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On parametric optimal control for a class of linear and quasilinear equations of parabolic type

by

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In the paper a problem of minimization of a cost functional, which is defined on a class of the solutions of quasilinear parabolic equation (state equation) is considered. Coefficients of the state equation depend on a functional parameter. An optimal parameter corresponds to the state trajectory which minimizes the cost functional over a set of admissible parameters.

In section 3 sufficient conditions for existence of an optimal parameter are given.

In section 4 sufficient conditions for differentiability of the cost functional are stated. In order to determine effectively the gradient of the cost functional so called generalized adjoint state equation is introduced. By such approach we do not need use Green formula to compute the gradient.

The generalized adjoint state is use to obtain necessary conditions of optimality.

Introduction

In the paper we consider a problem of parametric optimization for a class of quasilinear and linear partial differential equations of parabolic type.

Problems of such types appear in technology of solid state devices [6] — where coefficients of diffusion equation depend on temperature as a parameter.

In parametric optimization problem we look for optimal functional parameter (changes of temperature) which minimizes a certain cost functional $J(\Theta)$ over a set of admissible parameters. Functional $J(\Theta)$ is defined on a class of solution of a diffusion equation (state equation).

In our problem cost functional will depend on parameter Θ by means of the value of the solution at the terminal time of the diffusion process. It means that we are interested in final result of the process. In special case of one dimensional linear diffusion equation (heat equation) a problem of this type was solved in [6] by applying Fourier series technique, but such approach can be used only in the particular case. Some more general results for special type of cost functional and for linear state equations are given in [5]. The structure of the paper is the following:

In Section 1 we introduce some basic mathematical concepts which will be used throughout this paper.

In Section 2 we recall some results from theory of parabolic equation which are basic for our purposes.

In Section 3 a problem of parametric optimization is stated. Then an existence theorem for the problem is given.

In Section 4 sufficient conditions for differentiability of functional $J(\Theta)$ are considered. In this section we introduce so called generalized adjoint state equation and formulate necessary conditions of optimality. Usually a simple form of the gradient has been obtained using the Green formula, which in many cases has been applied only formally. The verification of the validity of the Green formula is particulary difficult in the case of weak solutions of parabolic equations. The proposed approach allows to avoid these difficulties.

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1. Some functional spaces

Let Ω be a bounded domain in R^n with smooth boundary $\Gamma=\partial\Omega$. For given number $T>0$ we define the following subsets of R^{n+1}:

$$Q=\Omega\times[0, T],$$
$$\Sigma=\Gamma\times[0, T],$$
$$S=\Sigma\cup\{(x, 0)\in R^{n+1}|x\in\Omega\}.$$

We start with short recalling of definitions [1] of some basic, for our purposes, functional spaces. As usual $L^2(\Omega)$ denotes Hilbert space of functions (equivalent classes) $u=u(x)$, $x\in\Omega$, with scalar product

$$(u, v)_{L^2(\Omega)}=\int_\Omega u(x)\,v(x)\,dx.$$

In the same manner as $L^2(\Omega)$ are defined Hilbert spaces $L^2(Q)$, $L^2(\Sigma)$.

Let

$$k=(k_1, \dots, k_n)$$

be a n-tuple of nonnegative integers, we put

$$|k|=k_1+\dots+k_n.$$

For given function $u=u(x)$, $D_x^k u$ denotes its derivative of order $|k|$

$$D_x^k u=\frac{\delta^{|k|} u}{\delta^{k_1} x_1 \dots \delta^{k_n} x_n}, \quad x=(x_1, \dots, x_n).$$

We will need some spaces of continuous functions in $\bar{\Omega}$ (for sets $[0, T]$, $\bar{Q}$, $\bar{\Sigma}$ such spaces are defined analogously), where $\bar{\Omega}$ denotes closure in R^n of the set Ω. $C(\bar{\Omega})$ denotes a Banach space of continuous functions on $\bar{\Omega}$ with the norm

$$\|u\|_{C(\bar{\Omega})}=\sup_{x\in\Omega}|u(x)|=|u|_\Omega^{(0)}$$

For given α, $0<\alpha<1$, $C^{\alpha}(\bar{\Omega})$ denotes Banach space of Hölder continuous funct ons (with exponent α) in $\bar{\Omega}$ with the norm

$$\sup_{\substack{x, x' \in \Omega \\ |x-x'|<\delta}} \frac{|u(x)-u(x')|}{|x-x'|^{\alpha}} = \langle u \rangle_{\Omega}^{(\alpha)}, \quad \delta>0.$$

Let m be an integer $m \geqslant 1$, $C^{m}(\bar{\Omega})$ is a Banach space of functions continuous in $\bar{\Omega}$ together with all derivatives of order $|k| \leqslant m$ and with the norm

$$\|u\|_{C^{m}(\bar{\Omega})} = |u|_{\Omega}^{(m)} = \sum_{|k| \leqslant m} |D_{x}^{k} u|_{\Omega}^{(0)}$$

we will need also spaces of functions

$$u=u(x, t), \ (x \ t) \in \bar{Q}$$

having different properties with respect to x and t.

Space $C^{l, l/2}(\bar{Q})$, $l>0$, $l \neq$ integer.

Remark: In [1] this space is denoted by $H^{l, l/2}(\bar{Q})$ but usual $H^{m}(Q)$ denotes Sobolev space of order m.

Definition 1 [1]. $C^{l, l/2}(\bar{Q})$ denotes a Banach space of functions $u=u(x, t)$ continuous in $\bar{Q}$ together with all derivatives $D_{t}^{r} D_{x}^{s} u$ of order $2r+|s|<l$ with the norm:

$$\|u\|_{C^{l, l/2}(\bar{Q})} = \langle u \rangle_{x, Q}^{(l)} + \langle u \rangle_{t, Q}^{(l, 2)} + \sum_{j=0}^{[l]} \langle u \rangle_{Q}^{(j)},$$

where

$$[l] = \text{entier}\,(l)$$

$$\langle u \rangle_{Q}^{(0)} = \max_{x \in Q} |u(x)| = |u|_{Q}^{(0)},$$

$$\langle u \rangle_{Q}^{(j)} = \sum_{2r+|s|=j} |D_{t}^{r} D_{x}^{s} u|_{Q}^{(0)},$$

$$\langle u \rangle_{x, Q}^{(l)} = \sum_{2r+|s|=l} \langle D_{t}^{r} D_{x}^{s} u \rangle_{x, Q}^{(l-[l])},$$

$$\langle u \rangle_{t, Q}^{(l/2)} = 0 < l-2r-|s| < 2 \, \langle D_{t}^{r} D_{x}^{s} u \rangle_{t, Q}^{\left(\frac{l-2r-|s|}{2}\right)},$$

for $0<\alpha<1$

$$\langle u \rangle_{x, Q}^{(\alpha)} = \sup_{\substack{(x, t), (x', t) \in Q \\ |x-x'|<\delta}} \frac{|u(x, t)-u(x, t')|}{|x-x'|^{\alpha}},$$

$$\langle u \rangle_{t, Q}^{(\alpha)} = \sup_{\substack{(x, t), (x, t') \in Q \\ |t-t'|<\delta}} \frac{|u(x, t)-u(x, t')|}{|t-t'|^{\alpha}}.$$

For $\bar{l}=2+\alpha$, $0<\alpha<1$, space $C^{\bar{l}, \bar{l}/2}(\bar{Q})$ we denote by $C^{2+\alpha, 1+\alpha/2}(\bar{Q})$.

If in Definition 1 the set $\bar{Q}$ is replaced by $\bar{\Sigma}$ we have the same definition for spaces $C^{\bar{l}, \bar{l}/2}(\bar{\Sigma})$.

$C^{2,1}(\bar{Q})$ denotes a Banach space of functions continuous in $\bar{Q}$ together with all derivatives $D_t^r D_x^s u$ order $2r+|s|\leqslant 2$ with norm

$$\|u\|_{C^{2,1}(\bar{Q})}=\sum_{2r+|s|\leqslant 2} |D_t^r D_x^s u|_Q^{(o)}.$$

2. Quasilinear parabolic equation

In this section we recall some results from theory of partial differential equations of parabolic type which will be useful for our purposes.

Let us consider the following quasilinear parabolic equation with associated boundary and initial conditions:

$$\frac{\partial u}{\partial t}-\sum_{i,j=1}^{n} a_{ij}(x,t,u)\frac{\partial^2 u}{\partial x_i\,\partial x_j}+b\left(x,t,u,\frac{\partial u}{\partial x_1},\dots,\frac{\partial u}{\partial x_n}\right)=0, \tag{2.1}$$

$$(x,t)\in Q,$$

$$\sum_{i,j=1}^{n} a_{ij}(x,t,u)\frac{\partial u}{\partial x_i}\gamma_j+\psi(x,t,u)=0, \quad (x,t)\in\Sigma, \tag{2.2}$$

$$u(x,0)=u_0(x),\ x\in\Omega. \tag{2.3}$$

In (2.2) $\gamma=(\gamma_1,\dots,\gamma_n)$ denotes the unit exterior normal to $\Gamma=\partial\Omega$

For n-tuple $p=(p_1,\dots,p_n)$ we denote

$$p=\left(\sum_{i=1}^{n} p_i^2\right)^{1/2}, \quad p^2=|p|^2.$$

Now we make several assumptions which will be referred to as Conditions Z1, Z2.

Condition Z1

(i) for $(x,t)\in\bar{\Omega}x[0,T]$, $u\in R$, $\xi\in R^n$:

$$0\leqslant\sum_{i,j=1}^{n} a_{ij}(x,t,u)\,\xi_i\,\xi_j\leqslant u_1\,\xi^2; \tag{2.4}$$

(ii) for $(x,t)\in\bar{Q}/S$, $u\in R$, $p\in R^n$:

$$-ub(x,t,u,p)\leqslant c_0 p^2+c_1 u^2+c_2;$$

(iii) for $(x,t)\in\Sigma$, $u\in R$, $\xi\in R^n$:

$$v_1\,\xi^2\leqslant\sum_{i,j=1}^{n} a_{ij}(x,t,u)\,\xi_i\,\xi_j,$$

$$-u\psi(x,t,u)\leqslant c_3\,u^2+c_4,$$

where u_1, $v_1=\text{const.}>0$, $c_i=\text{const.}$, $i=0,\dots,4$.

Condition Z2 (2.5)

(i) for $(x, t) \in \bar{Q}$, $|u| \leqslant M$, M — given constant, $\xi \in R^n$, $p \in R^n$:

$$\nu\xi^2 \leqslant \sum_{i,j=1}^{n} a_{ij}(x, t, u)\,\xi_i\xi_j \leqslant \mu\xi^2, \qquad \nu > 0,$$

$$\left|\frac{\partial a_{ij}}{\partial u}, \frac{\partial a_{ij}}{\partial x}, \frac{\partial a_{ij}}{\partial t}\right| \leqslant \mu,$$

$$\left|\frac{\partial^2 a_{ij}}{\partial u^2}, \frac{\partial^2 a_{ij}}{\partial u \partial t}, \frac{\partial^2 a_{ij}}{\partial u\,\partial x}, \frac{\partial^2 a_{ij}}{\partial x\,\partial t}\right| \leqslant \mu,$$

$$|b(x, t, u, p)| \leqslant \mu(1+p^2),$$

$$\left|\frac{\partial b}{\partial p}\right|(1+|p|+\left|\frac{\partial b}{\partial u}\right|+\left|\frac{\partial b}{\partial t}\right| \leqslant \mu(1+p^2);$$

(ii) for $(x, t) \in \bar{\Sigma}$, $|u| \leqslant M$:

$$\left|\psi, \frac{\partial \psi}{\partial x}, \frac{\partial \psi}{\partial t}, \frac{\partial \psi}{\partial u}\right| \leqslant \mu,$$

$$\left|\frac{\partial^2 \psi}{\partial u^2}, \frac{\partial^2 \psi}{\partial u\,\partial x}, \frac{\partial^2 \psi}{\partial u\,\partial t}\right| \leqslant \mu;$$

(iii) boundary $\Gamma = \partial\Omega$ is smooth enough — say of class C^2 (definition [1] p. 18)

The following two lemmas proved in [1] give some a priori bounds for solution of the problem (2.1)-(2.3).

Lemma 1. Let $u = u(x, t) \in C^{2,1}(\bar{Q})$ be a solution to the problem (2.1)-(2.3). Assume that Condition Z1 is satisfied. Then

$$\max_{Q} |u(x, t)| \leqslant \lambda_1 e^{\lambda T} \max\{\sqrt{c_2}, \sqrt{c_4}, \max |u(x, 0)|\}, \tag{2.6}$$

where constants λ_1, λ depend only on ν_1, μ_1, c_0, c_1, c_3 and boundary $\Gamma = \partial\Omega$. Proof is given in [1] p. 555.

Lemma 2. Let $u = u(x, t) \in C^{2,1}(\bar{Q})$ be a solution to the problem (2.1)-(2.3) with $\max_{Q} |u| \leqslant M$. Assume Condition Z2. Then

$$\max_{Q} |u_x| \leqslant M_1 \tag{2.7}$$

where

$$u_x = \left(\frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_n}\right).$$

Constant M_1 depend only on M, ν, μ, $\|u_0\|_{C^2(\bar{\Omega})}$ and boundary $\Gamma = \partial\Omega$.

Proof is given in [1] p. 554.

The following Theorem 1 gives sufficient conditions for existence of an solution to the problem (2.1), (2.2), (2.3)′

$$u(x, 0) = 0, \quad x \in \Omega. \tag{2.3$'$}$$

THEOREM 1. Assumptions:

(i) Assume conditions Z1, Z2 — then for all solution $u \in C^{2,1}(\bar{Q})$ of the problem (2.1)-(2.3) there exists (from Lemmas 1.2). Constants M, M_1 such that

$$\max_{Q} |u| \leqslant M, \quad \max_{Q} |u_x| \leqslant M_1 .$$

(ii) For $(x, t) \in \bar{Q}((x, t) \in \bar{\Sigma})$, $|u| \leqslant M$, $p \in R^n$ functions $a_{ij}(x, t, u)$, $b(x, t, u, p)$, $(\psi(x, t, u))$ are continuous together with all derivatives which were introduced in Conditions Z1, Z2.

(iii) For $(x, t) \in \bar{Q}$ $((x, t) \in \bar{\Sigma})$, $|u| \leqslant M$, $|p| \leqslant M_1$ the following functions are uniformly Hölder continuous (u.h.c):

$$\frac{\partial a_{ij}}{\partial x}(\cdot, t, u),\ b(\cdot, t, u, p),\ \left(\frac{\partial \psi}{\partial x}(\cdot, t, u)\right) \text{ u.h.c (exponent } \beta),$$

$$\frac{\partial \psi}{\partial x}(x, \cdot, u) \text{ u.h.c. (exponent } \beta/2).$$

$$\psi(x, 0, 0) = 0$$

(iv) Boundary Γ is of the class $C^{2+\beta}$ then there exists an unique solution

$$u = u(x, t) \in C^{2+\beta\ 1+\beta/2}(\bar{Q})$$

to the problem (2.1), (2.2), (2.3).

Proof is given in [1].

In particular case of linear parabolic equation:

$$\frac{\partial u}{\partial t} - \sum_{i,j=1}^{n} a_{ij}(x, t) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} a_i(x, t) \frac{\partial u}{\partial x_i} + $$
$$+ a(x, t)\, u = f(x, t),\ (x, t) \in Q \qquad (2.7)'$$

with boundary and initial conditions

$$\sum_{i=1}^{n} c_i(x, t) \frac{\partial u}{\partial x_i} + \psi(x, t) u = \varphi(x, t),\ (x, t) \in \Sigma \qquad (2.8)$$

$$u(x, 0) = u_0(x),\ x \in \Omega \qquad (2.9)$$

for existence we need weaker assumptions then in the case of the problem (2.1)-(2.3).

THEOREM 2. ([1], p. 364). Assume

(i) $c\gamma > 0$ for $(x, t) \in \bar{\Sigma}$, where $c = (c_1, \ldots, c_n)$;

(ii) for $(x, t) \in \bar{Q}$, $\xi \in R^n$:

$$\nu\xi^2 \leqslant \sum_{i,j=1}^{n} a_{ij}\, \xi_i\, \xi_j \leqslant \mu\xi^2,\ \nu > 0;$$

(iii) boundary $\Gamma=\partial\Omega$ belongs to the class C^{l+2}, for some $l>0$;
(iv) $a_{ij}, a_i, a, f \in C^{l,l/2}(\bar{Q})$

$$c_i, \psi, \varphi \in C^{1+l, \frac{1+l}{2}}(\bar{\Sigma})$$

$$u_0 \in C^{l+2}(\bar{\Omega})$$

and functions f, φ, u_0 satisfy to compability condition of order zero, then there exists the unique solution

$$u \in C^{l+2, 1+l/2}(\bar{Q})$$

to the problem (2.7)-(2.9) with an priori bound

$$\|u\|_{C^{l+2,1+l/2}(\bar{Q})} \leqslant C\left(\|f\|_{C^{l,l/2}(\bar{Q})} + \|\varphi\|_{C^{l+1,\frac{l+1}{2}}(\bar{\Sigma})} + \|u_0\|_{C^{l+2}(\bar{\Omega})}\right) \tag{2.10}$$

Proof is given in [1].

3. Quasilinear state equation

Let us consider a control system described by quasilinear parabolic equation (state equation)

$$A(\Theta, y) = \frac{\partial y}{\partial t} - \sum_{i,j=1}^{n} a_{ij}(x, t, \Theta, y)\frac{\partial^2 y}{\partial x_i \partial x_j} + $$

$$+ b\left(x, t, \Theta, y, \frac{\partial y}{\partial x_1}, \dots, \frac{\partial y}{\partial x_n}\right) = 0, \quad (x, t) \in Q \tag{3.1}$$

with boundary condition:

$$B(\Theta, y) = \sum_{i,j=1}^{n} a_{ij}(x, t, \Theta, y)\frac{\partial y}{\partial x_j}\gamma_i + \psi(x, t, \Theta, y) = 0, \quad (x, t) \in \Sigma \tag{3.2}$$

and initial condition:

$$y(x, 0) = y_0(x), \ x \in \Omega, \tag{3.3}$$

where $a_{ij}, b, \psi, y_0, \Theta$ are given functions.

Function $\Theta=\Theta(x, t)$, $(x, t) \in Q$ is a parameter which depends on control. In particular case parameter Θ may depend only on t. For a given parameter Θ we denote solution of (3.1)-(3.3) (state trajectory) by

$$y(\Theta) = y(\Theta)(x, t), \ (x, t) \in \bar{Q}.$$

Let U denote a given set of admissible parameters. Let $I(\cdot)$ denote a given continuous functional on Hilbert space $L^2(\Omega)$. We define cost functional as

$$J(\Theta) = I(y(\Theta)(x, T)), \tag{3.4}$$

where $y(\Theta)(x, T)$ denotes terminal state.

Functional (3.4) is well defined on the set U if state trajectory satisfies the following regularity conditions

$$y(\Theta)(x, T) \in L^2(\Omega). \tag{3.5}$$

The functional $J(\Theta)$ is required to be minimized over the set U. That is, we search for a $\hat{\Theta} \in U$ such that

$$J(\hat{\Theta}) = \min_{\Theta \in U} J(\Theta). \tag{3.6}$$

An element $\hat{\Theta} \in U$ at which the minimum is attained will be called an optimal parameter. Let the set $\mathcal{U} \subset U$ where $\mathcal{U}$ denotes linear topological space. Sufficient conditions for existence of an optimal parameter are for example the following:

(i) mapping given by the state equation

$$\mathcal{U} \supset U \ni \Theta \mapsto y(\Theta)(x, T) \in L^2(\Omega) \tag{3.7}$$

is continuous;

(ii) set U is compact in $\mathcal{U}$ topology.

In the case where parameter Θ depends on a control v, for example

$$\Theta = Lv, \tag{3.8}$$

where $v \in \tilde{U} \subset \tilde{\mathcal{U}}$, U is a Hilbert space of control functions, set $\tilde{U}$ is a given subset of admissible controls and L is a given linear mapping from $\tilde{\mathcal{U}}$ into $\mathcal{U}$, we can introduce another, cost functional $\tilde{J}(v)$ defined by equality

$$\tilde{J}(v) = J(Lv),\ v \in \tilde{U},\ Lv \in U. \tag{3.9}$$

If we assume that:

(i) mapping given by the state equation

$$\tilde{\mathcal{U}} \supset \tilde{U} \in v \mapsto y(Lv)(x, T) \in L^2(\Omega) \tag{3.10}$$

is continuous,

(ii) operator L is compact as a mapping from $\tilde{U}$ into U and set $\tilde{U}$ is compact in weak topology of the space $\tilde{\mathcal{U}}$ (we assume, that $\tilde{\mathcal{U}}$ is a Hilbert space), then the conditions (3.7) are satisfied, hence there exists an optimal control $\bar{v} \in \tilde{U}$ such that

$$\tilde{J}(v) = \min_{v \in U} J(Lv). \tag{3.11}$$

Note, that if assumptions (3.10), (3.11) hold then functional (3.9) is continuous in weak topology of the space $\tilde{\mathcal{U}}$.

Let us assume that Θ is a scalar function and denote by $K \subset R$ the set of all values of $\Theta(x, t)$ for all $(x, t) \in \bar{Q}$ and $\Theta \in U$. Then the following theorem gives sufficient conditions under which (3.7) is satisfied.

THEOREM 3. Assume

(i) $U \subset C(\bar{Q})$.

(ii) For all parameters $\Theta \in U$ state trajectory $y(\Theta) \in C^{2+\beta,\, 1+\beta/2}(\bar{Q})$ for some $\beta \geqslant 0$.

(iii) Assumptions Z1, Z2 are fulfilled for all $\Theta \in U$; For given $\bar{\Theta} \in U$ it means, that coefficients of quasilinear parabolic equation

$$\begin{aligned} &A_{\bar{\Theta}}\left(x, t, y, \frac{\partial y}{\partial x_i}, \frac{\partial^2 y}{\partial x_i, \partial x_j}\right) = A(\bar{\Theta}, y) = 0, \quad (x, t) \in Q, \qquad i, j = 1, \ldots, n \\ &B_{\bar{\Theta}}\left(x, t, y, \frac{\partial y}{\partial x_i}\right) = B(\bar{\Theta}, y) = 0, \qquad (x, t) \in \Sigma \\ &y(x, 0) = y_0(x), \qquad x \in \Omega \end{aligned} \tag{3.12}$$

satisfy Assumptions Z1, Z2.

(iv) Coefficients of state equation (3.1)-(3.3) are C^1 functions with respect to Θ on the set K for all $(x, t) \in \bar{Q}$, all u, p with $|u| \leqslant M$, $|p| \leqslant M_1$, M, M_1 are constants from Lemmas 1, 2.

Then the mapping given by state equation

$$C(\bar{Q}) \supset U \ni \Theta \mapsto y(\Theta) \in C(\bar{Q}) \tag{3.13}$$

is continuous.

Proof is given in Appendix.

4. Necessary conditions of optimality

In this section we assume for the sake of simplicity, that parameters Θ are functions of one variable t only and set U is a convex and bounded subset of Banach space $C^{1+\beta}(0, T)$ for some $\beta > 0$. Moreover it is assumed that for any given parameter $\Theta \in U$ state trajectory $y(\Theta)$ is an element of Banach space $C^{2+\beta, 1+\beta/2}(\bar{Q})$ for some $\beta > 0$.

Conditions under which the above regularity condition holds one can determine using Theorem 1.

In order to assure the existence of an optimal parameter it is assumed that U is a compact set in $C(0, T)$ topology.

In the first part of this section we will consider the problem of differentiability of functional (3.4). In the next part we will give the necessary conditions of optimality for problem (3.6).

It is easy to see that there exists the gradient of functional (3.4) at a point $\bar{\Theta} \in U$ if the following two conditions hold:

(i) functional $I(y)$ is differentiable and its gradient at point $y(\bar{\Theta})(x, T)$ fulfills the conditions:

$$\frac{dI}{dy}(y(\bar{\Theta})(x, T)) \in L^2(\Omega), \tag{4.1}$$

(ii) there exists Frechet derivative

$$\frac{dy}{d\Theta}(\bar{\Theta}): C^{1+\beta}(0, T) \ni \delta\Theta \mapsto \delta y \in C^{2+\beta, 1+\beta/2}(\bar{Q}) \tag{4.2}$$

of the mapping generated by state equation (3.1)-(3.3)

$$C^{1+\beta}(0, T) \supset U \ni \Theta \mapsto y(\Theta) \in C^{2+\beta, 1+\beta/2}(\bar{Q}). \tag{4.3}$$

If the Frechet differential δy exists, it is given by the solution of the linear parabolic equation:

$$\frac{\partial A}{\partial y}(\bar{\Theta}, y, (\bar{\Theta}))\, \delta y + \frac{\partial A}{\partial \Theta}(\bar{\Theta}, y(\bar{\Theta}))\, \delta\Theta = 0, \qquad (x, t) \in Q \tag{4.4}$$

with boundary condition:

$$\frac{\partial B}{\partial y}(\bar{\Theta}, y(\bar{\Theta}))\, \delta y + \frac{\partial B}{\partial \Theta}(\bar{\Theta}, y(\bar{\Theta}))\delta\Theta = 0, \qquad (x, t) \in \Sigma \tag{4.5}$$

and homogeneous initial condition:

$$\delta y(x, 0) = 0, \qquad x \in \Omega. \tag{4.6}$$

If the above conditions (i), (ii) hold, than the gradient of functional (3.4) at point $\bar{\Theta} \in U$ is given by

$$\left\langle \frac{dJ}{d\Theta}(\bar{\Theta}); \delta\Theta \right\rangle = \int_{\Omega} \frac{dI}{dy}(y(\bar{\Theta})(x, T)\left(\frac{dy}{d\Theta}(\bar{\Theta})\, \delta\Theta\right)(x, T) d\Omega \tag{4.7}$$

where the function

$$\delta y(x, T) = \left(\frac{dy}{d\Theta}(\bar{\Theta})\, \delta\Theta\right)(x, T)$$

is the value at the time $t = T$ of the solution of problem (4.4)-(4.6) for a given increment $\delta\Theta \in C^{1+\beta}(0, T)$.

To give sufficient conditions for existence of derivative (4.2) of mapping (4.3) we use the following general implicit function theorem:

THEOREM 4 ([3], p. 193). Let X, Y, Z be Banach spaces and let $\mathscr{H}$ maps an open subset $V \subset X \times Y$ into Z. Assume, that at the point $(\bar{\Theta}, \bar{y}) \in V$ we have

$$\mathscr{H}(\bar{\Theta}, \bar{y}) = 0.$$

Assume also that $\mathscr{H}$ is continuously Frechet differentiable in some neighbourhood of $(\bar{\Theta}, \bar{y})$. If there exists a bounded operator

$$\left(\frac{\partial \mathscr{H}}{\partial y}(\bar{\Theta}, \bar{y})\right)^{-1} \in \mathscr{L}(Z, Y) \tag{4.8}$$

then in some open ball $B(\tilde{\Theta}, r_1) \subset X$ the equation $\mathscr{H}(\Theta, y) = 0$ generates an implicit mapping

$$X \supset B(\bar{\Theta}, r_1) \ni \Theta \mapsto F(\Theta) \in B(\bar{y}, r_2) \subset Y.$$

Mapping F is differentiable at point $\tilde{\Theta}$ and its derivative is given by

$$\frac{dF}{d\Theta}(\bar{\Theta}) = -\left(\frac{\partial \mathscr{H}}{\partial y}(\bar{\Theta}, \bar{y})\right)^{-1} \circ \frac{\partial \mathscr{H}}{\partial \Theta}(\bar{\Theta}, \bar{y})$$

To make use of the above theorem we define:
(i) spaces:

$$X = \{\Theta \in C^{1+\beta}(0, T) | \Theta(0) = 0\},$$
$$Y = \{y \in C^{2+\beta, 1+\beta/2}(\bar{Q}) | y|_{t=0}\},$$
$$Z = Z_1 \times Z_2 \times Z_3,$$

where

$$Z_1 = C^{\beta, \beta/2}(\bar{Q}),$$
$$Z_2 = \{Z_2 \in C^{1+\beta, \frac{1+\beta}{2}}(\bar{\Sigma}) | Z_2|_{t=0} = 0\},$$
$$Z_3 = \{Z_3 \in C^{2+\beta}(\bar{\Omega}) | Z_3(x) = 0, \ x \in \Omega\};$$

(ii) mapping

$$\mathscr{H}(\Theta, y) = \{A(\Theta, y), B(\Theta, y), y|_{t=0}\}. \tag{4.9}$$

Remark. To assure that (4.9) maps $X \times Y$ into Z we have to assume that

$$A(\Theta, y) \in Z_1, \quad \forall \Theta \in U \in X, \quad \forall y \in Y, \tag{4.10}$$
$$B(\Theta, y) \in Z_2, \quad \forall \Theta \in U \subset X, \quad \forall y \in Y. \tag{4.11}$$

If nonlinear mapping $B(\Theta, y)$ has the following property:

$$\{\Theta(0) = 0 \wedge y(x, 0) = 0, \ x \in \Omega\} \Rightarrow B(\Theta, y)(x, 0) = 0, \ x \in \Gamma,$$

then to satisfy (4.10), (4.11) we need only the appropriate regularity of coefficient of operators A, B.

In the case where $\mathscr{H}$ is given by (4.9) the existence of linear mapping (4.8) is equivalent to the existence of the classical solution $\delta y \in Y$ to the following linear parabolic problem

$$\frac{\partial A}{\partial y}(\bar{\Theta}, y(\bar{\Theta}))\, \delta y = f_1, \quad (x, t) \in Q, \tag{4.12}$$

with boundary and initial conditions

$$\frac{\partial B}{\partial y}(\bar{\Theta}, y(\bar{\Theta}))\, \delta y = f_2, \quad (x, t) \in \Sigma, \tag{4.13}$$

$$\delta y(x, 0) = 0, \ x \in \Omega, \tag{4.14}$$

for any functions $f_1 \in Z_1, f_2 \in Z_2$.

Mapping (4.8) is bounded if for the solution y the following estimation holds:

$$\|y\|_Y \leqslant C(\|f_1\|_{Z_1} + \|f_2\|_{Z_2}). \tag{4.15}$$

By Theorem 2 condition (4.15) is satisfied if the following regularity conditions hold:

(i) coefficients of linear operator $\frac{\partial A}{\partial y}(\bar{\Theta}, \bar{y})$ are elements of space $C^{\beta, \beta/2}(\bar{Q})$;

(ii) coefficients of linear operator $\frac{\partial B}{\partial y}(\bar{\Theta}, \bar{y})$ are elements of space $C^{1+\beta, \frac{1+\beta}{2}}(\bar{\Sigma})$;

(iii) $\frac{\partial A}{\partial \Theta}(\bar{\Theta}, \bar{y})\, \delta\Theta \in C^{\beta, \beta/2}(\bar{Q})$, $\forall \delta\Theta \in X$;

(iv) $\frac{\partial B}{\partial \Theta}(\bar{\Theta}, \bar{y})\, \delta\Theta \in C^{1+\beta, \frac{1+\beta}{2}}(\bar{\Sigma})$, $\forall \delta\Theta \in X$.

Therefore from Theorem 4 we obtain the following.

LEMMA 3. Assume that (4.10), (4.11) and (4.16) are satisfied then mapping (4.3) is differentiable and for any given increment $\delta\Theta \in X$, its derivative may be obtained by solving the linear parabolic equation:

$$\frac{\partial A}{\partial y}(\bar{\Theta}, \bar{y})\, \delta y + \frac{\partial A}{\partial \Theta}(\bar{\Theta}, \bar{y})\, \delta\Theta = 0, \qquad (x, t) \in Q \tag{4.16}$$

with boundary condition:

$$\frac{\partial B}{\partial y}(\bar{\Theta}, \bar{y})\, \delta y + \frac{\partial B}{\partial \Theta}(\bar{\Theta}, \bar{y})\, \delta\Theta = 0, \qquad (x, t) \in \Sigma \tag{4.17}$$

and initial condition

$$\delta y(x, 0) = 0, \quad x \in \Omega. \tag{4.18}$$

From the numerical point of view it is inpractical to use formula (4.7) for computing the gradient of functional (3.4), because it would require solving equation (4.4)-(4.6) for every increment $\delta\Theta$.

To avoid this difficulty we will use Lagrangean function of our problem to obtain a simple formula for the derivative of functional (3.4).

We will define Lagrangian in the following way:

$$\mathscr{L}(y, \Theta, p, r) = J(\Theta) + \int_Q pA(\Theta, y)\, dQ + \int_\Sigma rB(\Theta, y)\, d\Sigma \tag{4.17}$$

where $(p, r) \in L^2(\Omega) \times L^2(\Sigma)$ are Lagrang multipliers.

To be sure that $\mathscr{L}(y, \Theta, p, r)$ is well defined we have to assume that:

$$A(\Theta, y) \in L^2(Q),$$

$$B(\Theta, y) \in L^2(\Sigma),$$

which in particular takes place for

$$\Theta \in U \subset X,$$

$$y \in Y.$$

If $y=y(\Theta)$ is the state trajectory then

$$\mathscr{L}(y(\Theta), \Theta, p, r)=J(\Theta), \quad \forall (p, r) \in L^2(Q) \times L^2(\Sigma) \tag{4.18}$$

whence along the trajectory we have:

$$\left\langle \frac{dJ}{d\Theta}(\bar{\Theta}); \delta\Theta \right\rangle = \left\langle \frac{d\mathscr{L}}{d\Theta}(y(\bar{\Theta}), \bar{\Theta}, p, r); \delta\Theta \right\rangle \tag{4.19}$$

$$\forall \delta\Theta \in X, \ \forall (p, r) \in L^2(Q) \times L^2(\Sigma)$$

Assume that for any pair of fixed functions $(f_1, f_2) \in L^2(Q) \times L^2(\Sigma)$ problem (4.12)-(4.14) has the unique weak solution δy satisfying the following regularity conditions

$$\delta y \in C(0, T; L^2(\Omega)) \tag{4.20}$$

and

$$\|\delta y\|_{C(0, T; L^2(\Omega))} \leqslant C(\|f_1\|_{L^2(Q)} + \|f_2\|_{L^2(\Sigma)}). \tag{4.20}$$

For given $\bar{\Theta} \in U$ we define the normed subspace of $L^2(Q)$ as the set:

$$\Phi_{\bar{\Theta}} = \left\{ v \in L^2(Q) \,\middle|\, \frac{\partial A}{\partial y}(\bar{\Theta}, y(\bar{\Theta}))\, v \in L^2(Q), \right.$$

$$\left. \frac{\partial B}{\partial y}(\bar{\Theta}, y(\bar{\Theta}))\, v \in L^2(\Sigma), \ v(x, 0)=0, \ x \in \Omega \right\} \tag{4.21}$$

with the norm

$$\|v\|_{\Phi_{\bar{\Theta}}} = \left\| \frac{\partial A}{\partial y} v \right\|_{L^2(Q)} + \left\| \frac{\partial B}{\partial y} v \right\|_{L^2(\Sigma)} \tag{4.22}$$

From (4.20) it follows that the set $\Phi_{\bar{\Theta}}$ is well defined normed space and the following inequality holds

$$\|v\|_{(C\,0, T; L^2(\Omega))} \leqslant C\|v\|_{\Phi_{\bar{\Theta}}}, \ C>0. \tag{4.23}$$

We define the generalized adjoint state at the point $\bar{\Theta} \in U$ as a function

$$(\bar{p}, \bar{r}) = (p(\bar{\Theta}), r(\bar{\Theta})) \in L^2(Q) \times L^2(\Sigma) \tag{4.24}$$

which satisfies the following generalized adjoint state equation:

$$\int_Q p\left(\frac{\partial A}{\partial y}(\bar{\Theta}, y(\bar{\Theta}))\, v\right) dQ + \int_\Sigma r\left(\frac{\partial B}{\partial y}(\bar{\Theta}, y(\bar{\Theta}))\, v\right) d\Sigma =$$

$$= -\int_\Omega \frac{dI}{dy}(y(\bar{\Theta})(x, T))\, v(x, T)\, d\Omega, \quad \forall v \in \Phi_{\bar{\Theta}}. \tag{4.25}$$

To obtain sufficient conditions for existence of the solutions to problem (4.25) we will use some general results given in [4], which can be formulated as follows:

Let W be a Hilbert space and Φ a normed linear subspace of W. We denote by Φ' the dual space to Φ which is obtained by extension of scalar product in W.

Let be given a bilinear form

$$E\colon W\times\Phi\mapsto R, \tag{4.26}$$

which is continuous with respect to the first argument.

We define a mapping

$$M\colon \Phi\to M[\Phi]\subset W \tag{4.27}$$

by the equality

$$(w, M\Phi)_W=E(w, \Phi), \qquad \forall w\in W, \qquad \forall\Phi\in\Phi \tag{4.28}$$

Let L be any given element of space Φ'.

THEOREM 5 [4]. Suppose that there exists a real constant $C>0$ such that

$$\|M\Phi\|_W\geqslant C\,\|\Phi\|_\Phi \tag{4.29}$$

then there exists a solution w_L to the variational problem:

$$E(w_L, \Phi)=L(\Phi), \ \forall\Phi\in\Phi \tag{4.30}$$

and

$$\|w_L\|_W\leqslant\frac{1}{C}\,\|L\|_{\Phi'}. \tag{4.31}$$

The element w_L is uniquely determined if the set $M[\Phi]$ is dense in W.

Proof is given in [4] (Theorems 2.1, 2.2).

To use the above theorem we assume

(i) W is Hilbert space $L^2(Q)\times L^2(\Sigma)$

(ii)
$$\Phi=\{\Phi, \Phi|_\Sigma)\in L^2(Q)\times L^2(\Sigma)|\Phi\in\Phi_{\bar{\Theta}}\} \tag{4.32}$$

with the norm

$$\|\Phi\|_\Phi=\|\Phi\|_{\Phi_{\bar{\Theta}}};$$

(iii) functional L is of the form:

$$L(\Phi)=L_{\bar{\Theta}}(\Phi)=-\int\limits_\Omega\frac{dI}{dy}\big(y(\bar{\Theta})(x, T)\big)\,\Phi(x, T)\,d\Omega; \tag{4.33}$$

(iv) bilinear form E is defined by:

$$E(w, \Phi)=\int\limits_Q p\left(\frac{\partial A}{\partial y}(\bar{\Theta}, y(\bar{\Theta}))\,\Phi\right)dQ+$$
$$+\int\limits_\Sigma r\left(\frac{\partial B}{\partial y}\bar{\Theta}, y(\bar{\Theta})\,\Phi|_\Sigma\right)d\Sigma, \qquad w=(p, r)\in W,\ \Phi\in\Phi_{\bar{\Theta}}. \tag{4.34}$$

In our case condition (4.29) is trivially fulfilled for $\|M\Phi\|_W=\|\Phi\|_\Phi$.

From (4.23) it follows that functional $L_{\bar{\Theta}}$ defined by (4.33) belongs to Φ'.

Then the generalized adjoint state $(\bar{p}, \bar{r})$ exists and is determined uniquely — it results from the existence of weak solution to problem (4.12)-(4.14), for any pair of functions $(f_1, f_2) \in W$. Since

$$\left\langle \frac{\partial \mathcal{L}}{\partial y}(\bar{y}, \bar{\Theta}, \bar{p}, \bar{r}); \delta y \right\rangle = \int_Q \bar{p}\left(\frac{\partial A}{\partial y} \delta y\right) dQ +$$

$$+ \int_\Sigma \bar{r}\left(\frac{\partial B}{\partial y} \delta y\right) d\Sigma + \int_\Omega \frac{dI}{dy}(\bar{y}(x, T))\, \delta y(x, T) d\Omega = 0,$$

$$\forall \delta y \in \Phi_{\bar{\Theta}} \quad \text{where} \quad \bar{y} = y(\bar{\Theta}) \tag{4.35}$$

by (4.19) we obtain the following simple form of the gradient

$$\left\langle \frac{dJ}{d\Theta}; \delta\Theta \right\rangle = \int_Q \bar{p}\left(\frac{\partial A}{\partial \Theta}(\bar{\Theta}, y(\bar{\Theta}))\, \delta\Theta\right) dQ +$$

$$+ \int_\Sigma \bar{r}\left(\frac{\partial B}{\partial \Theta}(\bar{\Theta}, y(\bar{\Theta}))\, \delta\Theta\right) d\Sigma, \tag{4.36}$$

where $(\bar{p}, \bar{r})$ satisfy (4.25).

If $\hat{\Theta}$ is an optimal parameter and functional $J(\Theta)$ is differentiable at point $\hat{\Theta}$ than the necessary condition of optimality takes on the form

$$\int_Q \bar{p}\left(\frac{\partial A}{\partial \Theta} \hat{\Theta}, y(\hat{\Theta}))\, (\Theta - \hat{\Theta})\right) dQ +$$

$$+ \int_\Sigma \bar{r}\left(\frac{\partial B}{\partial \Theta}(\hat{\Theta}, y(\hat{\Theta}))\, (\Theta - \hat{\Theta})\right) d\Sigma \geqslant 0, \qquad \forall \Theta \in U. \tag{4.37}$$

5. Linear state equation

Consider the problem of parametric optimization for the particular case of linear state equation of the form:

$$A(\Theta, y) \equiv \frac{\partial y}{\partial t} - \sum_{i,j=1}^{n} a_{ij}(x, t, \Theta) \frac{\partial^2 y}{\partial x_i \partial x_j} +$$

$$+ \sum_{i=1}^{n} a_i(x, t, \Theta) \frac{\partial y}{\partial x_i} + a(x, t, \Theta)\, y = f(x, t), \qquad (x, t) \in Q \tag{5.1}$$

with boundary conditions:

$$B(\Theta, y) \equiv \sum_{i=1}^{n} c_i(x, t, \Theta) \frac{\partial y}{\partial x_i} + \psi(x, t, \Theta) y = \varphi(x, t), \qquad (x, t) \in \Sigma \tag{5.2}$$

and initial condition

$$y(x, 0) = y_0(x), \qquad x \in \Omega. \tag{5.3}$$

In this case sufficient conditions for existence of an optimal parameter and for differentiability of functional $J(\Theta)$ are weaker then in general case of quasilinear state equation. On the basis of Theorem 2 we can find sufficient conditions for existence of the state trajectory

$$y(\Theta) \in C^{2+\beta, 1+\beta/2}(\bar{Q}), \qquad \beta > 0 \tag{5.4}$$

for all parameters $\Theta \in U$.

For simplicity let us assume like in section 4 that the scalar parameter Θ is a function of one variable t only. Substituting a fixed parameter $\Theta \in U$ to the coefficients of state equation (5.1)-(5.3) we obtain functions of variables $(x, t) \in \bar{Q}(\bar{\Sigma})$, for example

$$\tilde{a}_{ij}(x, t) \equiv a_{ij}(x, t, \Theta(t)), \; (x, t) \in \bar{Q},$$

for which the assumptions of Theorem 2 should be fulfilled. This way we get the following conclusions: for existence of state trajectory which satisfies (5.4) it is enough to assume:

(i) coefficients of state equation and given functions f, φ, y_0 satisfy assumptions of the Theorem 2 (with respect to x, t variables); (5.5)

(ii) $$U \subset C^{\frac{i+\beta}{2}}(0, T), \qquad \beta > 0; \tag{5.6}$$

(iii) $a_{ij}(x, t, .)$, $a_i(x, t, .)$, $a(x, t, .)$, $(c_i(x, t, .)$, $\psi(x, t, .))$ are C^1 functions on the set $K \subset R$ for all $(x, t) \in \bar{Q}(\bar{\Sigma})$. (5.7)

We state in form of two lemmas sufficient conditions for existence of an optimal parameter for the problem (3.6) and differentiability of functional (3.4) in the case of linear state equation.

LEMMA 4. Assume:

(i) conditions (5.5)-(5.7) hold,

(ii) set U is closed in $C(0, T)$ topology,

then there exists an optimal parameter for the problem (3.6) with linear state equation (5.1)-(5.3).

Proof. In similar way like in Theorem 3 one can proove that in this case mapping (3.13) is continuous. Hence functional (3.4) is continuous in $C(0, T)$ topology on the set U which is compact (it follows from assumptions (5.6) and (ii)) in $C(0, T)$ topology.

LEMMA 5. Assume:

(i) conditions (5.5)-(5.7),

(ii) $\frac{\partial a_{ij}}{\partial \Theta}(x, t, \Theta(t))$, $\frac{\partial a_i}{\partial \Theta}(x, t, \Theta(t))$, $\frac{\partial a}{\partial \Theta}(x, t, \Theta(t)) \in C^{\beta, \beta/2}(\bar{Q})$ for some $\beta > 0$ and all $\Theta \in U$,

(iii) $\dfrac{\partial c_i}{\partial \Theta}(x, t, \Theta(t)), \dfrac{\partial \psi}{\partial \Theta}(x, t, \Theta(t)) \in C^{1+\beta, \frac{1+\beta}{2}}(\bar{\Sigma}), \quad \forall \Theta \in U,$

iv) functional $I(y)$ is differentiable, then functional (3.4) is differentiable and its gradient is of the form (4.36).

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Appendix

Proof of Theorem 3.

Let y_k, $k=1, 2$, be the solution of the equation

$$\frac{\partial y_k}{\partial t} - \sum_{i, j=1}^{n} a_{ij}(x, t, \Theta_k, y_k) \frac{\partial^2 y_k}{\partial x_i \partial x_j} - \\ -b(x, t, \Theta_k, p_{k1}, \ldots, p_{kn})=0, \quad (x, t) \in Q, \quad \Theta_k \in U, \tag{A.1}$$

with boundary condition:

$$\sum_{i, j=1}^{n} a_{ij}(x, t, \Theta_k, y_k) p_{kj} \gamma_i + \psi(x, t, \Theta_k, y_k)=0, \ (x, t) \in \Sigma \tag{A.2}$$

and initial condition:

$$y_k(x, 0)=y_0(x), \ x \in \Omega, \tag{A.3}$$

where p_{ki} denotes derivative $\dfrac{\partial y_k}{\partial x_i}$.

By subtracting equations (A.1) for y_1, y_2 we obtain

$$\frac{\partial y_1}{\partial t} - \frac{\partial y_2}{\partial t} - \sum_{i, j=1}^{n} a_{ij}(\Theta_1, y_1) \frac{\partial^2 y_1}{\partial x_i \partial x_j} + \\ + \sum_{i, j=1}^{n} a_{ij}(\Theta_2, y_2) \frac{\partial^2 y_2}{\partial x_i \partial x_j} + b(\Theta_1, y_1, p_1) - b(\Theta_2, y_2, p_2)=0, \\ (x, t) \in Q, \tag{A.4}$$

where for simplicity we denote

$$p_k=(p_{k1}, \dots, p_{kn}),\ k=1,2,$$

$$a_{ij}(\Theta_1, y_1)=a_{ij}(x, t, \Theta_1, y_1),$$

$$b(\Theta_1, y_1, p_1)=b(x, t, \Theta_1, y_1, p_1, \dots, p_{1n}),$$

similarly substracting boundary conditions (A.2) we get

$$\sum_{i,j=1}^{n} a_{ij}(\Theta_1, y_1)p_{1j}\gamma_i+\psi(\Theta_1, y_1)-$$
$$-\sum_{i,j=1}^{n} a_{ij}(\Theta_2, y_2)p_{2j}\gamma_i-\psi(\Theta_2, y_2)=0, \quad (x,t)\in\Sigma. \tag{A.5}$$

Let us denote $\tilde{y}=y_1-y_2$. Then we can rewrite (A.4) in the form:

$$\frac{\partial\tilde{y}}{\partial t}-\sum_{i,j=1}^{n} a_{ij}(\Theta_1, y_1)\frac{\partial^2\tilde{y}}{\partial x_i\,\partial x_j}- \tag{A.6}$$
$$-\sum_{i,j=1}^{n}[a_{ij}(\Theta_2, y_1)-a_{ij}(\Theta_1, y_1)]\frac{\partial^2 y_2}{\partial x_i\,\partial x_j}-$$
$$-\sum_{i,j=1}^{n}[a_{ij}(\Theta_2, y_2)-a_{ij}(\Theta_2, y_1)]\frac{\partial^2 y_2}{\partial x_i\,\partial x_j}+$$
$$+[b(\Theta_1, y_1, p_{11}, \dots, p_{1n})-b(\Theta_1, y_1, p_{21}, p_{12}, \dots, p_{1n})]+\dots+$$
$$+[b(\Theta_1, y_1, p_{21}, \dots, p_{2n-1}, p_{1n})-b(\Theta_1, y_1, p_{21}, \dots, p_{2n})]+$$
$$+[b(\Theta_1, y_1, p_2)-b(\Theta_1, y_2, p_2)]+[b(\Theta_1, y_2, p_2)-b(\Theta_2, y_2, p_2)]=0$$

we consider the solution of (A.6) in a neighbourhood of a fixed parameter $\Theta_2\in U$. We will assume that $\Theta_1\in U$ is a point belonging to this neighbourhood.

Since $y(\Theta_2)\in C^{2+\beta, 1+\beta/2}(\bar{Q})$ we have

$$\left|y_2, \frac{\partial y_2}{\partial x_i}, \frac{\partial^2 y_2}{\partial x_i\,\partial x_j}\right|\leqslant C,\ (x,t)\in Q,\ i,j=1, \dots, n. \tag{A.7}$$

By Lemmas 1, 2 the following a priori estimation of the function $y_1=y(\Theta_1)$ takes place:

$$\left|y_1, \frac{y_1}{x_i}\right|\leqslant M_1, (x,t)\in Q, \quad \forall\Theta_1\in U. \tag{A.8}$$

By (A.7), (A.8) and the mean value theorem we can estimate subsequent summants in (A.6), for example

$$\left|(a_{ij}(\Theta_2, y_1)-a_{ij}(\Theta_1, y_1))\frac{\partial^2 y_2}{\partial x_i\,\partial x_j}\right|\leqslant C\max_{Q}\left|\frac{\partial a_{ij}}{\partial\Theta}\right||\Theta_1-\Theta_2|, \quad (x,t)\in\bar{Q}$$

and analogicaly

$$|b(\Theta_1, y_1, p_{11}, \ldots, p_{1n}) - b(\Theta_1, y_1, p_{21}, \ldots, p_{1n})| \leqslant C|\tilde{p}_1| \max_Q \left|\frac{\partial b}{\partial p_1}\right|, \quad (x, t) \in \bar{Q},$$

where $\tilde{p}_1 = p_{11} - p_{21} = \dfrac{\partial y_1}{\partial x_1} - \dfrac{\partial y_2}{\partial x_1}$.

Let us rewrite (A.6) in the form

$$\frac{\partial \tilde{y}}{\partial t} - \sum_{i,j=1}^{n} a_{ij}(\Theta_1, y_1) \frac{\partial^2 \tilde{y}}{\partial x_i \partial x_j} + B(x, t, \Theta_1, \Theta_2, y_1, y_2) = 0, \quad (x, t) \in Q, \tag{A.9}$$

since Θ_2, y_2 are fixed and

$$y_1 = y_2 + (y_1 - y_2) = y_2 + \tilde{y},$$

$$p_1 = p_2 + (p_1 - p_2) = p_2 + \tilde{p},$$

$$\Theta_1 = \Theta_2 + (\Theta_1 - \Theta_2) = \Theta_2 + \tilde{\Theta},$$

we can write

$$B(x, t, \Theta_1, \Theta_2, y_1, y_2) = B_1(x, t, \tilde{y}, \tilde{p}, \tilde{\Theta}).$$

To use Lemma 1 let us note that function B_1 may be estimated in the following way

$$|B_1(x, t, \tilde{y}, \tilde{p}, \tilde{\Theta})| \leqslant C_0'|\tilde{y}| + C_1'|\tilde{p}| + C_2'|\tilde{\Theta}|.$$

whence

$$-yB_1(x, t, \tilde{y}, \tilde{p}, \tilde{\Theta}) \leqslant C_0''\,|\tilde{y}|^2 + C_1''\,|\tilde{p}|^2 + C_2''|\tilde{\Theta}|^2$$

$$\leqslant C_0''\,|\tilde{y}|^2 + C_1''\,|\tilde{p}|^2 + C_2''\,(\max_Q|\tilde{\Theta}|)^2.$$

So the constant C_2 from the estimation (2.4) can be chosen as

$$C_2 = C_2''\,(\max_Q|\Theta_1 - \Theta_2|)^2$$

in the similar way we can determine the constant C_4 in the estimation (2.4) as

$$C_4 = C_4''\,(\max_Q|\Theta_1 - \Theta_2|)^2.$$

Then from Lemma we have

$$\max_{\bar{Q}}|\tilde{y}| \leqslant C \max_{\bar{Q}}|\Theta_1 - \Theta_2|$$

so the mapping (3.13) is continuous, q.e.d.

Problem optymalizacji parametrycznej dla pewnej klasy liniowych i quasi-liniowych równań typu parabolicznego

Rozważono problem minimalizacji pewnego funkcjonału jakości, określonego na klasie rozwiązań równania stanu, które jest quasi-liniowym równaniem parabolicznym. Współczynniki równania stanu zależne są od pewnego parametru funkcyjnego, zadanie optymalizacji polega na wyznaczaniu parametru optymalnego, któremu odpowiada rozwiązanie równania stanu minimalizujące funkcjonał jakości na zbiorze parametrów dopuszczalnych.

W punkcie 3 podane są warunki wystarczające dla istnienia parametru optymalnego.

W punkcie 4 podano warunki wystarczające dla różniczkowalności funkcjonału jakości. Dla efektywnego wyznaczenia gradientu funkcjonału jakości zdefiniowano tzw. uogólnione równanie sprzężone oraz podano warunki wystarczające dla istnienia jednoznacznego rozwiązania tego równania.

Wykorzystując uogólniony stan sprzężony podano warunki konieczne optymalności.

Проблема параметрической оптимализации некоторого класса линейных и квазилинейных уравнений параболическоготипа

В статье рассмотрена проблема минимизации некоторого функционала качества, определенного в классе решений уравнения состояния — которое является квазилинейным параболическим уравнением.

Коэффициенты уравнения состояния зависят от некоторого функционального параметра — задача оптимизации состоит в определении оптимального параметра, которому соответствует решение уравнения состояния, минимизирующее функционал качества на множестве допускаемых параметров.

В разд. 3 даны достаточные условия существования оптимального параметра.

В разд. 4 даны достаточные условия дифференцируемости функционала качества. Для эффективного вычисления градиента функционала качества дано определение так назыв. обобщенного сопряженниго уравнения и приведены достаточные условия существования однозначного решения этого уравнения.

Используя обобщенное сопряченное состояние даны необходимые условия оптимальности.