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Optimization of a completely decentralized production system

by

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The paper deals with a model of production system where a complete decentralization of management is assumed. The model consists of n production sectors and a sector of investment services. Production functions of all sectors are assumed in the Cobb-Douglas form. Each production sector maximizes its own net income with respect to maintenance and investment costs. The optimization problem is a dynamic one. The optimum strategies are found. It was shown that the optimum integrated maintenance costs and the optimum integrated capital expenditures are in proportion to the integrated production in each sector respectively.

1. Introduction

In [1] a model of complex development was considered where the capital expenditures were allocated by one decision center and the maintenance costs were independently optimised in each particular sector. In [3] a solution was given in the case at a central allocation of both capital expenditures and employment. It is interested to consider a system ahere the decision of maintenance cost and capital expenditures are undertaken independently in each production sector. The decision should result from optimization of a sectoral net income. The sector of investment services is separated. This paper deals with the model mentioned above. It is assumed that the production functions are of Cobb-Douglas form as in [1] and [3]. The optimization problems of production sectors are dynamic ones. The optimum strategies of maintenance costs and capital expenditures are obtained for particular sectors.

2. Model

The system considered (Fig. 1) consists of n production sectors and one sector of investment services.

Production functions of production sectors are as follows:

$$Y_{ii}(t)=F_i^{q_i}[I_i(t)]\prod_{\substack{j=1\\j\neq i}}^{n} Y_{ji}(t)^{\alpha_{ji}}, \tag{1}$$

where: $Y_{ii}(t)$ — global production intensity of the i-th sector; $Y_{ji}(t)$ — actual intersector flows from the sector j-th to i-th, $j=1, 2, ..., n$, $j \neq i$; $I_i(t)$ — intesity of capital investment in the sector i; α_{ji} — elasticity coefficients (fixed numbers);

$$q_i = 1 - \sum_{\substack{j=1 \\ j \neq i}}^{n} \alpha_{ji} > 0, \tag{2}$$

$$F_i[I_i(t)] = \int_0^t k_i(t-\tau) \cdot [I_i(\tau)]^{\beta_i} d\tau. \tag{3}$$

It is assumed that the parameters β_i, α_{ji} are constant in a given planning time interval $[0, T]$ and moreover $0 < \beta_i < 1$, $\alpha_{ji} > 0$, $j, i = 1, 2, ..., n$.

The function F_i describes the influence of the inertial investment process on the production of a sector. A typical function $k_i(t)$ is shown in Fig. 2. Appropriate

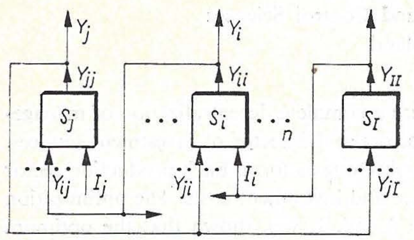

Fig. 1. Model of the system

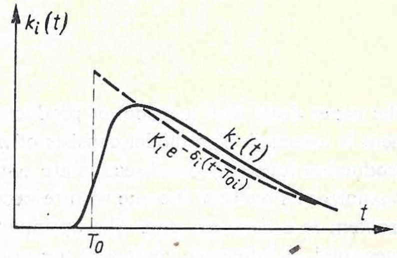

Fig. 2. The function $k_i(t)$ and its approximation

form of the k_i function allows to involve in the model such phenomena as delay and production increase with respect to the investment process and the production decrease according to the depreciation of the production base.

The description of the production function 1, 2, 3 is assumed as in [1]. The production sector takes resources for current production from other production sectors and the capital investments from the sector I. The current production intensity is allocated according to the expression:

$$Y_{ii}(t) = Y_i(t) + \sum_{\substack{j=1 \\ j \neq i}}^{n} Y_{ij}(t) + Y_{iI}(t), \tag{4}$$

where: $Y_{iI}(t)$ — the intensity of a part of production of the sector i, which is taken by the investment sector; $Y_{ij}(t)$ the intersector flows;

$Y_i(t)$ the rest of the production intensity assigned out of the system.

The investment system is described by the production function

$$Y_{ii}(t) = F_I(t)_{q_i} \prod_{\substack{j=1 \\ j \neq i}}^{n} Y_{jI}(t)^{\alpha_{jI}}, \tag{5}$$

$$q_i = 1 - \sum_{j=1}^{n} \alpha_{jI} > 0, \tag{6}$$

where: $F_I(t)$ — a given positive function of time; $Y_{jI}(t)$ the intensity of flows to the investment sector from the other ones.

Production of the investment sector is allocated among the production sectors as investments. The rest goes out of the system. It is expressed by the relation

$$Y_{II}(t) = Y_I(t) + \sum_{\substack{j=1 \\ j \neq i}}^{n} I_j(t), \tag{7}$$

where $Y_I(t)$ — the intensity of investment services assigned out of the system. All intensivities of the intersector flows and the production intensities are expressed in monetary units. The production of each sector takes no account the part of global production used at the same sector.

As the investment sector can be interpreted the investment building trade sector which assures the building services in other sectors and out of the system,

3. Problem statement

3.1. Investment sector S_I

Optimization problem consists in finding the functions $\hat{Y}_{iI}(t) \geqslant 0$, $i = 1, 2, ..., n$, that maximize the net income of the sector in a given time interval $[0, T]$

$$\max \left\{ D_I = \int_0^T Y_{II}(t) - \sum_{i=1}^{n} Y_{iI}(t)\, dt \right\} \tag{8}$$

subject to the production function described (5), (6).

The sectoral income (8) is computed as the production of the sector minus the resources costs taken from the other sectors.

3.2. Production sector S_i

The problem is to determine the investment strategies $\hat{I}_i(t)$ and the intensities of resources used $\hat{Y}_{ji}(t)$ that give the maximum net income of the sector

$$\max \left\{ D_i = \int_0^T w_i(t) \cdot Y_{ii}(t) - \sum_{\substack{j=1 \\ j \neq i}}^{n} w_{ji}(t) \cdot Y_{ji}(t) - w_{Ii}(t) \cdot I_i(t)\, dt \right\} \tag{9}$$

subject to the production function (1)—(3).

In the formula (9) the discount function were introduced in the form

$$w_{ji}(t) = (1 + \varepsilon_{ji})^{T-t}, \quad j = 1, 2, ..., n, I, \tag{10}$$

where ε_{ji} are given positive numbers. That way the credit possibly taken by the sector is considered.

The weight function

$$w_i(t) = (1 + \varepsilon_i)^{-t} \tag{11}$$

takes into account the depreciation of production value in time.

The sector considered pays the values of resources used $Y_{ji}(t)$ and the capital expeditures $I_i(t)$ from the value of the its own production.

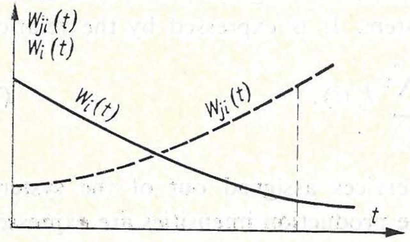

Fig. 3. The discount function $w_i(t)$ and $w_{ji}(t)$

4. Solution of the optimization problems

4.1. Investment sector S_I

The optimization problem is solved as presented in [1]. The net income of the sector can be maximized at each time moment independently.

The solution takes the form:

$$\hat{Y}_{iI}(t)=\alpha_{iI}(t)\cdot \bar{Y}_{II}(t), \tag{12}$$

$$\hat{Y}_{II}(t)=F_i(t)\cdot \prod_{i=1}^{n} [\alpha_{iI}]^{\alpha_{iI}/q_I}, \tag{13}$$

$$D_I=\int_0^T F_i(t)\,dt\cdot q_I\cdot \prod_{i=1}^{n} (\alpha_{iI})^{\alpha_{iI}/q_I}. \tag{14}$$

4.2. Production sector S_i

In the optimization problem (9) of the production system the investments influence on the sectoral production takes the form of a nonlinear operator $F_i[I_i(t)]$ (see expression (3)). It is a dynamic problem.

At first, the problem (9) subject to the constraints (1)—(3) will be solved in the parametric form. The parameters are the values of global integral expenditures in both investments and maintenance costs of the sector, that are denoted by $\bar{I}_i$, $\bar{Y}_{ji}$ respectively, $j, i=1, 2, ..., n,\ j\neq i$.

Parametric problem 1. The values $\bar{Y}_{ji}$, $\bar{I}_i$, $j, i=1, 2, ..., n$, are given. Find the function $\hat{I}_i(t)$, $\hat{Y}_{ji}(t)$ such that

$$\max \int_0^T w_i(t)\cdot Y_{ii}(t)\,dt-\sum_{\substack{j=1\\ j\neq i}}^{n}\int_0^T w_{ji}(t)\cdot Y_{ji}(t)\,dt-\int_0^T w_{Ii}(t)\cdot I_i(t)\,dt \tag{15}$$

holds subject to the constraints:

$$Y_{ii}(t)=\left\{\int_0^t k_i(t-\tau)\,[I_i(\tau)]^{\beta_i}\,d\tau\right\}^{q_i}\prod_{\substack{j=1\\ j\neq i}}^{n} Y_{ji}(t)^{\alpha_{ji}}, \tag{16}$$

$$\int_0^T w_{ji}(t)\cdot Y_{ji}(t)\,dt=\bar{Y}_{ji}. \tag{17}$$

$$\int_0^T w_{Ii}(t)\cdot I_i(t)\,dt=\bar{I}_i, \tag{18}$$

$$Y_{ji}(t)\geqslant 0, \quad I_i(t)\geqslant 0 \quad \text{for} \quad t\in[0,T]. \tag{19}$$

The solution is looked for in the parametric form with respect to $\bar{Y}_{ji}$ and $\bar{I}_i$. The above problem can be reduced to the form

$$\max\left\{\bar{Y}_{ii}=\int_0^T w_i(t)\cdot Y_{ii}(t)\,dt\right\} \tag{20}$$

subject to the constraints (16)—(19) because of the expenditures $\bar{Y}_{ji}$, $\bar{I}_i$ being given.

The following denotation is introduced:

$$f_i(t)=w_i(t)\cdot\prod_{\substack{j=1\\ j\neq i}}^{n} F_i[I_i(t)], \tag{21}$$

$$\varphi_j(t)=Y_{ji}(t)\cdot w_{ji}(t)^{\alpha_{ji}}. \tag{22}$$

Applying the generalized Hölder inequality one obtains

$$\bar{Y}_{ii}=\int_0^T f_i(t)\cdot\prod_{\substack{j=1\\ j\neq i}}^{n}\varphi_j(t)\,dt\leqslant\left\{\int_0^T f_i(t)^{1/q_i}\,dt\right\}^{q_i}\prod_{\substack{j=1\\ j\neq i}}^{n}\left\{\int_0^T \varphi_j(t)^{1/\alpha_{ji}}\right\}^{\alpha_{ji}}. \tag{23}$$

The relation (23) becomes the equality if

$$f_i(t)^{1/q_i}=C_{1j}\cdot\varphi_i(t)^{1/\alpha_{ji}}, \quad j=1,2,\ldots,n, \quad j\neq i \quad \text{for} \quad t\in[0,T],$$

where C_{1j} are constant, and $q_i+\sum_{\substack{j=1\\ j\neq i}}^{n}=1$ the last one holding according to the assumption (2).

The Hölder inequality is applied once again. The following consequence of the relation holds:

$$\int_0^T f_i(t)^{1/q_i}\,dt=\int_0^T\left[w_i(t)\cdot\prod_{\substack{j=1\\ j\neq i}}^{n} w_{ji}(t)^{-\alpha_{ji}}\right]^{1/q_i} F_i(I_i(t))=$$

$$=\int_0^T [w_{Ii}(\tau)\cdot I_i(\tau)]^{\beta_i}[w_{Ii}(\tau)]^{-\beta_i}\int_\tau^T \bar{w}_i(t)\,k_i(t-\tau)\,dt\,d\tau\leqslant$$

$$\leqslant\left\{\int_0^T w_{Ii}(\tau)\cdot I_i(\tau)\,d\tau\right\}^{\beta_i}\left\{\int_0^T g_i(\tau)\,d\tau\right\}^{1-\beta_i}, \tag{24}$$

where it was denoted

$$g_i(t)=\left[w_{Ii}(t)^{-\beta_i}\times\int_{\tau}^{T} w_i(t)\cdot k(t-\tau)\,d\tau\right]^{1/(1-\beta_i)}, \tag{25}$$

$$\bar{w}_i(t)=\left[w_i(t)\cdot\prod_{\substack{j=1\\ j\neq i}}^{n} w_{ji}(t)^{-\alpha_{ij}}\right]^{1/q_I}. \tag{26}$$

The relation (24) becomes equality if

$$I_i(t)=C_2\,g_i(t)\quad\text{for}\quad t\in[0,T],$$

where C_2 is a constant. Then the relation

$$\bar{Y}_{ii}=\bar{I}_i^{\beta_i q_i}\prod_{\substack{j=1\\ j\neq i}}^{n}\bar{Y}_{ji}^{\alpha_{ji}}\bar{F}_i^{\bar{q}_i}, \tag{27}$$

holds, where

$$\bar{F}_i=\left\{\int_0^T g_i(t)\,dt\right\}^{\bar{q}_i},$$

$$\bar{q}_i=1-\sum_{\substack{j=1\\ j\neq i}}^{n}\alpha_{ji}-\beta_i q_i. \tag{28}$$

The constants C_2, C_{1j}, $j=1, 2, ..., n$, are found from the relation (17), (18). Then the solution of the problem (20) is obtained in the following form

$$\hat{I}_i(t)=\frac{g_i(t)}{\int_0^T w_{Ii}(t)\,g(t)\,dt}\,\bar{I}_i, \tag{29}$$

$$\hat{Y}_{ii}(t)=\frac{\hat{f}(t)^{\alpha_{ij}/q_i}}{w_{ji}\cdot\int_0^T \hat{f}(t)^{\alpha_{ij}/q_i}\,dt}\,\bar{Y}_{ji}, \tag{30}$$

where

$$\hat{f}(t)=w_i(t)\prod_{\substack{j=1\\ j\neq i}}^{n} w_{ji}(t)^{-\alpha_{ji}}\times F_i[\hat{I}_i(t)].$$

The expressions (17)—(30) are the solution of the problems (20), however the formulae (29), (30) give the optimum strategies $\hat{I}_i(t)$, $\hat{Y}_{ji}(t)$, $j, i=1, 2, ..., n$, $j\neq i$, of the problem (15)—(19) as well. The aggregated production function (27) has the Cobb—Douglas form with respect to the maintenance costs and the capital investments integrated in the time interval $[0, T]$.

The idea of applying the Hölder inequality was given in [2] where the similar optimization problem was solved for $n=1$.

The solution of parametric problem 1 allows to continue the consideration in the static case.

Parametric problem 2. The number $\bar{Y}_c$ is given. The problem is to find the numbers $\tilde{\bar{Y}}_{ji}$ and $\tilde{I}_i$ such that

$$\max \bar{F}_i^{q_i} I^{\beta_i q_i} \cdot \prod_{\substack{j=1 \\ j\neq i}}^{n} \bar{Y}_{ij}^{\alpha_{ji}} \tag{31}$$

is obtained subject to the constraint

$$\sum_{\substack{j=1 \\ j\neq i}}^{n} \bar{Y}_{ji} + I_i = \bar{Y}_c . \tag{32}$$

In order to solve the problem the Kuhn—Tucker conditions were analysed. The solution obtained is in the form

$$\tilde{\bar{Y}}_{ji} = \bar{Y}_c \, \alpha_{ji}/\gamma_i , \tag{33}$$

$$\tilde{I}_i = \bar{Y}_c \, \beta_i \, q_i/\gamma_i , \tag{34}$$

$$\tilde{\bar{Y}}_{ii} = M_i \, \bar{Y}_c^{\gamma_i} , \tag{35}$$

where

$$\gamma_i = q_i \, \beta_i + \sum_{\substack{j=1 \\ j\neq i}}^{n} \alpha_{ji} , \tag{36}$$

$$M_i = \bar{F}_i^{q_i} \left(\frac{\beta_i \, q_i}{\gamma_i} \right)^{\beta_i q_i} \prod_{\substack{j=1 \\ j\neq i}}^{n} \left(\frac{\alpha_{ji}}{\gamma_i} \right)^{\alpha_{ij}} . \tag{36}$$

Then the following optimization problem is consideted:

$$\max \{ \tilde{\bar{Y}}_{ii} - \bar{Y}_c = M_i \, \bar{Y}_c^{\gamma_i} - \bar{Y}_c \} , \tag{37}$$

with respect to the parameter $\bar{Y}_c$. The derivative of (37) is set to zero and that way the optimal $\bar{Y}_c$ is obtained.

The final solution of the problem (15)—(19) takes the form

$$\hat{\bar{Y}}_{ii} = \bar{F}_i \, (\beta_i \, q_i)^{\beta_i q_i/q_i} \prod_{\substack{j=1 \\ j\neq j}}^{n} [\alpha_{ji}]^{\alpha_{ij}/q_i} \tag{38}$$

$$\hat{\bar{Y}}_{ji} = \alpha_{ji} \, \hat{\bar{Y}}_{ii} , \tag{39}$$

$$\hat{I}_i = \beta_i \, q_i \, \hat{\bar{Y}}_{ii} , \tag{40}$$

$$\hat{\bar{D}}_i = \hat{\bar{Y}}_{ii} - \bar{q}_i , \tag{41}$$

where

$$q_i = 1 - \sum_{\substack{j=1\\ j\neq i}}^{n} \alpha_{ji},$$

$$\bar{q}_i = 1 - \sum_{\substack{j=1\\ j\neq i}}^{n} \alpha_{ji} - \beta_i q_i .$$

The solution (38)—(41) is given for the values integrated in the time interval $[0, T]$. In order to find the stategies of the investments and maintenance costs as a functions of time, the values $\hat{Y}_{ji}$ and $\hat{I}_i$ should be introduced in the relations (29), (30).

4. Final remarks

We can find the similar relation (39) and the results obtained in [1]. The optimum maintenance costs and production of the system i are in proportion. The ratio of them is equal to the elasticity coefficient α_{ji}. In [1] the proportions hold at each time moment whereas in this paper for the values integrated in the whole planning time interval. The similar proportion holds for the investments, and the ration is equal to $\beta_i q_i$. In the planning period the investment costs are concentrated in the first part of the period — relations (29), (25).

The maintenance costs increase with the increasing production capacity of a sector. The aggregated costs are relatively equally distributed in time. There is no solution of the bang-bang type as it is characteristic in the case of the linear dynamic Leontief models. However the relation of the intersectoral flows described by (39), (40), being characteristic for those models, are hold.

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Optymalizacja systemu produkcyjnego całkowicie zdecentralizowanego

Rozważono system produkcyjny o całkowicie zdecentralizowanym zarządzaniu. Model zawiera n sektorów produkcyjnych i jeden sektor świadczący usługi inwestycyjne. Funkcje produkcji wszystkich sektorów są postaci Cobba—Douglasa. Każdy sektor produkcyjny maksymalizuje swój własny dochód względem kosztów bieżących i nakładów inwestycyjnych. Problem optymalizacji jest dy-

namiczny. Znaleziono optymalne strategie. Pokazano, że optymalne scałkowane koszty bieżące i nakłady inwestycyjne są proporcjonalne do scałkowanej produkcji, odpowiednio w poszczególnych sektorach.

Оптимизация полностью децентрализованной производственной системы

В работе рассматривается производственная система с полностью децентрализованным управлением. Модель содержит *n* производственных секторов и один сектор по капиталовложениям. Функции производства всех секторов имеют вид Кобба—Дугласа. Каждый производственный сектор максимизирует свою прибыль по отношению к текущим затратам и капиталовложениям. Задача оптимизации является динамической. Найдены оптимальные стратегии. Показано, что оптимальные интегрированные текущие затраты и капиталовложения пропорциональны интегрированному производству, соответствующие по отдельным секторам.