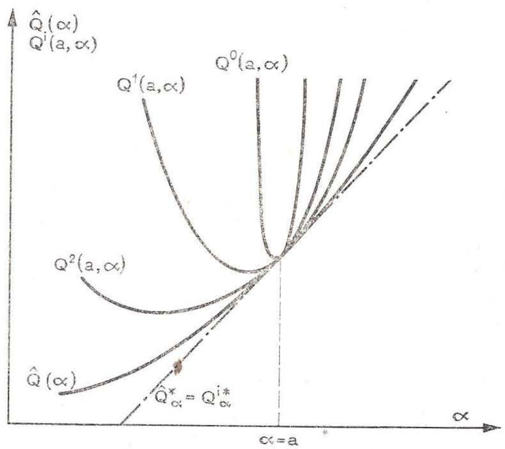

Fig. 4. An example of the performance characteristics $\hat{Q}(\alpha)$ and $Q^i(a, \alpha)$ by given a

ideal, optimal performance $\hat{Q}(\alpha)$ on the parameters of the problem. Nevertheless, the real performance $Q^i(a, \alpha)$ depends strongly on the structure i; the differences between the performance in various structures can be astonishingly large — see [12] — even for small $\alpha-a$. A simple interpretation of the fact is given in Fig. 4. Therefore, in order to investigate the sensitivity of various optimal control system structures, it is necessary to compute either the sensitivity measures $S^i(a, \alpha)=Q^i(a, \alpha)-\hat{Q}(\alpha)$, or at least the second-order sensitivity operators $S_{aa}^i=S_{\alpha\alpha}^i$. A determination of the sensitivity measure $S^i(a, \alpha)$ in an analytical form is an extremely complicated task even in the simplest examples — see [12]; hence the importance of the Lemma 4.

4. Sensitivity operators of several optimal control systems structures

The Lemma 4 provides for a basis for sensitivity computations: if we only know the structural variations δx^i and δu^i, it is easy to determine the approximation of $S^i(\alpha+\delta a, \alpha)$ given by the relation (34).

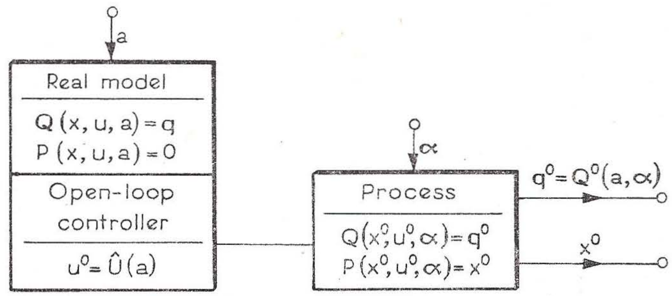

Fig. 5. Open-loop optimal control system

$i=0$ (*open-loop*)

In the *open-loop structure* — see Fig. 5 — we have

$$\delta u^0 = \delta \hat{u};\ \delta x^0 = -\hat{P}_x^{-1}\,\hat{P}_u\,\delta\hat{u};\ U_a^0 = \hat{U}_a;\ X_a^0 = -\hat{P}_x^{-1}\,\hat{P}_u\,\hat{U}_a. \tag{35a}$$

Sometimes it may be useful to determine δx^0, X_a^0 with the help of $\delta\hat{x}$, $\hat{X}_a$:

$$\delta x^0 = \delta\hat{x} + \hat{P}_x^{-1}\,\hat{P}_a\,\delta a;\ X_a^0 = \hat{X}_a + \hat{P}_x^{-1}\,\hat{P}_a. \tag{35b}$$

$i=1$ (*closed-loop*).

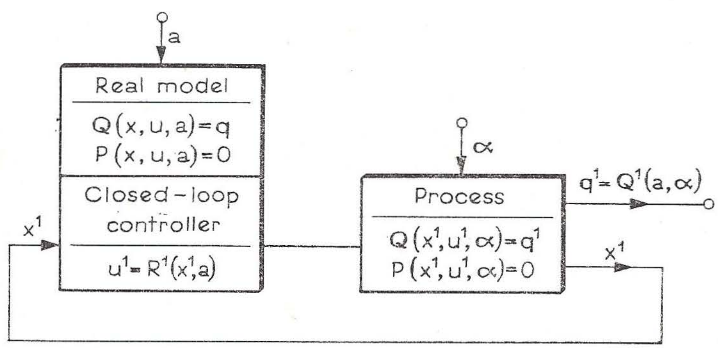

Fig. 6. Closed-loop optimal control system

In the *closed-loop structure* — see Fig. 6 — we obtain

$$\delta u^1 = \hat{R}_x^1\,\delta x^1 + \hat{R}_a^1\,\delta a;\ \delta x^1 = -\hat{P}_x^{-1}\,\hat{P}_u\,\delta u^1 \tag{36a}$$

where — see (18)

$$\hat{R}_x^1 = -\hat{L}_{uu}^{-1}\,(\hat{L}_{ux} + \hat{L}_{u\eta}\,K);\ \hat{R}_a^1 = -\hat{L}_{uu}^{-1}\,(\hat{L}_{ua} + \hat{L}_{u\eta}\,M) \tag{36b}$$

and K, M are determined by (17a, 17b). We can get also closed expressions for U_a^1, X_a^1

$$U_a^1 = -\big(I+\hat{L}_{uu}^{-1}(\hat{L}_{ux}+\hat{L}_{u\eta}K)(\hat{L}_{\eta x}-\hat{L}_{\eta u}\hat{L}_{uu}^{-1}(\hat{L}_{ux}+ \\ +\hat{L}_{u\eta}K))^{-1}\hat{L}_{\eta u}\big)\hat{L}_{uu}^{-1}(\hat{L}_{ua}+\hat{L}_{u\eta}M), \tag{36c}$$

$$X_a^1 = (\hat{L}_{\eta x}-\hat{L}_{\eta u}\hat{L}_{uu}^{-1}(\hat{L}_{ux}+\hat{L}_{u\eta}K))^{-1}\hat{L}_{\eta u}\hat{L}_{uu}^{-1}(\hat{L}_{ua}+\hat{L}_{u\eta}M) \tag{36d}$$

but in most computational applications it is more simple to determine R_x^1, R_a^1 with the help of predetermined K, M, and then to solve (36a) numerically.

$i=2$ (*optimal trajectory tracking*).

If the operator P_u has an inverse (or a pseudoinverse in some sense), another structure can be applied — see Fig. 7. The structure is called *optimal trajectory tracking* and is denoted by $i=2$. The optimal trajectory tracking structure is in

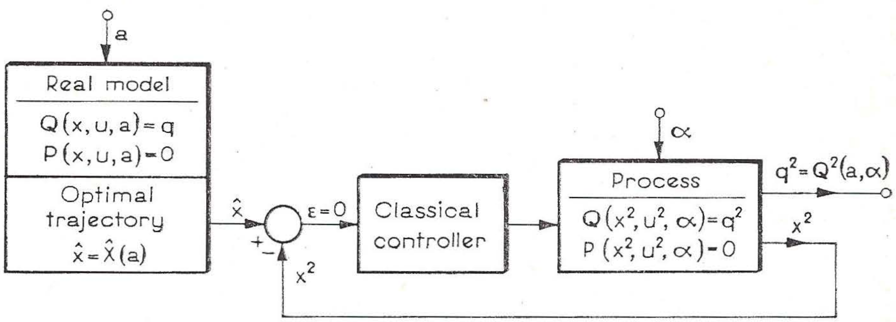

Fig. 7. Optimal trajectory tracking control system

a way dual to the open-loop structure: in the open-loop, we keep the optimal control for the model, $\hat{U}(a)$, independently of the process changes, whereas in the optimal trajectory tracking we induce the process to realize the optimal state trajectory for the model, $\hat{X}(a)$, independently of the process changes. The structural variations and sensitivity operators are[4]):

$$\delta x^2 = \delta\hat{x};\ \delta u^2 = -\hat{P}_u^{-1}\hat{P}_x\,\delta\hat{x};\ X_a^2 = \hat{X}_a;\ U_a^2 = -\hat{P}_u^{-1}\hat{P}_x\hat{X}_a. \tag{37a}$$

A much more useful formulae for δu^2, U_a^2 are

$$\delta u^2 = \delta\hat{u} + \hat{P}_u^{-1}\hat{P}_a\,\delta a;\ U_a^2 = \hat{U}_a + \hat{P}_u^{-1}\hat{P}_a. \tag{37b}$$

$i=3$ (*open-loop optimizing feedback*)

Suppose the process equation has the form

$$P(x, u, \alpha) = P_1(x) + P_2(x, u, \alpha) = \Theta,$$

where $P_2(x, u, \alpha)$ can be actually measured in the real process. Let the optimization problem be (locally) convex so that the optimal control can be determined by minimizing the Lagrange functional; we may also assume that a maximum principle is valid for the optimal control problem and the control can be determined by maximizing a Hamiltonian function. Let $\hat{\eta} = \hat{N}(a)$ be computed on the basis

[4]) Note that in this case the assumption (ii) of Lemma 4 is not fulfilled. Nevertheless, the Lemma remains valid in this special case with slight changes in the proof.

of the model. We can require that the control be determined by a peak-holding controller, performing the operation

$$\min_u L(\hat{\eta}, x, u, a) = \langle \hat{\eta}, P_1(x)\rangle + \min_u (Q(x, u, \alpha) + \langle \hat{\eta}, P_2(x, u, \alpha)\rangle . \tag{38}$$

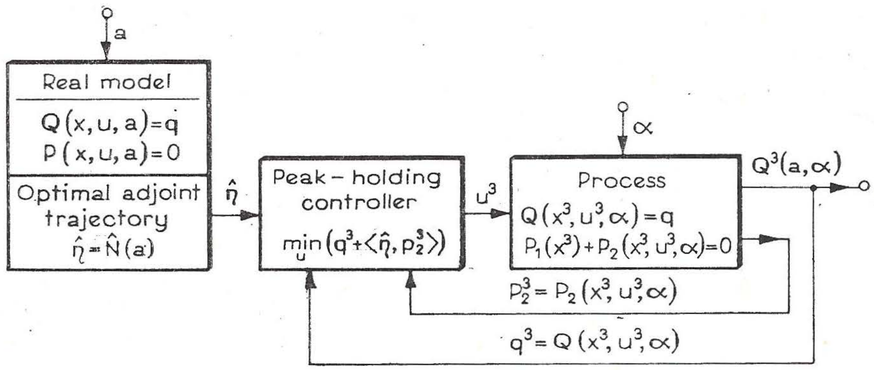

Fig. 8. Optimizing feedback control system (open-loop determination of optimal adjoint trajectory)

Such a structure — see Fig. 8 — is called the (open-loop) optimizing feedback, see [12]. It is not allways applicable, since we must measure not only $P_2(x, u, \alpha)$ but also $Q(x, u, \alpha)$ in order to construct an optimizing feedback controller. Howver, in some applications (particularly if a Hamiltonian function can be maximized at each instant of time) the structure is particularly effective. The structural variations and sensitivity operators control, are:

$$\begin{aligned}
&\delta u^3 = -(\hat{L}_{uu} + \hat{L}_{ux}\hat{L}_{\eta x}^{-1}\hat{L}_{\eta u})^{-1}\hat{L}_{u\eta}, \quad U_a^3 = -(\hat{L}_{uu} + \hat{L}_{ux}\hat{L}_{\eta x}^{-1}\hat{L}_{\eta u})^{-1}\hat{L}_{u\eta}\hat{N}_a \\
&\delta x^3 = \hat{P}_x^{-1}\hat{P}_u(\hat{L}_{uu} + \hat{L}_{ux}\hat{L}_{\eta x}^{-1}\hat{L}_{\eta u})^{-1}\hat{L}_{u\eta}\delta\hat{\eta}; \\
&X_a^3 = \hat{L}_{\eta x}^{-1}\hat{L}_{\eta u}(\hat{L}_{uu} + \hat{L}_{ux}\hat{L}_{\eta x}^{-1}\hat{L}_{\eta u})^{-1}\hat{L}_{u\eta}\hat{N}_a,
\end{aligned} \tag{39a}$$

where $\delta\hat{\eta}$ and $\hat{N}_a$ are the basic adjoint variation (11c) and the corresponding basic optimal adjoint sensitivity operator. Computationaly, we do not use the closed expressions (39), but solve the system of equations

$$\delta u^3 = -\hat{L}_{uu}^{-1}(\hat{L}_{ux}\delta x^3 + \hat{L}_{u\eta}\delta\hat{\eta}); \; \hat{L}_{\eta x}\delta x^3 + \hat{L}_{\eta u}\delta u^3 = \Theta . \tag{39b}$$

$i=4$ (*closed-loop optimizing feedback*).

The adjoint variable $\hat{\eta}$ in the optimizing feedback structure can be determined as well in the closed-loop, as a function of the current state. Since the basic variation $\delta\hat{\eta}$ is then determined by (15), we get

$$\begin{aligned}
&\delta u^4 = -(\hat{L}_{uu} + (\hat{L}_{ux} + \hat{L}_{u\eta}K)\hat{L}_{\eta x}^{-1}\hat{L}_{\eta u})^{-1}\hat{L}_{u\eta}M\delta a; \\
&U_a^4 = -(\hat{L}_{uu} + (\hat{L}_{ux} + \hat{L}_{u\eta}K)\hat{L}_{\eta x}^{-1}\hat{L}_{\eta u})^{-1}\hat{L}_{u\eta}M; \\
&\delta x^4 = \hat{L}_{\eta x}^{-1}\hat{L}_{\eta u}(\hat{L}_{uu} + (\hat{L}_{ux} + \hat{L}_{u\eta}K)\hat{L}_{\eta x}^{-1}\hat{L}_{\eta u})^{-1}\hat{L}_{u\eta}M\delta a; \\
&X_a^4 = \hat{L}_{\eta x}^{-1}\hat{L}_{\eta u}(\hat{L}_{uu} + (\hat{L}_{ux} + \hat{L}_{u\eta}K)\hat{L}_{\eta x}^{-1}\hat{L}_{\eta u})^{-1}\hat{L}_{u\eta}M;
\end{aligned} \tag{40a}$$

or, equivalently, we solve the system of equations

$$\delta u^4 = -\hat{L}_{uu}^{-1} \left((\hat{L}_{ux} + \hat{L}_{u\eta} K) \delta x^4 + \hat{L}_{u\eta} M \delta a \right);$$
$$\hat{L}_{\eta x} \delta x^4 + \hat{L}_{\eta u} \delta u^4 = \Theta. \tag{40b}$$

The structures presented above are only examples of many possible structures. It is impossible to compare their general sensitivity properties. However, some heuristic rules exist. If the performance functional is related mostly to the cost of the open-loop structure (or, if applicable, the open-loop optimizing feddback) are usually the least sensitive. If the cost related to the state trajectory dominates in the performance functional, the optimal trajectory tracking structure is usually the best. There are cases when closedloop or the closed-loop optimizing feedback are very effective; but we cannot say in general that the closed-loop has lower performance sensitivity than the openloop (see e.g. [12]).

5. An example

The general approach presented above has been applied to several optimal control systems of industrial importance. For example, the optimizing feedback structures have low sensitivity when applied to the optimal control of energy supply to an arc furnace in steel industry [3]; when investigating the optimal control of a natural gas pipe-line supply system, described by partial differential equations, a special structure of optimal boundary conditions tracking system is advantageous. However, each optimization problem of industrial importance has its own, specific properties; to present such a problem in full detail would require much space. Therefore, we consider here only a very simplified example of an environment control problem [2].

The problem consists of minimizing the effects of industrial pollution in a geographical area, taking into account the cost of anti-pollution equipement. The model has the form

$$\dot{x}(t) = Ax(t) + Bu(t); \; x(t_0) = x_0,$$
$$Q(x, u) = \int_{t_0}^{t_1} [0.5 (u(t) - u_m)^T C (u(t) - u_m) + \exp(h^T x(t))] \, dt, \tag{41}$$

where $x = (x_1, \ldots, x_n)$ represents the accumulated pollutants, $u = (u_1, \ldots, u_k)$ corresponds to the industrial waste generated by each factory, and $u_m = (u_{m1}, \ldots, u_{mk})$ is the waste generated without any anti-pollution equipement. The matrix $A = \mathrm{diag}\,\{a_i\}$ reflects the dynamic properties of pollutant accumulation and decomposition. The matrix B represents the content of pollutants in the industrial waste; the matrix $C = \mathrm{diag}\,\{c_i\}$ is related to the cost of anti-pollution equipement, and the vector h consists of coefficients estimating the social losses related to each type of pollutant.

The problem was solved numerically by a decomposition and coordination method [2]. The convergence properties of the optimization procedure and the optimal solutions are illustrated in Fig. 9 for $n=2$, $k=2$. In the sensitivity analysis, the parameters u_{mi}, a_i were considered as the least certain; hence $\alpha = (u_{m1}, \ldots, u_{mk}, a_1, \ldots, a_n)$. Since it was possible to determine the matrices $\hat{X}_a(t)$, $\hat{U}_a(t)$, $\hat{N}_a(t)$, $K(t)$, $M(t)$ at the optimal solution, the corresponding matrices of the second-order sensitivity coefficients S^i_{aa} were found for several optimal system struc-

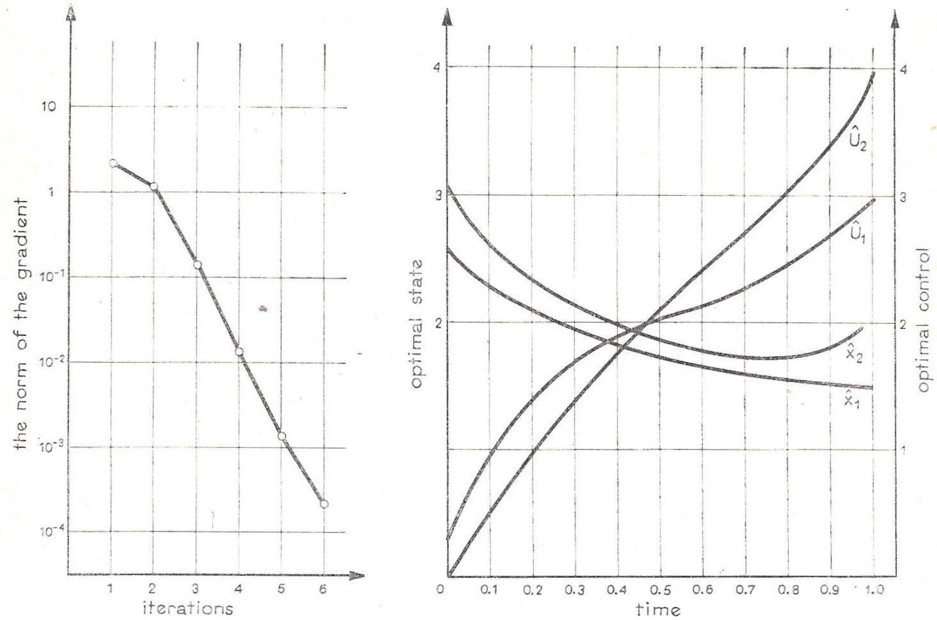

Fig. 9. The convergence of optimization procedure and the optimal solutions for the problem of environmental control

tures by suitable integration. For a given vector $\Delta\alpha$, the corresponding performance loss were determined:

$$L^i = \frac{0.5}{\hat{Q}} \Delta\alpha^T S^i_{aa} \Delta\alpha. \tag{42}$$

The results obtained for two cases of assumed parameters values are given in Table. In both cases the best structure was the closed-loop optimizing feedback. This result has the following interpretation for the problem considered: the most effective (the least sensitive) control system structure consists of predetermining the dependence of the adjoint variables η on the state x, measuring the state in the real process and maximizing the Hamiltonian function in the real process:

$$H(\eta, x, u, t) = -0.5(u-u_m)^T C(u-u_m) + \eta^T Bu + \eta^T [Ax - \exp(h^T x)]. \tag{42}$$

But the matrix C is diagonal; hence each factory can minimize its own goal-function

$$g_i(u_i) = 0.5\, c_i (u_i - u_{mi})^2 - (\eta^T b_i)\, u_i, \tag{43}$$

where b_i is a column of B and η can be interpreted as the vector of shadow-prices related to pollutants; they change in time and in dependence of the measured amount of the accumulated pollutants x. The exact dependence of the shadow-prices η on x may be difficult to determine; however, given $\hat{x}$, $\hat{\eta}$, we can use the approximation $\eta(t)-\hat{\eta}(t)=K(t)\left(x(t)-\hat{x}(t)\right)$. These considerations result in a hierarchical structure of the control system, presented in Fig. 10. The coordination level defines the

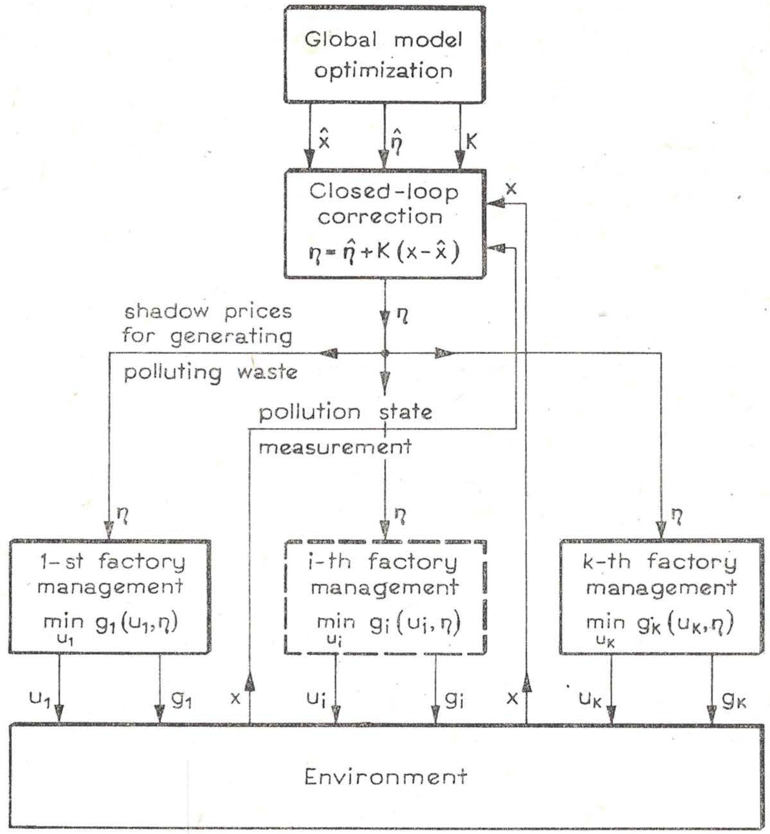

Fig. 10. The hierarchical structure of environmental control (closed-loop optimizing feedback structure) resulting from sensitivity considerations

optimal trajectory $\hat{x}$, the optimal adjoint trajectory $\hat{\eta}$ (the optimal shadow prices) and the Riccati matrix K (the correction coefficients for shadow prices) on the basis of a global model. The state x is measured and the corrected shadow prices are determined, also in the coordination level. The local controllers determine the actual control u_i on the basis of the given shadow-prices, by minimizing the local goal-functions g_i.

The example presented here is clearly rather simplified: we could represent in the model the accumulation of antipollution investments by introducing additional state variables, choose more adequate performance functionals etc. However, the main goal of the example was to illustrate the comparison and interpretation of several control system structures, resulting from the general sensitivity analysis approach.

Table. Sensitivity coefficients and estimators of sensitivity measure for the problem of environment control ($t_0=0$, $t_1=1$, $n=2$, $k=2$)

$A=\begin{bmatrix}1.0 & 0\\ 0 & 1.5\end{bmatrix}$; $B=\begin{bmatrix}0.5 & 0\\ 0 & 1.0\end{bmatrix}$; $C=\begin{bmatrix}1.0 & 0\\ 0 & 1.5\end{bmatrix}$; $h=\begin{bmatrix}0.5\\ 0.8\end{bmatrix}$; $u_m=\begin{bmatrix}3.0\\ 4.0\end{bmatrix}$; $x(t_0)=\begin{bmatrix}2.5\\ 3.0\end{bmatrix}$ $\Delta\alpha=[5\ 4\ 30\ 20]^T$(in %)						$A=\begin{bmatrix}2.0 & 0\\ 0 & 8.0\end{bmatrix}$; $B=\begin{bmatrix}5.0 & 0\\ 0 & 1.0\end{bmatrix}$; $C=\begin{bmatrix}8.0 & 0\\ 0 & 2.0\end{bmatrix}$; $h=\begin{bmatrix}0.1\\ 0.5\end{bmatrix}$; $u_m=\begin{bmatrix}4.0\\ 1.0\end{bmatrix}$; $x(t_0)=\begin{bmatrix}2.0\\ 1.0\end{bmatrix}$ $\Delta\alpha=[5.0\ 1.25\ 1.25\ 5.0]^T$ (in %)				
Control law	S^i_{aa}				L^i in %	S^i_{aa}				L^i in %
Open-loop	2.18	—0.34	0.39	0.39		9.89	—0.03	0.05	0.10	
	—0.34	0.39	0.40	0.70	5.28	—0.03	10^{-4}	$3\cdot10^{-5}$	$2\cdot10^{-4}$	4.40
	0.39	0.40	0.94	0.14		—0.05	$3\cdot10^{-5}$	0.03	0.08	
	0.39	0.70	0.14	0.22		0.10	$2\cdot10^{-4}$	0.08	$2\cdot10^{-3}$	
Closed-loop	2.18	—0.34	0.29	0.22		9.89	—0.03	0.04	0.08	
	—0.34	0.39	0.34	0.57	4.26	—0.03	10^{-4}	$5\cdot10^{-5}$	$7\cdot10^{-5}$	4.36
	0.29	0.34	0.63	0.90		0.04	$5\cdot10^{-5}$	0,03	0.03	
	0.22	0.57	0.90	0.13		0.08	$7\cdot10^{-5}$	0.03	0.04	
Optimal trajectory tracking	2.18	—0.34	3.86	3.16		9.89	—0.03	10.3	0.57	
	—0.34	0.39	—0.41	—0.08	15.2	—0.03	10^{-4}	0.03	10^{-3}	12.2
	3.86	—0.41	15.8	1.33		10.3	0.03	1.47	0.19	
	3.16	—0.08	1.33	0.87		0.57	10^{-3}	0.19	$3\cdot10^{-3}$	
Open-loop optimizing feedback	2.55	—0.82	0.53	0.78		10.2	—1.94	0.02	0.05	
	—0.82	3.22	0.78	1.13	8.80	—1.92	2.00	0.06	0.01	3.73
	0.53	0.78	0.84	1.24		0.02	0.06	0.05	0.09	
	0.78	1.13	1.24	0.18		0.05	0.01	0.09	0.01	
Closed-loop optimizing feedback	0.06	0.15	0.15	0.26		$3\cdot10^{-4}$	$2\cdot10^{-4}$	$9\cdot10^{-5}$	$2\cdot10^{-4}$	
	0.15	0.39	0.39	0.66	3.51	$2\cdot10^{-4}$	10^{-4}	$8\cdot10^{-5}$	$2\cdot10^{-4}$	0.04
	0.15	0.39	0.84	1.24		$9\cdot10^{-4}$	$8\cdot10^{-5}$	$5\cdot10^{-2}$	$9\cdot10^{-2}$	
	0.26	0.66	1.24	0.18		$2\cdot10^{-4}$	$2\cdot10^{-4}$	$9\cdot10^{-2}$	$2\cdot10^{-3}$	

6. Conclusions

The paper presents a general methodology of the ideal and real sensitivity analysis of optimal control systems in various structures. Processes described by ordinary — and partial differential equations, difference, difference-differential and integral equations can be analysed uniformly in this general approach. The approach has been recently applied by one of the authors to solve two important infinite-dimensional problems: the problem of changing the structure of a difference-differential model by neglecting small time delays and the problem of neglecting small parameters changing a system of differential equations in Banach space — see [14], [15].

Moreover, the computational methods of sensitivity analysis resulting from the general theory are closely related to the known computational methods of optimization. Once the optimal solution is found, the determination of the basic control, state and adjoint variations is equivalent to the solution of a quadratic — linear approximation of the original problem. The determination of structural variations and thus of the second-order sensitivity coefficients is usually rather easy; a possible exception is the computation of closed-loop structural variations for complicated (partial differential, difference-differential) processes, where the solution of the corresponding Riccati equation is rather a difficult task. However, the local sensitivity analysis of closed-loop systems requires less effort than the actual, global synthesis of such systems because it is necessary to perform only the synthesis of a linear approximation of the control law in order to determine the local sensitivity coefficients. Since it is not a priori known whether a closed-loop system would be less sensitive than an open-loop or other structures, it may be advantageous to perform the local sensitivity analysis before deciding to synthesise the closed-loop control law. Moreover, the local sensitivity analysis can provide data for choosing hierarchical control system structures, investigating their feasibility etc.

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Podstawowe zależności w analizie wrażliwości optymalnych układów sterowania

Przedstawiono podstawowe pojęcia i zależności w analizie wrażliwości pierwszego i drugiego rzędu optymalnych układów sterowania opisanych modelami w przestrzeni Banacha. Omówiono dwa podstawowe sformułowania problemu analizy wrażliwości: idealnej analizy wrażliwości na podstawie modelu matematycznego oraz analizy wrażliwości układu sterowania na podstawie dwu modeli matematycznych i pojęcia sterownika w danej strukturze. Podstawowe lematy odnoszące się do tego pojęcia i własności miary wrażliwości przedstawiono dla przypadku ogólnego rozważanego w tym artykule. Omówiono pojęcie operatorów wrażliwości drugiego rzędu, zbadano ich istnienie oraz warunki niewrażliwości pierwszego rzędu. Podano metody obliczania wartości operatorów wrażliwości drugiego rzędu dla kilku struktur układów sterowania. Metody te umożliwiają efektywne porównanie wrażliwości różnych struktur sterowania optymalnego i wybór struktury najlepszej (najmniej czułej) przy niepewnych wartościach parametrów. Przykład ilustruje zastosowanie tych metod do problemu sterowania środowiskiem, co w wyniku daje wybór hierarchicznej struktury układu sterowania.

Основные зависимости при анализе чувствительности оптимальных систем управления

Статья содержит основные понятия и зависимости анализа чувствительности первого и второго порядка оптимальных систем управления, описываемых моделями в банаховом пространстве. Рассмотрены две основные формулировки проблемы анализа чувствительности: идеального анализа чувствительности на основе математической модели и понятия управления в данной структуре. Для общего случая, рассмотриваемого в данной статье, представлены основные леммы, касающиеся этого понятия и свойств меры чувствительности. Рассмотрено понятие операторов чувствительности второго порядка, исследованы условия их существования и условия нечувствительности первого порядка. Даны методы вычисления значений операторов чувствительности второго порядка для нескольких структур систем управления. Эти методы позваляют эффективно сравнивать чувствительность разных структур оптимального управления и выбирать наилучшие структуры (наименее чувствительные) при неопределенных значениях параметров. Дан пример, иллюстрирующий применение этих методов к проблеме управления средой, что в результате дает возможность выбора иерархической структуры системы управления.

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