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Modelling of production, utility structure, process and technological change

by

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The paper deals with modelling of complex development system including production, consumption and environment. The production subsystem consists on n given sectors described by CES production functions. The decentralized decision systems allocates the GNP among the given spheres of activity including investments, consumption, gouvernment expenditures, environment, etc. in such a way that the given utility function attains the maximum value. The utility function changes along with GNP per capita. Then the price indices can be computed. The price changes result in the change of the production function technological coefficients. The future projections of development processes can be derived in the iterative form. All the model parameters can be derived from historical data.

1. Introduction

The complex normative model, shown in Fig. 1, has been studied in Refs. [2]—[5]. The model consists of three main subsystems: Production (P), Consumption

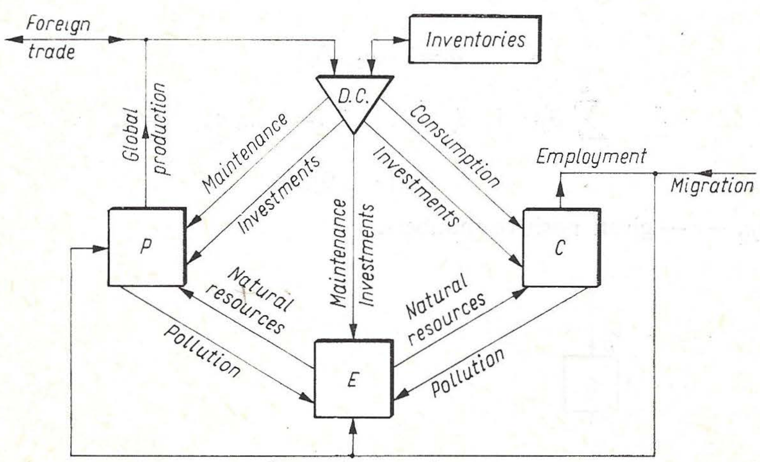

Fig. 1. The complex normative development model

(C) and the Environment (E). The global production (GNP) is splitted by the decision center (DC) between the capital investments and consumption; including the individual consumption (food, clothing etc.) and the public consumption (gouvernment expenditures such as education, health service, science etc.). The public consumption in the form of education, scientific and organizational progress stimulates the system development rate. Part of global production is also used for financing maintenance and investments in the environment E. This includes the expenditures for cleaning and preservation of the environment as well as the discoveries and excavation of natural resources.

The development goal (utility), which is adopted in the model takes into account such factors as individual and public consumption levels, environment quality, etc.

In the present paper the following extension of the model will be investigated:

a) modelling and optimization of CES-production sectors;

b) modelling of utility function change;

c) modelling of price indices;

d) modelling of technological change.

In order to use the model for forecasting the future development an identification technique, based on past observation has been applied.

2. Production sub-system

Consider the n-sector production model P (Fig. 2) described by the equations (compare [4]):

$$Y_i = Y_{ii} - \sum_{\substack{j=1\\ j\neq i}}^{n} Y_{ij}, \tag{1}$$

$$Y_{ii} = F_i^{q_i} \left\{ \sum_{\substack{j=1\\ j\neq i}}^{n} \vartheta_{ji} Y_{ji}^{-\nu} \right\}^{-\alpha_i/\nu}, \quad i=1, \ldots, n, \tag{2}$$

where

$$q_i = 1 - \alpha_i > 0, \tag{3}$$

$$\sum_{j=1}^{n} \vartheta_{ji} = 1, \ \vartheta_{ji} > 0, \quad j, i = 1, \ldots, n, \quad j \neq i, \tag{4}$$

$$\nu \in [-1, 0],$$

α_i, F_i, ϑ_{ji}, $-\nu$ — given positive numbers.

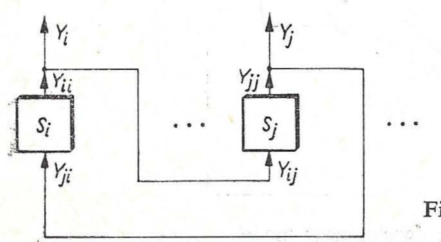

Fig. 2. The production model

Assume that outputs Y_i, Y_{ii} as well as the intersector flows Y_{ij}, $i, j=1, \dots, n$, are expressed in the monetary units. The first set of equations (1) describes the flow balances at the sector S_i output terminals. The sectors input-output relation (2) is the so called CES production function and $s=1+v$ is the elasticity of substitution factor. According to (3) no increasing of scale production effect is possible.

In the model under consideration it is assumed that the decentralized system of management exists. Namely, each production sector S_i, $i=1, \dots, n$, maximizes the corresponding profit (value added):

$$D_i = Y_{ii} - \sum_{\substack{j=1 \\ j \neq i}}^{n} Y_{ji}, \quad i=1, \dots, n. \tag{5}$$

In order to solve that problem one should find first of all the optimum input mix which maximizes the output (2). That problem is equivalent to maximization of

$$\bar{Y}_{ii} = Y_{ii}^{-v/\alpha_i} F^{q_i v/\alpha_i} = \sum_{\substack{j=1 \\ j \neq i}}^{n} (\bar{\vartheta}_{ji})^{1+v} Y_{ji}^{-v}, \tag{6}$$

where $(\bar{\vartheta}_{ji})^{1+v} = \vartheta_{ji}$.

Subject to the constrained total input cost

$$\bar{Y}_i \geqslant \sum_{\substack{j=1 \\ j \neq i}}^{n} Y_{ji}, \quad \text{and } Y_{ji} \geqslant 0, \; j=1, \dots, n. \tag{7}$$

Since (6) is concave and (7) defines a convex bounded set the unique optimization strategy $\hat{Y}_{ji}$ exists (one can find it by solving the equations $\delta \bar{Y}_{ii}/\delta Y_{ji}=0$, $j=1, \dots, n$, $j \neq i$):

$$\hat{Y}_{ji} = \frac{\bar{\vartheta}_{ji}}{\sum_{\substack{j=1 \\ j \neq i}}^{n} \bar{\vartheta}_{ji}} \bar{Y}_i, \; j=1, \dots, n, \; j \neq i, \; i=1, \dots, n. \tag{8}$$

Setting that strategy into (5) one finds easily

$$D_i(\hat{Y}_{ji}) = \bar{F}_i^{q_i} \bar{Y}_i^{\alpha_i} - \bar{Y}_i, \tag{9}$$

where

$$\bar{F}_i = F_i \left\{ \sum_{\substack{j=1 \\ j \neq i}}^{n} \bar{\vartheta}_{ji} \right\}^{-\frac{\alpha_i}{q_i} \frac{1+v}{v}}.$$

Now it is possible to derive the optimum value of $\bar{Y}_i = \hat{\bar{Y}}_i$ which maximizes the profit (9). Since (9) is concave the unique optimum strategy can be derived by solving the equation

$$D_i' = \alpha_i \bar{F}_i^{q_i} \bar{Y}_i^{\alpha_i - 1} - 1 = 0, \tag{10}$$

which yields

$$\hat{\bar{Y}}_i = \{\alpha_i \bar{F}_i^{q_i}\}^{1/1-\alpha_i} = \alpha_i^{1/q_i} \bar{F}_i. \tag{11}$$

Then the corresponding optimum output Y_{ii} becomes:

$$\hat{Y}_{ii}=\bar{F}_i^{q_i}\,\hat{Y}_i^{\alpha_i}=\bar{F}_i\,\alpha_i^{\alpha_i/q_i}=F_i\left\{\sum_{\substack{j=1\\ j\neq i}}^{n}\vartheta_{ji}\right\}^{-\frac{\alpha_i}{q_i}\,\frac{1+\nu}{\nu}}\alpha_i^{\alpha_i/q_i},\qquad i=1,\dots,n. \tag{12}$$

Multiplying (10) by Y_i one also obtains $\alpha_i\,\hat{Y}_{ii}=\bar{Y}_i$, or — using (8)—

$$\hat{Y}_{ji}=\frac{\vartheta_{ji}}{\sum\limits_{\substack{j=1\\ j\neq i}}^{n}\vartheta_{ji}}\,\alpha_i\,\hat{Y}_{ii},\qquad \begin{matrix} j=1,\dots,n, & j\neq i,\\ i=1,\dots,n. & \end{matrix} \tag{13}$$

For optimum strategy one gets

$$\hat{D}_i=\hat{Y}_{ii}\left(1-\sum_{\substack{j=1\\ j\neq i}}^{n}\frac{\vartheta_{ji}}{\sum\limits_{j}\vartheta_{ji}}\,\alpha_i\right)=q_i\,\hat{Y}_{ii}. \tag{14}$$

Changing the summation order one obtains also:

$$\sum_{i=1}^{n}\hat{Y}_i=\sum_{i=1}^{n}\left[\hat{Y}_{ii}-\sum_{\substack{j=1\\ j\neq i}}^{n}\hat{Y}_{ij}\right]=\sum_{i=1}^{n}q_i\,\hat{Y}_{ii}=\sum_{i=1}^{n}\hat{D}_i.$$

It is interesting to observe that under optimum strategy the outputs $\hat{Y}_{ii}$ do not depend on the inputs $\hat{Y}_{ji}, j=1,\dots,n, j\neq i$, or in other words, the interactions between sectors S_i, S_j, represented by $\hat{Y}_{ji}$ have been completely relaxed.

The formulae (12)—(14) can be treated as an extension of the relations obtained in [3] and [4] for the Cobb-Douglas production function

$$Y_{ii}=F_i^{q_i}\prod_{\substack{j=1\\ j\neq i}}^{n}Y_{ji}^{\alpha_{ji}}, \tag{15}$$

(which can be obtained from (2) when $\nu\to 0$ and $\vartheta_{ji}\,\alpha_i=\alpha_{ji}$, $j, i=1,\dots,n$, $j\neq i$).

In the last case one gets instead of (12)—(14)

$$\hat{Y}_{ii}=F_i\prod_{\substack{j=1\\ j\neq i}}^{n}\alpha_{ji}^{\alpha_{ji}/q_i}, \tag{16}$$

$$\hat{Y}_{ji}=\alpha_{ji}\,\hat{Y}_{ii}, \tag{17}$$

$$\hat{D}_i=q_i\,\hat{Y}_{ii},\qquad i=1,\dots,n. \tag{18}$$

The present method can be also easily extended for the production functions which are composed of Cobb-Douglas and CES functions, i.e.:

$$Y_{ii}=F_i^{q_i}\prod_{k=0}^{N-1}\left\{\sum_{\substack{j=\bar{n}k\\ j\neq i}}^{\bar{n}(k+1)}\vartheta_{ji}\,Y_{ji}^{-\nu_k}\right\}^{-\alpha_{ik}/\nu_k},$$

where

$$N\bar{n}=n, \ \nu_k \in [-1, 0], \quad \sum_{k=1}^{N} \alpha_{ik}=\alpha_i,$$

or

$$Y_{ii}=F_i^{q_i}\left\{\sum_{j=0}^{N-1} \vartheta_{ji} \prod_{k=\bar{n}j}^{\bar{n}(j+1)} Y_{ki}^{-\alpha_{ki}\nu}\right\}^{-\alpha_i/\nu},$$

where $N\bar{n}=n$, $\sum_{k=\bar{n}j}^{\bar{n}(j+1)} \alpha_{ki}=1$, $j=0, ..., N-1$.

The advantage of the present decomposition approach consists in the possibility of representation of the total production (GNP):

$$Y=\sum_{i=1}^{n} q_i \hat{Y}_{ii} \tag{19}$$

in the form which does not depend on the intersector production flows. It depends, however, on the exogeneous factors, such as capital investments, employment, technical and organizational progress etc., which enter into F_i factors (production capacities).

In the present model it is assumed that Y_{ij}, $i, j=1, ..., n$, are functions of time and[1])

$$F_i(t)=\int_{-\infty}^{t} k_i(t-\tau)\left[\prod_{\nu=0}^{N} [Z_{\nu i}(\tau-T_{\nu i})]^{\beta_\nu} d\tau, \qquad i=1, ..., n, \tag{20}$$

$Z_{\nu i}(\tau)$ — expenditures intensities in investment ($\nu=0$), education and training, research and development health and social care etc.; $k_i(t)$ — given nonnegative functions; β_ν, $T_{\nu i}$ — given nonnegative numbers; $\sum_{\nu=1}^{N} \beta_\nu \leqslant 1$.

The factors $F_i(t)$ combine, according to (20) all the inertial effects in the production processes, such as, e.g. the depreciation of capital and technical and scientific progress, depreciation of education and training etc. For that purpose the $k_i(t)$ function in (20) is assumed in to be

$$K_i(t)=\begin{cases} K_i \exp(-\delta_i t), & t \geqslant 0, \\ 0, & t<0, \end{cases}$$

where K_i, δ_i — positive constants.

The delays T_{oi} correspond to construction delays in investment processes. In the similar way $T_{\nu i}$ (for $\nu \geqslant 1$) correspond to education, research and development etc. delays.

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[1]) The integrals instead of sums are used in the present paper (when dealing with dynamic problems) for convenience in notation mainly.

As shown in [4] the production system parameters α_{ij}, $i,j=0,1,\ldots,n$, $i\neq j$, K_i, δ_i, T_{oi} can be estimated from historical data using the relations (16), (17), (20). Assuming that the total amount of expenditures $Z_{\nu i}$ is limited, i.e.

$$\sum_{i=1}^{n} Z_{\nu i}(t)\leqslant Z_\nu(t),\quad 0\leqslant t\leqslant T,\quad \nu=0,1,\ldots,N. \tag{21}$$

The problem of optimum allocation of Z_ν among the sectors S_i, $i=1,\ldots,n$, which yields the maximum value of total integrated production (19)

$$\bar{Y}=\int_0^T (1+\varepsilon)^{-t}\, Y(t)\,dt, \tag{22}$$

where ε — given discount rate, can be formulated. As shown in Sec. 3, the unique optimum strategy $Z_{\nu i}=\hat{Z}_{\nu i}$, $\nu=0,1,\ldots,N$, $i=1,\ldots,n$, exists and the corresponding value of Y becomes

$$\bar{Y}=\int_0^T f^q(\tau)\prod_{\nu=0}^{N} Z_\nu^{\beta_\nu}(\tau)\,d\tau,\quad q=1-\sum_{\nu=0}^{N}\beta_\nu. \tag{23}$$

When $Z_\nu^{\beta_\nu}(\tau)=Z_\nu^{\beta_\nu}=\text{const}$ for $\tau\in[0,T]$ the last expression can be written in the "static" form:

$$\bar{Y}=A\prod_{\nu=0}^{N} Z_\nu^{\beta_\nu},\qquad A=\int_0^T f^q(\tau)\,d\tau.$$

In the model under consideration the labour was not introduced explicitely yet. A possible way to do that is to consider the labour force as exogeneous factor and assume instead of (20):

$$F_i(t)=X_i^\alpha(t)\int_{-\infty}^{t} k_i(t-\tau)\prod_{\nu=0}^{N}[Z_{\nu i}(\tau-T_{\nu i})]^{\beta_\nu}\,d\tau,\qquad i=1,\ldots,n, \tag{24}$$

and

$$\sum_{i=1}^{n} X_i(t)\leqslant X(t),\qquad 0\leqslant t\leqslant T,$$

where $X(t)$ — the total labour.

Another extension of (24) one obtains when $X_i^\alpha(t)$ represents a group of noninertial factors $X_{\nu i}^{\alpha_\nu}$, i.e.

$$X_i^\alpha(t)=\prod_{\nu\in N} X_{\nu i}^{\alpha_\nu}(t),\quad \sum_{\nu}\alpha_i=\alpha,$$

and

$$\sum_{i=1}^{n} X_{\nu i}\leqslant X_\nu(t),\qquad 0\leqslant t\leqslant T.$$

In the case (24) when $\hat{X}_i(t)=X_i=\text{const}$, $t\in[0,T]$, one gets instead of (23):

$$\tilde{Y}=AX^\alpha\prod_{\nu=0}^{N} Z_\nu^{\beta_\nu}. \tag{25}$$

It is interesting to compare (25) with the macroeconomic Cobb-Douglas production function [1]:

$$Y = a \exp(\mu T)\, K^{\beta_0} X^{\alpha}, \tag{26}$$

where K — productive capital, μ — coefficients of technical and organizational progress; $\alpha + \beta_0 \approx 1$, $a = \text{const}$.

Obviously in the model (25) μT corresponds to $\ln\left[\prod_{\nu=1}^{N} Z_\nu^{\beta_\nu}\right]$ which gives more detailed description of the factors (Z_ν) which contribute to the technical and organizational progress. Besides $\bar{Y} \to Y$ when T increases and $Z_o = \omega_1 K$.

Introducing the notion of the growth rate ρ_x of a differentiable function $x(t)$

$$\rho_x = \dot{x}/x$$

the relations (25), (26) can be written

$$\rho_y = \alpha \rho_x + \sum_{\nu=0}^{N} \beta_\nu \rho_{z_\nu} \tag{27}$$

$$\rho_y = \mu + \alpha \rho_x + \beta_o \rho_k, \tag{28}$$

respectively.

Since $\rho_x \approx x^t - 1$, where $x^t = x(t)\, x(t-1)$ the last relations (for the case $\alpha + \sum_{\nu=0}^{N} \beta_o = 1$, $\alpha + \beta_\nu = 1$) can be also written as

$$Y^t = \alpha X^t + \sum_{\nu=0}^{N} \beta_\nu Z_\nu^t, \tag{29}$$

$$Y^t = \mu + \alpha X^t + \beta_o K^t. \tag{30}$$

It should be also noted that when the values of Y^t, X^t, K^t are known the value of the technical and organizational progress can be derived by (28) or (30).

As shown in [4] a possibility also exists to consider the labour as the output of an additional sector S_0, which cooperates with the production system $(S_1, \ldots, S_n)$. The production function of that sector (for $\nu = 0$)

$$Y_{oo} = F_o^q \prod_{j=1}^{n} Y_{jo}^{\alpha_{jo}}, \quad q_o = 1 - \sum_{j=1}^{n} \alpha_{ji} > 0,$$

where Y_{jo} — cost of goods produced by sectors S_j and consumed by sector S_0, $Y_{oo} = p_o X_{oo} + \bar{Y}_{oo}$ — total value of employment in monetary units, p_o — average net wage, X_{oo} — number of employies denoted by X in (26), $\bar{Y}_{oo}$ — other means of income.

The sector S_0 can be treated as a productive sector with the value added (savings)

$$D_o = Y_{oo} - \sum_{j=1}^{n} Y_{jo}.$$

One can assume that S_o objective is to maximize D_o (i.e. to maximize wages subject to the given consumption; the savings obtained in that way can be used for purchases of durable goods). Then according to (12)—(14) one gets:

$$\hat{Y}_{oo}=F_o \prod_{j=1}^{n} \alpha_{jo}^{\alpha_{jo}/q_o},$$

$$\hat{Y}_{jo}=\alpha_{jo}\, \hat{Y}_{oo},$$

$$\hat{D}_o=q_o\, \hat{Y}_{oo}.$$

In the similar way as for productive sectors (S_i, $i=1, \ldots, n$) it is also possible to introduce in F_o the intensities of purchases of durable goods.

The output $\hat{Y}_{oo}$ is divided among n productive and N nonprodutive sectors, i.e.

$$\hat{Y}_{oo}=\sum_{j=1}^{n+N} \hat{Y}_{oj},$$

and as a result it is necessary to replace the lower summation limit ($j=1$) in (1), (2), (4)—(8), (12)—(17) by $j=0$.

Using that approach the total individual consumers expenditures and savings become equal the total wages. Indeed

$$\hat{D}_o+\sum_{j=1}^{n} \hat{Y}_{jo}=\hat{Y}_{oo}\left(q_o+\sum_{j=1}^{n} \alpha_{jo}\right)=\hat{Y}_{oo}.$$

It is also possible to observe that the present approach is equivalent to the method which introduces labour X as the egzogeneous factor (compare (24)).

3. Utility and consumption-structure change

Assume the gros national product (GNP) Y_o generated by the productive system in the year $t-1$ to be allocated in the year t to the different spheres of activity according to the multi-level hierarchic structure shown in Fig. 3.

At the first level Y_o should be alloted to the two spheres of activity: accumulation of capital K (by investments Z^0) and consumption Z^1 (by labour L). The factors K, L are responsible for creation of the new GNP: $Y(t)$ according to the known macroeconomic relation [1]:

$$Y=AK^{\beta} L^{1-\beta}, \quad 0<\beta<1, \quad A \text{ — positive constant.} \tag{31}$$

Following the known imputation approach (see e.g. [1]) we assure also that the prices ω_1, ω_2 can be attached to the capital and labour respectively, so that

$$\omega_1 K+\omega_2 L=Y_o. \tag{32}$$

As shown in [1] ω_1 can be treated as the discount rate of capital used and ω_2 — the average gross wage [2]).

[2]) J. e. $\omega_2=(1+\Omega)p_0$, where Ω — wage tax, p_0 — average net wage.

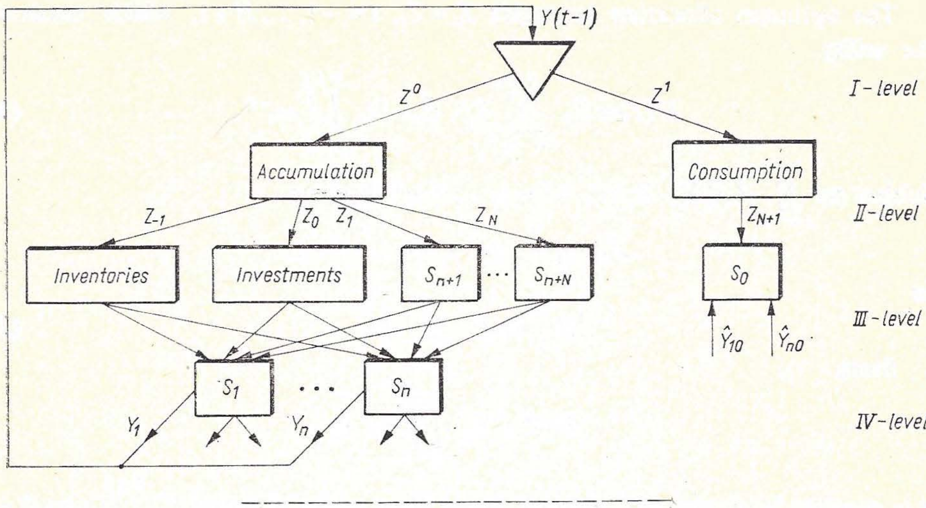

Fig. 3. The multi-level structure of allocation of GNP

The basic I-level optimization problem consists in finding $K=\hat{K}$, $L=\hat{L}$, such that Y attains maximum subject to the constraints[3])

$$\omega_1 K+\omega_2 L \leqslant Y_o, \quad K \geqslant 0, \quad L \geqslant 0. \tag{33}$$

It is easy to show that

$$\hat{K}=(\beta/\omega_1)\, Y_o, \quad \hat{L}=[(1-\beta)/\omega_2]\, Y_o$$

or

$$\hat{K}\omega_1=\hat{Z}^0=\beta Y_o, \quad \hat{L}\omega_2=\hat{Z}^1=(1-\beta)\, Y_o.$$

The I-level problem can be easily extended to the II-level case of allocation of Z^0, Z^1 in the structure of Fig. 3. For that purpose consider the utility function

$$U=U_0 \prod_{i=-1}^{N+1} X_i^{\gamma_i},$$

where

$$\sum_{i=-1}^{N+1} \gamma_i=1, \quad \gamma_i>0, \quad i=-1, \ldots, N+1, \quad U_0>0,$$

$X_i=Z_i/\omega_1$, ω_i — prices attached to X_i, Z_{-1} — inventories, Z_0 — productive investments, Z_1-Z_N — gouvernment expenditures on education, B+R, health and social care, administration, environment, etc. Z_{N+1} — individual consumption (net wages)$=\hat{Z}^1-p_0\,\Omega+\bar{Y}_{oo}$, where $\bar{Y}_{oo}$ — other means of in come (retirement pay, social benefits, etc.).

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[3]) That strategy based on the imputation principle (32) is some times called "the standard strategy" [1]. As a result one gets $p=\omega\text{E}$ [1].

The optimum allocation strategies $Z_i=\hat{Z}_i$, $i=-1, ..., N+1$, which maximize the utility

$$U(Z)=\bar{U}_o \sum_{i=-1}^{N+1} Z_i^{\gamma_i},\ \bar{U}_o=U_o \prod_{i=-1}^{N+1} \omega_i^{-\gamma_i}. \tag{34}$$

Subject to $\sum_{i=-1}^{N+1} Z_i \leqslant Y_o$, $Z_i \geqslant 0$, $i=-1, ..., N+1$, become

$$\hat{Z}_i=\gamma_i Y_o,\ i=-1, 0, ..., N+1,$$

or

$$\hat{Z}_i(t)=\gamma_i(t)\, Y(t-1). \tag{35}$$

Since

$$U(\hat{Z})=\bar{U}_o \prod_{i=-1}^{N+1} \gamma_i^{\gamma_i} Y_o$$

it is possible to choose the arbitrary factor U_o in such a way that $U_o \prod_{i=-1}^{N+1} (\gamma_i/\omega_i)^{\gamma_i}=1$ and $U(\hat{Z})=Y_o$. In that case the utility under optimum allocation strategy can be measured in monetary units and is equal to the GNP attained. When the allocation strategy differs from (35) utility is less Y_o.

We shall assume that the decisions concerning the allocation of Y_o are optimum. In the centrally planned economies they are derived by the central planning systems, in the form of annual budget, and proposed for approval to the parliament.Then from the historical data it is possible to identify the numerical values of $\gamma_i(t)= =\hat{Z}_i(t)/Y(t-1)$, $i=-1, ..., N+1$, $t=0, 1, ...$. These values change usually in time as a result of GNP, prices, population etc. changes. In the present section the simple models of allocation structure changes will be considered only. In a class of model it is assumed that $\hat{Z}_i$ depends mainly on the GNP per capita (in constant prices) in the preceeding year, i.e.

$$\hat{Z}_i(t)=a_i\,[\bar{Y}(t-1)/\bar{L}(t-1)]_j^{\varepsilon_i}\, Y(t-1), \tag{36}$$

$$i=-1, ..., N+1,$$

where a_i, ε_i — constant coefficients, $\bar{Y}(t)$ — GNP in constant prices, $\bar{L}(t)$ — population.

Since $\hat{Z}_i(t-1)$ are assumed to be known it is convenient to introduce the indices

$$Z_i^t=\hat{Z}_i(t)/\hat{Z}_i(t-1),\ \bar{Y}^t=\bar{Y}(t)/\bar{Y}(t-1),\ \bar{L}^t=\bar{L}(t)/\bar{L}(t-1)$$

and express (36) in the form

$$Z_i^t=\gamma_i^t\, Y^{t-1}=[\bar{Y}^{t-1}/\bar{L}^{t-1}]^{\varepsilon_i}\, Y^{t-1} \tag{37}$$

where $\gamma_i^t=\gamma_i(t)/\gamma_i(t-1)$.

The expression (37) does not satisfy, however, the balance condition

$$\sum_{i=-1}^{N+1} \hat{Z}_i(t)=\sum_{i=-1}^{N+1} Z_i^t\, \hat{Z}_i(t-1)=Y(t-1),$$

and in the model under consideration we can assume instead

$$Z_i^t = \frac{Y(t-1)}{\hat{Z}_i(t-1)} \frac{[\bar{Y}^{t-1}/\bar{L}^{t-1}]^{\varepsilon_i}}{\sum_{v=-1}^{N+1} [\bar{Y}^{t-1}/\bar{L}^{t-1}]^{\varepsilon_v}}, \quad i=-1, ..., N+1, \tag{38}$$

and

$$\gamma_i(t) = \frac{[\bar{Y}^{t-1}/\bar{L}^{t-1}]^{\varepsilon_i}}{\sum_{v=-1}^{N+1} [\bar{Y}^{t-1}/\bar{L}^{t-1}]^{\varepsilon_v}}, \quad i=-1, ..., N+1. \tag{39}$$

Similar relations can be obtained using $t=0$ as the base year. In that case one can write:

$$\tilde{Z}_i^t = Z_i(t)/Z_i(0), \; \tilde{Y}^t = \bar{Y}(t)/\bar{Y}(0), \; \tilde{L}^t = \bar{L}(t)/\bar{L}(0),$$

$$\tilde{\gamma}_i(t) = Z_i(t)/Y(0).$$

Obviously

$$\tilde{\gamma}_i(t) = \gamma_i(t)\, \tilde{Y}^{t-1}, \; \tilde{Y}^t/\tilde{L}^t = \prod_{k=0}^{k=t-1} \bar{Y}^{t-k}/\bar{L}^{t-k}.$$

$$\bar{\gamma}_i(t) = Z_i(t)/Y(t-1).$$

Using the least square method and statistical data it is possible to estimate the values of ε_i. As an example consider calculation of the models of consumption share in GNP: $\tilde{\gamma}(t) = [\gamma(t)/\gamma(0)]\,[\tilde{Y}^{t-1}/T^{t-1}]$ in Poland (for the base year $t=1960$) given in Table 1 and Fig. 4.

Table 1

	Year					
	1950	1960	1965	1970	1972	1972
Y^t/L^t	0.563	1	1.249	1.601	1.744	1.950
$\gamma(t)$	0.79	0.758	0.732	0.721	0.707	0.684
$\tilde{\gamma}(t)$	1.04	1	0.965	0.950	0.932	0.902
$\ln \tilde{\gamma}(t)$	0.041	0	—0.035	—0.051	—0.070	—0.093
$\ln \tilde{Y}^t/\tilde{L}^t$	—0.570	0	0.222	0.470	0.555	0.669
$\ln \gamma(t)/\ln \tilde{Y}^t/\tilde{L}^t$	—0.0703	—	—0.158	—0.108	—0.126	—0.139

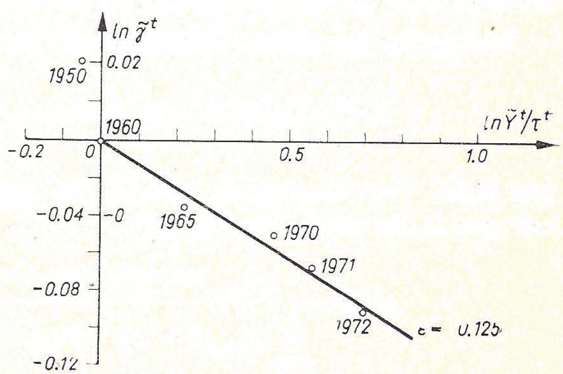

Fig. 4. The consumption share in GNP

As follows from Fig. 4 in the period 1960—1972 elasticity of consumption share ε was arround -0.125 with comparatively small square error. The corresponding value of $\ln \tilde{\gamma}(t)/\ln \tilde{Y}^t/\tilde{L}^t$ for $t=1950$ differs much as compared to -0.125. It corresponds however to the past and obrolete situations.

In order to attach less weight to the past data a weighted least square method can be used. According to that method the estimate $\tilde{\varepsilon}_i$ can be derived by solving

$$\min_{\varepsilon_i} \sum_{t=-T}^{t=0} w(t)\,[\ln \gamma_i(t)-\varepsilon_i \ln \tilde{Y}^{t-1}/\tilde{L}^{t-1}]^2,$$

where $w(t)=(1-\delta)^t$, $0<\delta<1$, T — estimation time.

Since the „aging" of an economy depends on the discount rate η it was proposed to assume $\delta \approx \eta$, which is in Poland arround 0.1.

Using estimates of ε_i derived by that method, i.e.

$$\tilde{\varepsilon}_i=\frac{\sum\limits_{t=-T}^{0} w(t) \ln \gamma_i(t) \ln \tilde{Y}^{t-1}/\tilde{L}^{t-1}}{\sum\limits_{t=-T}^{0} w(t)\,[\ln \tilde{Y}^{t-1}/\tilde{L}^{t-1}]^2}, \quad i=-1, \ldots, N+1 \tag{40}$$

one attaches to all the data before $T=12$ years the weight less $0.9^{12}=0.28$.

Another model of allocation of GNP takes into account the change of price indices attached to Z_v. It is especially suitable when one studies the consumers expenditures $\hat{Y}_{io}$, $i=1, \ldots, n+N$, with the budget (income) Z_{N+1} (see Fig. 3) on the market.

The consumers strategy $\hat{Y}_{io}$, $i=1, \ldots, n+N$, depends not only on the wages per capita

$$Z_{N+1}/\bar{L}=\gamma_{N+1}(t)\,[Y(t-1)/\bar{L}(t-1)]$$

but as well on the prices p_i of the goods Y_{io}.

Then instead of (36), one can assume

$$\hat{Y}_{io}(t)=b_i\,[\gamma_{N+1}(t)\,Y(t-1)/\bar{L}(t-1)]^{\varepsilon_i}\,[p_i(t-1)]^{E_i}\,\gamma_{N+1}(t)\,Y(t-1) \tag{41}$$

where ε_i, E_i — income and price elasticities respectively.

Introducing the relative factors $p_i^t=p_i(t)/p_i(t-1)$

$$\gamma_i^1(t)=Y_{io}(t)/Z_{N+1}(t), \quad i=0, 1, \ldots, n+N, \quad \sum_{i=1}^{n+N} \gamma_i^1(t)=1,$$

(the index $i=0$ corresponds here to the savings $\hat{D}_o=q_o\,\hat{Y}_{oo}$ in the model S_0), one gets in the similar way as (39)

$$\gamma_i^1(t)=\frac{[\gamma_{N+1}^t\,Y^{t-1}/\bar{L}^{t-1}]^{\varepsilon_i}\,[p^{t-1}]^{E_i}}{\sum\limits_{i=0}^{n} [\gamma_{N+1}^t\,Y^{t-1}/\bar{L}^{t-1}]^{\varepsilon_i}\,[p_i^{t-1}]^{E_i}}, \quad i=0, 1, \ldots, n+N, \tag{42}$$

Obviously

$$\sum_{i=0}^{n+N} \hat{Y}_{io}(t) = \sum_{i=0}^{n} \gamma_i^1(t)\, Z_{N+1}(t) = Z_{N+1}(t).$$

Since on the other hand $\hat{Y}_{io}(t) = \alpha_{io} \hat{Y}_{oo}(t) = \alpha_{io} Z_{N+1}(t)$ then $\alpha_{io} = \gamma_i^1(t)$, $i=1, \ldots$ $\ldots, n+N$, $q_o = \gamma_0^1(t)$. In other words, the change of $\tilde{Y}^{t-1}/L^{t-1}$, γ_{N+1}^t and price p_i^{t-1} changes the consumption structure expressed by α_{io} coefficients (elasticities of utility function) in S_0 sector.

It remains to consider the allocation of $Z_i, i=-1, 0, \ldots, N$, resources among the sectors S_i, $i=1, \ldots, n$, of the production subsystem (III-level in Fig. 3).

As already mentioned in Sec. 2 that problem consists in maximizing the functional (22), i.e.

$$\bar{Y}(z) = \int_0^T (1+\varepsilon)^{-t} \sum_{i=1}^{n} q_i \hat{Y}_{ii}(t)\, dt =$$

$$= Y_0 + \int_0^T (1+\varepsilon)^{-t} \sum_{i=1}^{n} \int_0^T k_i(t-\tau) \prod_{\nu=-1}^{N} Z_{\nu i}^{\beta_\nu}(\tau - T_{\nu i})\, d\tau\, dt,$$

where

$$Y_0 = \int_0^T (1+\varepsilon)^{-t} \sum_{i=1}^{t} \int_{-\infty}^{0} k_i(t-\tau) \prod_{\nu=-1}^{N} Z_{\nu i}^{\beta_\nu}(\tau - T_{\nu i})\, d\tau\, d\tau,$$

represents the inertial effect of allocation of resources $Z_{\nu i}(\tau)$, $i=1, \ldots, n$, $\nu=-1, \ldots, N$, in the past $(\tau < 0)$.

Since Y_0 is a given number the present optimization problem can be formulated as follows. Find the nonnegative strategies $Z_{\nu i}(t) = \hat{Z}_{\nu i}(t)$, $\nu=1, \ldots, N$, $i=1, \ldots, n$, $t \in [0, T]$ such that the functional

$$Y(Z) = \sum_{i=1}^{n} \int_0^T \prod_{\nu=-1}^{N} Z_{\nu i}^{\beta_\nu}(\tau - T_{\nu i}) f_i^q(\tau)\, d\tau, \tag{43}$$

where

$$f_i^q(\tau) = \int_\tau^T k_i(t-\tau)(1+\varepsilon)^{-t}\, dt,$$

$$q = 1 - \sum_{\nu=-1}^{N} \beta_\nu > 0,$$

is maximum subject to the constraints

$$\sum_{i=1}^{n} Z_{\nu i}(t) \leqslant Z_\nu(t), \quad \nu=-1, 0, \ldots, N, \quad t \in [0, T] \tag{44}$$

where $Z_\nu(t)$ — given functions of time.

The functional (43) will attain the maximum value, when the function

$$\varphi(\tau) = \sum_{i=1}^{n} \prod_{\nu=-1}^{N} Z_{\nu i}^{\beta_\nu}(\tau - T_{\nu i}) f_i^q(\tau)$$

attains the maximum value for each $t \in [0, T]$ subject to the constraints (44). As shown in [5] the optimum strategy in that case becomes

$$\hat{Z}_{\nu i}(\tau - T_{\nu i}) = \frac{f_i(\tau)}{f(\tau)} Z_\nu(\tau), \qquad \nu = -1, 0, \ldots, N, \quad i = 1, \ldots, n, \tag{45}$$

$$f(\tau) = \sum_{i=1}^{n} f_i(\tau),$$

and

$$Y(\hat{Z}) = \int_0^T f^q(\tau) \prod_{\nu=-1}^{N} Z_\nu^{\beta_\nu}(\tau)\, d\tau. \tag{46}$$

Then when $T \to \infty$, $Y_0 \to 0$, and $\bar{Y} \to Y$, the aggregated production function (46) assumes for given $\hat{Z}_\nu(\tau) = Z_\nu = \bar{Z}_\nu(t)$, where $Z_\nu = \int_0^T \hat{Z}_\nu(\tau)\, d\tau$, the statical form similar to (34) and it is possible to assume $\beta_\nu = \gamma_\nu(t)$, $\nu = -1, 0, \ldots, N$. Since generally γ_ν is a function of time one can also assume $\beta_\nu(t) = \gamma_\nu(t)$, $t \in [0, T]$, $\nu = -1, \ldots, N$, or assume that β_ν is equal the averaged (t), i.e. $\beta_\nu = \frac{1}{T} \int_0^T \gamma_\nu(\tau)\, d\tau$.

Since $Z_\nu(t)$ can be determined in the iterative manner by (36), (39) it is also possible to derive the $\hat{Z}_{\nu i}(t)$, $\nu = -1, 0, \ldots, N$, $i = 1, \ldots, n$, in the given planning interval $[0, T]$.

As shown in [5] the optimum allocation problem can be solved effectively also in the case when $F_i(t)$ contain a group (N_1) of noninertial (e.g. the above) as well as inertial (N_2) factors (compare (24) of Sec. 2) i.e.

$$F_i(t) = \prod_{\nu=1}^{N_1} X_{\nu i}^{\alpha_\nu}(t) \int_{-\infty}^{t} k_i(t-\tau) \prod_{\nu=1}^{N_2} [Z_{\nu i}(\tau - T_{\nu i})]^{\beta_\nu}\, d\tau, \quad i = 1, \ldots, n, \; N_1 + N_2 \leqslant N + 1,$$

and when besides the amplitude constraints (44) the integral type of constraints

$$\int_0^T \sum_{i=1}^{n} X_{\nu i}(t)\, dt \leqslant X_\nu, \qquad \int_0^T \sum_{\nu=1}^{n} Z_{\nu i}(t)\, dt < Z_\nu,$$

where X_ν, $\nu \in N_1$, Z_ν, $\nu \in N_2$ — given numbers; should be valid.

It should be observed that in the centrally planned economics the optimization of total investment strategy is a common practice. The optimization of the gouvernment expenditures $Z_{\nu i}$, $\nu \neq 0$, is not so evident. The efforts are being done, however, to increase the training, R+D, etc. effect on the production efficiency. That can be done by a better allocation of resources $Z_{\nu i}$ among the S_{Z_ν} sectors which help to increase production in S_i sectors. Fore example the part of budget spent on research and development can be assigned to these areas which maximize the total production output. Then the "production function" of the $S_{Z\nu}$ sectors can be written as

$$Y_{Z_\nu} = F_{Z_\nu}^{q_\nu} \prod_{j=0}^{n} X_{jZ_\nu}^{\alpha_{jZ\nu}}, \qquad q = 1 - \sum_{j=0}^{n} \alpha_{jZ_\nu} > 0.$$

The value added

$$\hat{D}_{Z_\nu} = q_\nu X_{Z_\nu}$$

corresponds here to the expenditures spent on sectors S_{Z_ν} own activity (e.g. the basic research in the case of R+D sector).

Summarizing the results obtained in the present section one can observe that the allocation strategies in the multi-level structure of Fig. 3 have been completely determined.

They can be used for prediction purposes when the system parameters such as ε_i, E_i have been estimated by historical data. However in order to derive the future projections one have to determine also the price indices.

4. Price indices

Studying the sector of aggregated M different products it is convenient to deal with the aggregated proce p instead of detailed products prices, p_i, $i=1, ..., M$. When the sector production in natural units (e.g. in tons) is $X=\sum_{i=1}^{M} X_i$, the aggregated price is defined as

$$p=\sum_{i=1}^{M} \frac{X_i}{X} p_i. \tag{47}$$

When that prices changes with respect to the previous year it is convenient to introduce the price index $p^t = p(t)/p(t-1)$.

Using the price index it is possible to derive the value $\bar{Y}(t)$ of production $X(t)$ in the prices of the year $t-1$, i.e.

$$\bar{Y}(t)=Y(t)/p^t, \tag{48}$$

or in the prices of the basic year $t=0$

$$\tilde{Y}(t)=Y(t)/p^0, \quad p^0=\prod_{\tau=1}^{t} p^\tau.$$

Now we shall investigate the price-change mechanism resulting from the change of demand and supply in the model of Secs. 2, 3.

It is necessary to observe that since the model flows $Y_{ji}, j, i=1, ..., n$, are specified in monetary units the net outputs (i.e. the supplies) Y_i, $i=1, ..., n$, are determined in an unique manner. When the demands Y_i change in time as a result of consumption structure change, described in Sec. 3, the prices attached to $Y_{ji}, j, i=1, ..., n$, should change in such a way that $Y_i = Y_i$, $i=1, ..., n$.

Consider the n-sector production system with the Cobb-Douglas production functions in natural units (compare (2) for $v \to 0$):

$$X_{ii}(t)=[F_i^*(t)]^{q_i} \prod_{\substack{j=0 \\ j\neq i}}^{n} X_{ji}^{\alpha_{ji}}(t), \qquad i=1, ..., n.$$

Introducing prices $p_j(t) = Y_{ji}(t)/X_{ji}(t)$, $j=0, 1, ..., n$, one gets

$$Y_{ii}(t)=[F_i^*(t)]^{q} p_i(t) \prod_{\substack{j=0 \\ j\neq i}}^{n} p_i(t)^{-\alpha_{ji}} \prod_{\substack{j=0 \\ j\neq i}}^{n} Y_{ji}^{\alpha_{ji}} = [F_i(t)]^{q_i} \prod_{\substack{j=0 \\ j\neq i}}^{n} Y_{ji}^{\alpha_{ji}}$$

where

$$F_i(t)=F_i^*(t)\left[p_i(t)\prod_{\substack{j=0\\j\neq i}}^{n} p_j(t)^{-\alpha_{ji}}\right]^{1/q_i}. \tag{49}$$

Introducing the indices $p_i^t=p_i(t)/p_i(t-1)$, $Y_{ii}=\hat{Y}_{ii}(t)/\hat{Y}_{ii}(t-1)$, $F_i^{*t}=$ $=F_i^*(t)/F_i^*(t-1)$, $i=0, 1, ..., n$, one gets by (49)

$$p_i^t \prod_{\substack{j=0\\j\neq i}}^{n} [p_j^t]^{-\alpha_{ji}}=[F_i^t/F_i^{*t}]^{\alpha_i}, \quad i=1, ..., n. \tag{50}$$

As follows from (46)

$$F_i(t)=\hat{Y}_{ii}(t)\prod_{\substack{j=0\\j\neq i}}^{n}(\alpha_{ji})^{-\alpha_{ji}/q_i},$$

so $F_i(t)$ is measured in monetary units. On the other hand (49) suggests that a price p_i^* can be attached to the production capacity F_i^* so that

$$F_i(t)=F_i^*(t)\,p_i^*(t). \tag{51}$$

The price $p_i^*(t)$ can be treated as the price of productive capacity. It may change in time as a result of changes in depreciation rate of productive capacity, i.e. the change of discount rate, deterioration or improvement in training, R+D, etc. [4]).

Taking into account (50), (51) and observing that $F_i^t=\hat{Y}_{ii}^t=Y_i^t$ (see (16)), one gets

$$\ln p_i^t-\sum_{\substack{j=0\\j\neq i}}^{n}\alpha_{ji}\ln p_j^t=q_i\ln\left[\frac{Y_{ii}^t p_i^{*t}}{F_i^t}\right]+\alpha_{oi}\ln p_0^t, \quad i=1, 2, ..., n. \tag{52}$$

where according to Sec. 3

$$p_0^t=\frac{\gamma_{N+1}^t\, Y^{t-1}}{X_{oo}^t},$$

$X_{oo}^t=X_{oo}(t)/X_{oo}(t-1)$ — employment index (egzogeneous variable).

Assuming that the determinant

$$D=\begin{vmatrix} 1, & -\alpha_{21}, & ..., & -\alpha_{ni}\\ -\alpha_{12}, & 1, & ..., & -\alpha_{n2}\\ \dots & \dots & \dots & \dots\\ -\alpha_{1n}, & ..., & ..., & 1\end{vmatrix}\neq 0 \tag{53}$$

it is easy to observe that a system of positive price indices p_i^t, $i=1, ..., n$, exists.

In the stationary situation when $Y_i^t p_i^{*t}/F_i^t=p_0^t=1$, $i=1, ..., n$, the prices derived by (52) $p_i^t=1$, $i=1, ..., n$.

---

[4]) Taking into account relations (34), (46) it is also possible to express $p_i^*(t)$ in terms of prices ω_ν, $\nu=-1, ..., N+1$, attached to the $X_\nu=Z_\nu/\omega_\nu$. As shown in Ref. [5] the present prince model can be explained in terms of the general economic equilibrium theory.

The factor $Y_i^t p_i^{*t}/F_i^t$ expresses the ratio of the final demand change (Y_i^t) to the supply change F_i^t/p_i^*.

As shown in Sec. 3 the demand, specified by the $\gamma_\nu(t)$ coefficients, changes as a result of GNP-change. In other words, a system of nonnegative coefficients $\lambda_{i\nu}$, $i=1, \ldots, n$, $\nu=-1, .., N+1$, exists, such that

$$\sum_{\nu=-1}^{N+1} \lambda_{i\nu} Z_\nu = Y_i, \quad i=1, \ldots, n, \quad \text{where } Y_i = \hat{Y}_{ii} - \sum{}' \alpha_{ij} \hat{Y}_{jj}.$$

Since $Z_\nu(t)=\gamma_\nu(t)\, Y(t-1)$, $\nu=-1, \ldots, N+1$, the last relation can be solved with respect to $\hat{Y}_{ii}$, $i=1, \ldots, n$, and the solution $Y_{ii}(\gamma)$ can be written also as

$$Y_{ii}(\gamma)=l_i(t)\, Y(t-1)$$

or

$$Y_{ii}^t = l_i^t\, Y^{t-1}, \quad i=1, \ldots, n. \tag{54}$$

The relation (54) specifies the demand indices Y_i^t for the year t in term of GNP and consumers structure change.

On the other hand the investments and other gouvernment expenditures $(Z_{\nu i})$ change F_i^t, according to the formula (24)

$$F_i^t = \frac{\int_{-\infty}^{t} k_i(t-\tau) \prod_{\nu=-1}^{N} Z_{\nu i}^{\beta_\nu}(\tau - T_{\nu i})\, d\tau}{\int_{-\infty}^{t-1} k_i(t-1-\tau) \prod_{\nu=-1}^{N} Z_{\nu i}^{\beta_\nu}(\tau - T_{\nu i})\, d\tau}, \quad i=1, \ldots, n.$$

The numerical values of p_i^{*t} can be estimated from historical data.

In the simpliest case of one sector economy when the numbers $p_1^t, p_0^t, F_i^t/Y^t$ are given and there is no consumption structure change $(l_1^t=1)$ the price index of productive capacity can be derived by formula

$$p_1^{*t} = \frac{F_1^t}{Y^t} \left[\frac{p_1^t}{(p_0^t)^{\alpha_{01}}} \right]^{1/q_1}, \quad q_1 = 1 - \alpha_{01}. \tag{55}$$

When the productive capacity F_i^t increases witg a faster rate than Y^t the p_1^{*t} goes up. The increase in wages decrease the price index P_1^{*t}.

The values of Y^t, F_1^t, p_0^t, p_1^t for polish economy in the time period (1961—1972) according to [6]) are given in Table 2.

Table 2

	Year				
	1961	1966	1970	1971	1972
Y^t	1.082	1.071	1.052	1.081	1.101
F_1^t	1.042	1.052	1.065	1.055	1.059
p_1^t	1.009	1.012	1.011	1.004	1.000
p_0^t	1.040	1.041	1.031	1.057	1.067
p_1^{*t}	0.9097	0.9375	0.9818	0.8814	0.8422

Assuming $q_1=0.32$ (according to [7] that is the average value of capital elasticity in the production function (β_0 in formula (26)) in Table 2 the corresponding values of p_1^{*t} have been also derived. Strictly speaking the data F_1^t used in Table 2 correspond to the productive capital only and they do not reflect the changes in Z_ν, $\nu \neq 0$. Then the values of p_1^{*t} derived in Table 2 take into account as well the effects of technical and organizational progress, R+D, education, etc. The wage indix p_0^t has been derived using the gross average value of wages per capita (the private sectors of economy have been neglected). The average of the values p_1^{t*} derived in Table 2 is 0.91.

Since the allocation strategies $Z_{\nu i}(t)$ are specified, according to formula (61) by $Z_\nu(t)$, and the last functions depend on $Y(t-1)$ (according to (35)) is is possible to derive by (52) the price indices p_i^t, $i=1, \ldots, n$, by an iterative process starting with the basic year ($t=0$) statistical data.

So far the pricing model has concerned the closed economy. It is, however, possible to extend that model to the case when one or more external foreign trade markets are present.

Consider as an example the n-sector production system (the domestic system) with one foreign trade sector (market) S_z (Fig. 5), where $\hat{Y}_{zz}$ represents the total export of the market S_z, $\bar{D}=d\hat{\bar{Y}}_{zz}$ — the balance of payment and $\bar{E}$, $\bar{I}$ expoert and import to the domestic system (in S_z currancy).

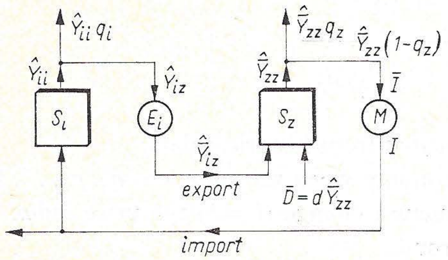

Fig. 5. The model of foreign-trade system

The production function of S_z sector (in market S_z currency) can be written in similar form to the S_i-sectors functions, i.e.

$$\bar{Y}_{zz}=F_z^{q_z}\prod_{i=1}^{n}\bar{Y}_{iz}^{\alpha_{iz}}\bar{D}^d,\; q_z=1-\sum_{i=1}^{n}\alpha_{iz}-d>0. \tag{56}$$

It is also possible to write down the sector S_z production function in S_i-currency

$$Y_{zz}=F_z^{q_z}\prod_{i=1}^{n}Y_{iz}^{\alpha_{iz}}D^d. \tag{57}$$

Since Y_{iz}, $i=1, \ldots, n$, Y_{zz} are measured in domestic currency it is necessary to introduce the exchange rates:

a. The export exchange rate

$$E_i=\bar{Y}_{iz}/Y_{iz}=\bar{p}_{iz}/p_i, \tag{58}$$

where p_i is the domestic and $\bar{p}_{iz}$ — the market, S_z price index for goods produced by S_i.

b. The import exchange rate

$$M = Y_{zz}/Y_{zz}\,. \tag{59}$$

It is assumed that sector S_z maximizes the net output and as a result the total export (in S_z currency) becomes

$$\bar{E} = \hat{\bar{Y}}_{zz} \sum_{i=1}^{n} \alpha_{iz} = \hat{\bar{Y}}_{zz}\,[1-(q_z-d)]\,, \tag{60}$$

or

$$E = \sum_{i=1}^{n} \hat{Y}_{iz} = \sum_{i=1}^{n} (\hat{\bar{Y}}_{iz}/E_i) = \hat{\bar{Y}}_{zz} \sum_{i=1}^{n} \alpha_{iz}/E_i) \tag{61}$$

(in domestic currency).

In the similar way the total import $\bar{I}(I)$ can be written as

$$I = \sum_{i=1}^{n} \alpha_{zi}\, Y_{ii}\,. \tag{62}$$

$$\bar{I} = M^{-1}\, I = \hat{\bar{Y}}_{zz}(1-q_z)\,. \tag{63}$$

In the system of Fig. 5 the following balance of payment (in market S_z currency) is observed

$$\bar{E} - \bar{I} = \hat{\bar{Y}}_{zz}\, d = D\,. \tag{64}$$

As follows from (63), (62)

$$M = \frac{I}{\bar{I}} = \frac{\sum\limits_{i=1}^{n} \alpha_{zi}\, \hat{Y}_{ii}}{\hat{\bar{Y}}_{zz}(1-q_z)} \tag{65}$$

where by (16)

$$\hat{Y}_{ii} = F_i \prod_{\substack{j=1\\ j\neq i}}^{n} \alpha_{ji}^{\alpha_{ji}/q_i}\, \alpha_{zi}^{\alpha_{zi}/q_i}, \quad q_i = 1 - \sum_{\substack{j=1\\ j\neq i}}^{n} \alpha_{ji} - \alpha_{zi} > 0\,, \qquad i = 1, \ldots, n\,,$$

$$\hat{\bar{Y}}_{zz} = \bar{F}_z \prod_{j=1}^{n} \alpha_{jz}^{\alpha_{jz}/q_z}\, d^{d/q_z}\,.$$

According to (65) the import exchange rate M increases along with the increase of demands for import ($\alpha_{zi}\, \hat{Y}_{ii}$) claimed by S_i and decreases when the supply of import (from the market S_z) $(1-q_z)\, \hat{\bar{Y}}_{zz}$ increases. From the domestic point of view the lower is the M the bigger are the benefits.

The value of M decreases also along with the increase of balance of payment $d = \bar{D}/Y_{zz}$.

Using in addition a more restrictive balance of payment condition $D=0$ and

$$I - E = 0 \tag{66}$$

one gets by (65), (61)

$$M=\frac{E}{\bar{I}}=\frac{1}{1-q_z}\sum_{i=1}^{n}\frac{\alpha_{iz}}{E_i}. \tag{67}$$

Then (67) can be written as

$$M=\sum_{i=1}^{n}\bar{\alpha}_{iz}/E_i, \tag{68}$$

where $\bar{\alpha}_{iz}=\hat{Y}_{iz}/\sum_{i=1}^{n}Y_{iz}$ is the products of S_i share in the total export E.

As follows from (68) the resulting import exchange rate M is determined by the export exchange rates E_i and the shares $\bar{\alpha}_{iz}$. In order to decrease M (i.e. increase the value of import $\bar{I}$) the domestic system should increase E_i by decreasing local prices p_i with respect to $\bar{p}_{iz}$. That can be done by investments and technical and organizational progres in these sectors S_i which yield the greatest values of E_i.

In the case when $E_i=E=$const for all $i=1, ..., n$, one gets by (67) $M=1/E$.

It should be observed that in the case when (66) holds the following condition (obtained by (65), (67)) relating the parameters of domestic and market S_z systems

$$\sum_{i=1}^{n}[\alpha_{zi}(\hat{Y}_{ii}/\hat{\hat{Y}}_{zz})-\alpha_{iz}(p_i/\bar{p}_{zi})]=0,$$

where

$$\frac{\hat{Y}_{ii}}{\hat{\hat{Y}}_{zz}}=\frac{F_i}{\bar{F}_z}\frac{\prod_{\substack{j=1\\ j\neq i}}^{n}\alpha_{ji}^{\alpha_{ji}/q_i}}{\prod_{j=1}^{n}\alpha_{jz}^{\alpha_{jz}/q_z}}\frac{\alpha_{zi}^{\alpha_{zi}/q_i}}{d^{d/q_z}}$$

is valid.

Now we can return to the main problem of the present section which is the solution of price equations (52). When the term Y_{zi} is added to the standard production function one gets instead of (49)

$$F_i(t)=F_i^*(t)\left[p_i(t)\,p_{zi}(t)^{-\alpha_{zi}}\prod_{\substack{j=0\\ j\neq i}}^{n}p_j(t)^{-\alpha_{ji}}\right]^{1/q_i} \tag{69}$$

$$q_i=1-\prod_{\substack{j=0\\ j\neq i}}^{n}\alpha_{ji}-\alpha_{zi}>0, \qquad i=1, ..., n,$$

where p_{zi} represents the aggregates price of imported goods, which can be written $p_{zi}=M\bar{p}_{zi}$, $\bar{p}_{zi}$ being the imported goods price index. It should be observed that the export and import aggregated prices $\bar{p}_{iz}$, $\bar{p}_{zi}$ differ generally as a result of different shares (X_i/X in (47)) for exported and imported goods.

Consequently the right side of eqs. (52) should be supplemented with the term

$$\sum_{z=1}^{s}\alpha_{zi}\ln M^t\bar{p}_{zj}^t, \tag{70}$$

where s — the number of different markets, $M^t = M(t)/M(t-1)$, $\bar{p}^t_{zi} = \bar{p}_{zi}(t)/\bar{p}_{zi}(t-1)$ — the import exchange and price indices.

As follows from (67), (68) M^t depends also on the domestic prices p^t_i. That fact complicates the solution of egs. (52). Taking into account (68) we can assume hoever that $E_i(t) \approx E(t) = 1/M(t)$, $i = 1, \ldots, n$, which can be called the "assumption of rational foreign trade policy".

Then introducing into (69)

$$p_{zi}(t) = p_i(t)/T_{zi}(t), \tag{71}$$

where

$$T_{zi}(t) = \bar{p}_{iz}(t)/\bar{p}_{zi}(t), \qquad i = 1, \ldots, n,$$

can be called the "terms of trade" for sector S_i, one gets instead of (52)

$$\left(1 - \sum_{z=1}^{s} \alpha_{zi}\right) \ln p^t_i - \sum_{\substack{j=1 \\ j \neq i}}^{n} \alpha_{ji} \ln p^t_j = q_i \left[\frac{Y^t_{ii} p^{*t}_i}{F^t_i}\right] + \alpha_{oi} \ln p^t_0 - \sum_{z=1}^{s} \alpha_{zi} \ln T^t_{zi}, \tag{72}$$

where

$$q_i = 1 - \sum_j \alpha_{ji} - \sum_z \alpha_{zi} > 0, \quad T^t_{zi} = T_{zi}(t)/T_{zi}(t-1), \quad i = 1, \ldots, n, \tag{73}$$

Now it is possible to derive p^t_i from (72) in the iterative manner, starting with the base year $t = 0$. It should be observed that the increase of T^t_{zi}, decreases the domestic prices p^t_i, $i = 1, \ldots, n$.

As follows from (71), (74) the domestic price indices p^t_i depend much on T^t_{zi}, i.e. the export and import indices $\bar{p}_{zi}$, $\bar{p}^t_{iz}$, which should be introduced into the modell exogeneously.

5. Technological change

Consider again the production function (16) and relations (17) and (18)

$$\alpha_{ji} = \hat{Y}_{ji}/\hat{Y}_{ii} = p_j \hat{X}_{ji}/p_i \hat{X}_{ii}, \qquad i, j = 0, 1, \ldots, n, \ i \neq j, \tag{74}$$

$$D_i = (1 - \alpha_i) \hat{Y}_{ii}, \qquad i = 0, 1, \ldots, n. \tag{75}$$

The relation (74) represents the input j share in the net output $\alpha_i \hat{Y}_{ii}$, $\alpha_i = \sum_{\substack{j=0 \\ j \neq i}}^{n} \alpha_{ji}$.

The production sectors S_i, $i = 0, 1, \ldots, n$, maximize D_i by choosing the best mix of inputs $\hat{Y}_{ji}$, $j, i = 0, 1, \ldots, n$. When the prices p_j, p_i change the substitution process should follow in the ideal Cobb-Douglas production function in such a way that $\alpha_{ji} = c_{ji} = \text{const.}$, i.e. the sectors should change $\hat{X}_{ji}/\hat{X}_{ii}$ in order to obtain $\hat{X}_{ji}/\hat{X}_{ii} = c_{ji} p_i/p_j$.

That requires, generally speaking, an adaptation delay during which the change of technologies, investments, etc. follows.

On the other hand in many cases the technology of production requires the input shares $\hat{X}_{ji}/\hat{X}_{ii}$ to be kept constant i.e.

$$\hat{X}_{ji}/\hat{X}_{ii}=c_{ij}, \quad i,j=0,1,\ldots,n, \quad i\neq j,$$

and in order to satisfy (74) it is necessary to change α_{ji}, i.e.

$$\alpha_{ji}=(p_j/p_i)\, c_{ij}, \quad i,j=0,1,\ldots,n, \quad i\neq j.$$

In that case no substitution is possible.

Since each sector S_i is an aggregate of different production processes it is reasonable to assume that

$$\alpha_{ji}(t)=c_{ij}\frac{[p_j(t-T_{ji})]^{\sigma_j}}{[p_i(t-T_{ji})]^{\sigma_i}}, \quad i,j=0,1,\ldots,n, \quad i\neq j,$$

where T_{jt}-adaptation delay; c_{ji}, σ_j, σ_i — given numbers.

When $\sigma=0$ the nonrestrictive substitution follows and when $\sigma=1$ no substitution is possible.

Then one can expect that in typical situation σ_i, $\sigma_j\in[0,1]$. In the model under consideration one is interested mainly in the indices $\alpha^t_{ji}=\alpha_{ji}(t)/\alpha_{ji}(t-1)$

$$\alpha^t_{ji}=[p_j^{t-T_{ji}}]^{\sigma_j}/[p_i^{t-T_{ji}}]^{\sigma_i}, \quad i,j=0,1,\ldots,n, \tag{76}$$

When the values $\alpha_{ji}(t)=Y_{ji}(t)/Y_{ii}(t)$ and the corresponding price indices $p_j^{t-T_{ji}}$ are known from historical data it is possible to find the estimates $\tilde{\sigma}_j$, $\tilde{\sigma}_i$, $j,i=0,1,\ldots,n$, by minimalization of the weighted square errors

$$E(T_{ji})=\min_{\sigma_j,\sigma_i}\sum_{t=-T}^{0} w(t)\sum_{j=0}^{n}[\ln\alpha^t_{ji}-\sigma_j\ln p_j^{t-T_{ji}}+\sigma_i\ln p_i^{t-T_{ji}}]^2, \quad i=0,1,\ldots,n. \tag{77}$$

Computing the values of $E(T_{ij})$ for different $T_{ij}=0,1,\ldots$ it is possible to find such a value $T_{ij}=\tilde{T}_{ij}$ which yields the minimum value of E.

Then the model of technological change assumes the following form

$$\alpha_{ji}(t)=\alpha_{ji}(t-1)\frac{[p_j^{t-\tilde{T}_{ji}}]^{\sigma_j}}{[p_i^{t-\tilde{T}_{ji}}]^{\sigma_i}}, \quad j,i=0,1,\ldots,n, \quad j\neq i. \tag{78}$$

It should be observed that the process of technological change introduced into the production system (S_i, $i=1,\ldots,n$) changes the value of production outputs and prices. The prices change again $\alpha_{ji}(t)$ etc. but in a delayed fashion which allows the consecutive computations in an iterative manner [5]). The process of iterative computations is explained by Fig. 5, where the production is influenced by two main feedbacks. One is connected with the utility structure change and allocation of

[5]) It is assumed also that $\alpha_{ji}(t)$ do not change fast so it is possible to assume in Sec. 4 $a_{ji}(t)\approx a_{ji}(t+1)$.

investments and other gouvernment expenditures (β_ν, $Z_{\nu i}$). The second feedback influences the production by means of prices which change the technological coefficients. The model can be used for forecasting the future development processes starting with a given base year $t=0$, when all the parameters have been determined from historical data. All the processes of interest such as e.g. sectors inputs and outputs, individual and public consumption, investments employments, price indices,

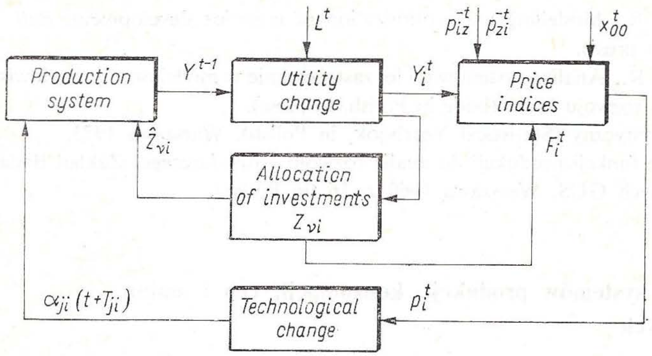

Fig. 6. Feedbacks and iterative computations

wages, foreign trade etc. can be computed (in actual or constant prices) in the iterative manner. Only a few parameters such as population, employment and foreign price indices should be introduced exogeneously.

Besides a number of relating variables can be also derived such as e.g.:

a) the labour efficiences

$$w_i=\hat{Y}_{ii}/X_{oi}=p_0(t)/\alpha_{oi}(t),\quad i=1, ..., n;$$

b) the "input in output" shares

$$s_{ji}=\hat{X}_{ji}/\hat{X}_{ii}=\alpha_{ji}(t)\,p_j(t)/p_i(t),\qquad i,j=1, ..., n;$$

c) the shares of the components of technical and organizational progress in the GNP growth (compare (30))

$$b_\nu=\beta_\nu\, Z^t_\nu/Y^t,\qquad \nu=1, ..., N;$$

d) the components of the aggregated (into one sector) production function (25), (26), etc.

It should be observed that the main advantage of the model discussed in the present paper consists in the possibility of aggregation or desaggregation (decomposition) of the complex multi-sector and multi-level normative structure, shown in Figs 1—3.

It is also possible [5] to extent the decomposition process along the IV and lower levels of multi-level structure shown in Fig. 3. In particular the fourth level can take into account the regional sub-models or the more detailed organization of production and consumption sectors.

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Modelowanie systemów produkcji, konsumpcji, cen i zmian technologicznych

Praca dotyczy modelowania złożonych systemów rozwoju zawierających podsystem produkcji, konsumpcyjny i środowiska. Podsystem produkcji zawiera n danych sektorów opisanych funkcją produkcji typu CES. Zdecentralizowany system decyzyjny rozdziela dochód narodowy między dane sfery aktywności (zawierające inwestycje, konsumpcję indywidualną i zbiorową oraz środowisko) w taki sposób, aby dana funkcja użyteczności osiągnęła wartość maksymalną. Funkcja ta zmienia się wraz ze zmianą dochodu na głowę ludności. Następnie modeluje się zmiany indeksów cen. Zmiany cen powodują zmiany współczynników technologicznych funckji produkcji. Procesy rozwoju można obliczać w sposób iteracyjny umożliwiający dokonanie projekcji w przyszłość (tj. prognoz rozwojowych). Wszystkie parametry modelu można obliczać na podstawie danych statystycznych.

Моделированне производства структуры полезности расходов и технологических изменений

Работа касается моделирования сложных систем развития содержащих производственную, потребительскую подсистемы и подсистемы среды. Производственная подсистема содержит n данных секторов описанных функциами производства типа CES. Децентрализованная система принятия решений распределяет национальный доход на данные среды активности (содержащие капиталовложения, личное и коллективное потребление также среду) таким образом, чтобы данная функция полезности достигла максимального значения. Эта функция изменяется с изменением дохода на душу населения. Затем моделируются изменения показателей расходов. Изменения расходов вызывают изменения показателей технологических функций производства. Процессы развития можно рассчитать итерационным методом предоставляющим возможность сделать проекцию в будущее (т.е. поставить прогнозы развития). Все параметры модели можно рассчитать на основе статистических данных.