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Observability tests for constant time-lag systems

by

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An algebraic test for checking observability of a constant time-lag-lag system is is derived. The test is of the form of a finite-step algorithm with coefficients of the system as initial data and depends on the length of an observation interval. Two types of observability are considered one with initial function in the space L^1, second with continuous initial function.

1. Introduction

The purpose of the paper is to give computable criteria for observability of time-lag systems. Such criteria were recently obtained only for very special definitions of observability. In [6] an observability problem for the system

$$\dot{x}(t)=Ax(t)+A_1 x(t-h),\ y(t)=Cx(t),\quad x(t)\in R^n,\ y(t)\in R^m, \tag{1.1}$$

is studied. The problem is to determine initial point $x(0)=x_0$ provided that the initial function $x(t)=0$ on $[-h, 0)$, the point $x_0=Hz$ for some $z\in R^n$, H is a given matrix, and the output $y(t)$, $t\in[0, t_1]$ is known. The solution is obtained by so called defining equation

$$X_{k+1}(j)=AX_k(j)+A_1 X_k(j-1),\ Y_k(j)=CX_k(j) \tag{1.2}$$

$X_0(j)=I$ if $j=0$ and zero matrix otherwise.

Each x_0 of the given from can be determined if and only if

$$\operatorname{rank}\{HY_k(j),\ 0\leqslant j\leqslant k-1,\ 0\leqslant k\leqslant n-1\}=\operatorname{rank} H.$$

Some minor generalizations of this problem are done in [7] where the initial function is assumed to be a polynomial with unknown coefficients to be determined. This formulation is more reasonable from practical point of view since it practically never occurs that the initial function is known with unknown jump at $t=0$ only. Some special cases of two dimensional systems were studied in [8].

Since the state space for (1.1) is infinite — dimensional it is possible to introduce an equivalent differential equation (without delays) defined in Banach space [15]. However the infinitesimal generator of the corresponding strongly continuous semigroup is unbounded in this case and the general observability theory for such cases is not at the satisfactory level. The closest to practical applications seems to be the results of [12] expressed in terms of an infinite sequence generated by the infinitesimal and the output operator. The paper [14] contains duality result between the controllability of a system with delays and the observability of its adjoint in Banach space.

Another approach is related to evolution equations in abstract spaces, defined by a given time-lag system. The general duality results with possible relevance to functional-differential equations were obtained in [3], [5]. The results of [3] were utilized by the authors in [4]. Observability conditions are obtained in terms of integral symmetric matrix (operator if $x(t) \in H$ — a Hilbert space) constructed with the aid of fundamental matrix (resolvent map). Both the restricted definition of observability similar to that of [6] and the general case are considered.

In this paper we restrict our attention to constant systems of the form (1.1) although the method presented is applicable to systems with many commensurable delays and analytically depending on time coefficients. The main idea is to transform a given time-lag system to a system without delay but with state space of greater dimension (R^{nk} instead of R^n) and additional two-points boundary condition. Accordingly the observability problem is reformulated and preliminary lemmas are given in section 2. In section 3 the algebraic criteria for observability are proved. Both the case of integrable and continuous initial function are considered yielding two types of observability (strong observability and observability). The criteria for these two types differ slightly. All the terms appearing in the criteria are computable from given coeffitiens of the system, either directly or through an algorithm converging in finite number of steps. The construction of an algorithm is given in section 4 where also illustiating example is computed.

Notation. For an operator (matrix) A the image, the kernel, the adjoint and the generalized inverse will be denoted respectively by im A, ker A, A^*, $A^\dagger$. For given $n \times n$ and $n \times m$ matrices A and B the controllability subspace we denote as $\{A|B\} = \text{im}\,[B; AB; \ldots; A^{n-1}B]$, where $[B; A]$ is the matrix obtained by writing the columns of B and then the columns of A from the right. If $\mathscr{B} = \text{im}\,B$ the controllability subspace will be also written $\{A|\mathscr{B}\}$.

2. Definitions and preliminary results

The system considered is of the form

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bx(t-1), \quad t \geq 0 \\ y(t) &= Cx(t) \end{aligned} \tag{2.1}$$

with initial conditions

$$x(t-1)=f(t), \qquad t \in [-1, 0].$$

The trajectory of the system $x(t) \in R^n$, the output $y(t) \in R^p$. Matrices A, B, C are constant and of suitable dimension. The initial function $f \in L^1(0, 1)$ in general. In practical applications this function satisfies as a rule the system equation i.e. it is a piece of trajectory of (2.1) for $t \leqslant 0$. Therefore the case of continuity of f at $t=0$ will be distinguished in the definitions below by assuming $f \in C(0, 1)$.

Recall that usually a function $x_t: [-1, 0] \mapsto R^n$, $x_t(\theta) = x(t+\theta)$, $\theta \in [-1, 0]$ is taken as a state for (2.1) at time t. Such definition is simple and conveniant but cannot serve as far as observability problem is considered and ker $B \neq 0$. In this case one may add any function g with $g(t) \in \ker B$ to an initial function f and the trajectory will be the same as that corresponding to f. This fact is rather rarely employed by the authors [9]. The true state which contains necessary and sufficient information to solve equation (2.1) is the following.

Definition 2.1. A pair $\big(x(t), Bx_t\big) = \big(x(t), Bx(t+\theta),\ \theta \in [-1, 0]\big)$ is said to be a state of system (2.1) at instant t.

In contradistinction to that x_t will be called redundant state.

Proceed to definitions of observability.

Definition 2.2. An initial state $(x(0), Bf)$ of system (2.1) is observable on $[0, T]$ (observable) iff the corresponding output $y(t)$, $t \in [0, T]$ (respectively $t \in [0, \infty]$) does not vanish identically unless $(x(0), Bf) = (0, 0)$.

By this definition zero initial state is always observable.

Definition 2.3. System (2.1) is strongly observable on $[0, T]$ (strongly observable) if all initial states $\big(x(0), Bf\big)$ with $f \in L^1(0, 1)$ are observable on $[0, 1]$ (are observable).

Definition 2.4. System (2.1) is observable on $(0, T)$ (observable) if all initial states are observable on $[0, T]$ (observable) with $f \in C(0, 1)$.

Obviously all these definitions make sense for $T \geqslant 1$. Observability property of system (2.1) means that initial states of the system are distinguishable, i.e. there exists one-to-one correspondence: an output on $[0, T] \mapsto$ an initial state. An important question arises whether this mapping is continuous in the given topologies of the output function and the initial state spaces. This question however will not be treated in this paper.

The following property can be obtained directly from the definitions and from the stationarity of the system.

Remark 2.5. Let $T_2 > T_1 \geqslant 1$. Then observability on $[0, T_1]$ of a state (system) implies observability on $[0, T_2]$. Conversely if a state (system) is not observable on $[0, T_2]$ then it is not observable on $[0, T_1]$. Similar implications are valid for strong observability of a system on $[0, T_1]$ and $[0, T_2]$. If system (2.1) is strongly observable (on $[0, T]$) then it is observable (on $[0, T]$).

So as to show the essential difference between observability and strong observability let us introduce an example.

Example 2.1. Let in eqs. (2.1)

$$A=\begin{bmatrix}-2 & 1\\ 0 & -1\end{bmatrix}; \quad B=\begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix}; \quad C=[0, 1]$$

It can be shown that the system (2.1) with coeffitients as above is not strongly observable on [0, 1] but it is observable on [0, 1].

Suppose the system is not observable on [0, 1]. Then there exist a nonzero initial state $(x(0), Bf)$ with f continuous such that

$$y(t)=Cx(t)=0 \text{ on } [0, 1].$$

Thus $x(t)$ must have the form

$$x(t)=\begin{bmatrix}a(t)\\ 0\end{bmatrix}$$

and satisfy on [0, 1] the eqs. (2. 1)

$$\begin{bmatrix}\dot{a}(t)\\ 0\end{bmatrix}=\begin{bmatrix}-2a(t)\\ 0\end{bmatrix}+\begin{bmatrix}0\\ f_1(t)\end{bmatrix}.$$

This gives $f_1(t)=0$ on [0, 1] and by continuity of f

$$x_1(0)=f_1(1)=0.$$

Hence $(x(0), Bf)=(0, 0)$ which is a contradiction to the hypothesis above. This argument proves observability.

Now choose an initial function $f \in L^1(0, 1)$, $f=(f_1, f_2)$ as defined below

$$f_1(t)=\begin{cases} a_0 \neq 0, & t=1\\ 0, & t \in [0, 1]\end{cases}, \quad f_2(t)=\begin{cases} 0, & t=1\\ \text{arbitrary}, & t \in [0, 1]\end{cases}.$$

Such initial conditions give nonzero trajectory $x(t)$ on [0, 1]

$$x(t)=\begin{bmatrix}a_0 \exp(-2t)\\ 0\end{bmatrix}$$

and identically zero output $y(t)$.

The system considered is not strongly observable on [0, 1].

Now preliminary results, essential for the proofs of main results, will be quoted.

LEMMA 2.6 [2], [10]. Let k be an integer. To each trajectory $x(t)$, $t \in [0, k]$ of the system (2.1) there corresponds uniquely a trajectory $z_k(s) \in R^{kn}$, $s \in [0, 1]$ of a system without delay defined below:

$$\dot{z}_k(s)=A_k z_k(s)+B_k f(s), \quad s \in [0, 1], \tag{2.2}$$

and satisfying additional constraints.

$$z_k(0)=z_k^o+J_k\, z_k(1)\,. \tag{2.3}$$

The new state vector is of the form

$$z_k(s)=\begin{bmatrix} x_1(s-1) \\ \vdots \\ x_k(s-1) \end{bmatrix}; \qquad z_k^o=\begin{bmatrix} x(0) \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{2.4}$$

where $x_i(\cdot)$ is the redundant state of (2.1) at $t=i$.

The matrices A_k, B_k, J_k are respectively $kn\times kn$, $kn\times n$, $kn\times kn$ and have the form

$$A_k=\begin{bmatrix} A & 0 & \dots & 0 & 0 \\ B & A & & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ & & & A & 0 \\ 0 & 0 & \dots & B & A \end{bmatrix}, \quad B_k=\begin{bmatrix} B \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}, \quad J_k=\begin{bmatrix} 0 & 0 & \dots & \cdot & 0 \\ I & 0 & \dots & \cdot & 0 \\ 0 & I & \dots & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \dots & I & 0 \end{bmatrix} \tag{2.5}$$

Conversely, if z_k (s) is a solution to (2.2) and (2.3) then using formula (2.4) a a trajectory $x(t)$, $t\in[0, k]$ satisfying equation (2.1) can be obtained.

By the Lemma 2.6., any problem concerning the system (2.1) may be replaced with a new transformed problem in terms of the system (2.2) and equation (2.3). In case of observability the transformed problem is as follows.

Lemma 2.7. A nonzero initial state $(x(0), Bf)$ of the system (2, 1) is not observable on $[0, k]$ iff the correcponding solution $x(t)\in\mathcal{N}\overset{\text{df}}{=}\ker\, C$ $\forall t\in[0, k]$, or equivalently, iff the sclution $z_k(s)$ to (2.2) and (2.3) corresponding to z_k^o, f satisfies $\forall s\in[0, 1]$ $z_k(s)\in\mathcal{N}_k\overset{\text{df}}{=}\mathcal{N}\times\dots\times\mathcal{N}$ (the product of k-subspaces). The system (2.1) is strongly observable (observable) on $[0, k]$ iff for all nonzero initial states $(x, (0), Bf)$ with $f\in L^1$ (with $f\in C$) the condition $z_k(s)\in\mathcal{N}_k$ on $[0, 1]$ cannot be satisfied by a corresponding solution to (2, 2), (2, 3).

Proof. The proof follows directly from Definitions 2.2, 2.3, 2. 4 and Lemma 2.6.

The question of observability for time-lag system (2.1) is now the question of the existence of a nonzero trajectory of system (2.2) satistying (2.3) and with values in the subspace $\mathcal{N}_k$ for all time interval [0 ,1]. If such a trajectory exists the system is not strongly observable (not observable if $f\in C(0, 1)$) and vice versa. Therefore the problem considered has been converted into a kind of controllability problem. Therefore let us recall some basic facts concerning controlability with restriction $x(t)\in\mathcal{N}_k$ (see [1], [13]).

Lemma 2.8.

(i) Given an initial vector $z_k(0)$ of the system (2.2), there exists a function $f\in L^1(0, 1)$ (or equivalently $f\in C(0, 1)$) such that the solution $z_k(s)\in\mathcal{N}_k$ for all $s \in [0, 1]$ if and only if $z_k(0) \in \mathcal{M}_k \subset \mathcal{N}_k$, where $\mathcal{M}_k$ is the greatest subspace contained in $\mathcal{N}_k$ satisfying

$$A_k \mathcal{M}_k \subset \mathcal{M}_k + \mathrm{im}\, B_k. \tag{2.6}$$

(ii) Condition (2.6) is satisfied if and only if there exists $n \times kn$ matrix D_k such that

$$(A_k + B_k D_k) \mathcal{M}_k \subset \mathcal{M}_k. \tag{2.7}$$

(iii) The subspace $\mathcal{M}_k$ may be computed from the following algorithms.

Algorithm I.

$$X^o = \mathcal{N}_k,\ X^i = X^{i-1} \cap A_k^{-1} (\mathrm{im}\, B_k + X^{i-1}).$$

Set $\mathcal{M}_k = X^p$ where $p = \dim \mathcal{N}_k$ or set $\mathcal{M}_k = X^i$ if $X^{i-1} = X^i$.

Algorithm II.

$$Y^o = \mathcal{N}_k^{\perp},\ Y^i = Y^o + A_k^* (Y^{i-1} \cap (\mathrm{im}\, B_k)^{\perp}).$$

Set $\mathcal{M}_k = (Y^p)^{\perp}$ or set $\mathcal{M}_k = (Y^i)^{\perp}$ if $Y^{i+1} = Y^i$.

A trajectory of (2.2) with $z_k(t) \in \mathcal{N}_k$ we shall call $\mathcal{N}_k$ — maintainable trajectory. By Lemma 2.8 each $\mathcal{N}_k$ — maintainable trajectory has to start from a smaller subspace $\mathcal{M}_k$ named in [1] a maximal (A_k, C_k) — invariant (controlled invariant) contained in $\mathcal{N}_k$. Hence it is obvious by stationarity that each $\mathcal{N}_k$, — maintainable trajectory is $\mathcal{M}_k$ — maintainable. The more explicit form of $\mathcal{N}_k$-manitainable trajectory is as follows [11].

LEMMA 2.9. Each $\mathcal{N}_k$ — maintainable trajectory of (2.2) has the form.

$$z_k(t) = e^{\bar{A}_k t} z_k(0) + \bar{z}_k(t),\ \bar{z}_k(t) \in \mathcal{P}_k,\ z_k(0) \in \mathcal{M}_k, \tag{2.8}$$

where

$$\bar{A}_k \overset{\mathrm{df}}{=} A_k + B_k D_k \text{ for arbitrary } D_k \text{ satistying (2.7)} \tag{2.9}$$

and

$$\mathcal{P}_k \overset{\mathrm{df}}{=} \{\bar{A}_k | \mathcal{M}_k \cap \mathrm{im}\, B_k\} \tag{2.10}$$

is of the form of a controllability subspace.

By the lemma above the following description of the set of all points $z_k(t)$ attainable from $z_k(0)$ by a trajectory in $\mathcal{N}_k$ holds.

$$z_k(t) \in e^{\bar{A}_k t} z_k(0) + \mathcal{P}_k. \tag{2.11}$$

COROLLARY 2.10. If $\mathcal{P}_k = 0$ (i.e. $\mathcal{M}_k \cap \mathrm{im}\, B_k = 0$ or equivalently rank B + rank $\mathcal{M}_k$ = = rank $[B_k; M_k]$ where columns of M_k form a basis for $\mathcal{M}_k$) then for given $z_k(0) \in \mathcal{M}_k$ there is only one $\mathcal{N}_k$ — maintainable trajectory of (2.2) starting from $z_k(0)$.

Proof. It suffices to prove the uniqueness of the mapping $\bar{A}_k$ restricted to invariant subspace $\mathcal{M}_k$. Let D_k', D_k'' be the two different matrices satisfying (2.7). Then for any $m \in \mathcal{M}_k$:

$\bar{A}'_k m=(A_k+B_k D'_k) m\in\mathcal{M}_k$ and $\bar{A}''_k m=(A_k+B_k D''_k) m\in\mathcal{M}_k$. Hence $B_k(D'_k-D''_k) m \in\mathcal{M}_k$. Since $\mathcal{M}_k\cap \operatorname{im} B_k=0$ this yields $B_k(D'_k-D''_k)\ m=0$ and then $\bar{A}'_k m=\bar{A}''_k m$ for any $m\in\mathcal{M}_k$. By (2.8) a trajectory has unique form $z_k(t)=e^{\bar{A}_k t} z_k(0)$.

LEMMA 2.11. Let $z\in R^{(k-1)n}$, $x\in R^n$. If $(z, x)\in\mathcal{M}_k\subset R^{kn-n}\times R^n$ then $z\in\mathcal{M}_{k-1}$.

Proof. $(z, x)\in\mathcal{M}_k$ implies by Lemma 2.8 that starting from $z_k(0)=(z, x)$ one can choose f such that the solution to (2.2) satisfies $z_k(t)\in\mathcal{N}_k=\mathcal{N}_{k-1}\times\mathcal{N}$. Writing explixitly A_k, B_k in the form (2.5) one concludes immediately that the soluton to $\dot{z}_{k-1}(t)=A_{k-1} z_{k-1}(t)+B_{k-1} f(t)$ with $z_{k-1}(0)=z$ satisfies $z_{k-1}(t)\in\mathcal{N}_{k-1}$ on $[0, 1]$. Applying again Lemma 2.8. one obtains $z\in\mathcal{M}_{k-1}$.

3. Main results

First consider the interval $[0, T]=[0, k]$, k — an integer. By Lemma 2.7 and 2.8 a simple sufficient conditions for strong observability and, by Remark 2.5, for observability follows.

THEOREM 3.1. If $\mathcal{M}_k=0$ then the system (2.1) is strongly observable (observable) on $[0, k]$.

Proof. If $\mathcal{M}_k=0$ then, by Lemma 2.9, $\mathcal{P}_k=0$ and by Corollary 2.10 the only $\mathcal{N}_k$ — maintainable trajectory of (2.2) is $z_k(t)=0$. By Lemma 2.7 the system (2.1) is clearly (strongly) observable.

The next theorem gives less simple but in compact form necessary condition.

THEOREM 3.2. If the system (2.1) is observable (strongly observable) on $[0, k]$ then $\mathcal{P}_k=0$ (or equivalently $\mathcal{M}_k\cap\operatorname{im} B_k=0$).

Proof. Suppose $\mathcal{P}_k\neq 0$. Let $z_k(0)=z_k(1)=0$, $x(0)=0$. Hence by (2.4) $z_k^0=0$ and (2.3) holds. Take arbitrary $\bar{A}_k$ defined by (2.9) and put $f(t)=D_k z_k(t)+g(t)$ with $B_k g(t)\in\mathcal{M}_k\cap\operatorname{im} B_k$. The equation (2.2) now is equivalent to

$$\dot{z}_k(t)=\bar{A}_k z_k(t)+B_k g(t).$$

Since $\mathcal{P}_k$ is invariant der $\bar{A}_k$ (see (2.10)) this system is equivalent to a completely controllable systems with $\mathcal{P}_k$ as a state space. So one can join the points $z_k(0)=0$ and $z_k(1)=0$ through a nonzero trajectory in $\mathcal{P}_k\subset\mathcal{N}_k$ by using a continuous "control" $t\mapsto g(t)$ with $g(1)=0$. This implies that $f(t)=D_k z_k(t)+g(t)$ is continuous and $f(1)=0=x(0)$. By Lemmas 2.7, 2.6 there corresponds to $z_k(t)$ a nonzero trajectory $x(t)\in\mathcal{N}$ with continuous initial function f and this implies that the system (2.1) is not observable on $[0, k]$.

The constraint $g(1)=0$ is immaterial in the controllability problems "from point to point in R^n" since the space of controls remains infinite dimensional.

The condition $\mathcal{P}_k=0$ may have the equivalent form $\operatorname{rank} B+\operatorname{rank}\mathcal{M}_k=\operatorname{rank}[B_k; M_k]$ (see Corollary 2.10). Theorems 3.1 and 3.2 has been specified since it is reasonable that a real algorithm for checking observability starts to verify simple alternatives which may give answer at early steps. The checking of general conditions should take place at the end of an algorithm. This general criterion is as follows.

THEOREM 3.3. The system (2.1) is strongly observable on $[0, k]$ if and only if the two following conditions hold.

(i) $\operatorname{rank} B+\operatorname{rank} M_k=\operatorname{rank}\,[B_k;\, M_k]$,

(ii) $n+\operatorname{rank}\, M_k=\operatorname{rank}\,[E_{1k};\,(I-J_k\, e^{\bar{A}_k})\, M_k]$,

where M_k is a matrix whose columns span the subspace $\mathscr{M}_k$ ($\operatorname{rank} M_k=\dim \mathscr{M}_k$), $E_{1k}^{*}=[I, 0, \dots 0]$ is $n\times kn$ matrix, $\bar{A}_k$ is defined by (2.9).

Proof. Necessity. Observe that $\operatorname{rank} B=\operatorname{rank} B_k$ and therefore condition (i) is equivalent to $\operatorname{im} B_k\cap \operatorname{im} M_k=0$, that is $\mathscr{P}_k=0$. Now necessity of (i) is clear in view point of Theorem 3.2.

Note that always

$$n+\operatorname{rank} M_k=\operatorname{rank} E_{1k}+\operatorname{rank} M_k\geqslant \operatorname{rank} E_{1k}+\operatorname{rank}\,(I-J_k\, e^{\bar{A}_k})\, M_k\geqslant$$
$$\geqslant \operatorname{rank}\,[E_{1k};\;(I-J_k\, e^{\bar{A}_k})\, M_k]=\dim\left(\operatorname{im} E_{1k}+(I-J_k\, e^{\bar{A}_k})\, \mathscr{M}_k\right). \tag{3.1}$$

Therefore if (ii) is not satisfied then there exists $z'\in \mathscr{M}_k$, $z'\neq 0$ such that $z''\overset{\mathrm{df}}{=}(I-J_k\, e^{\bar{A}_k})\, z'\in \operatorname{im} E_{1k}$. Choose $z_k(0)=z'$, $z_k^o=z''$. It may be assumed $\mathscr{P}_k=0$ (the condition (i) is already proved). By Lemma 2.9 and Corollary 2.10 the trajectory starting from $z_k(0)$ and $\mathscr{N}_k$ — maintainable is unique and can be obtained by setting $f(t)=D_k\, z_k(t)$. Hence $z_k(1)=e^{\bar{A}_k}\, z_k(0)$ and by definition of $z_k(0)$ and z_k^o it is easily seen that (2.3) is satisfied. Moreover, the trajectory $z_k(t)=e^{\bar{A}_k t}\, z_k(0)$ of (2.2) is a nonzero $\mathscr{N}_k$ — maintainable trajectory since $z_k(0)=z'\neq 0$. By Lemma 2.7 the system (2.1) is not strongly observable on $[0, k]$. Therefore condition (ii) is necessery for strong observability.

Sufficiency. Suppose (i), (ii) are satisfied and the system (2.1) is not strongly observable. By Lemma 2.7 and 2.8 there exists a nonzero trajectory of (2.2) $z_k(t)\in \mathscr{M}_k\subset\mathscr{N}_k$ satisfying condition (2.3). Since by (i) $\mathscr{P}_k=0$ Lemma 2.9 implies that such trajectory has the form $z_k(t)=e^{\bar{A}_k t}\, z_k(0)$, $0\neq z_k(0)\in \mathscr{M}_k$. Hence by substituting $z_k(1)$ to (2.3) we get

$$z_k^o=(I-J_k\, e^{\bar{A}_k})\, z_k(0), \tag{3.2}$$

where by (2.4) $z_k^o\in \operatorname{im} E_{1k}$. This means that either $(\operatorname{im} E_{1k})\cap(I-J_k\, e^{\bar{A}_k})\, \mathscr{M}_k\neq 0$ (in this case z_k^o may be different from 0) or $(I-J_k\, e^{\bar{A}_k})\, z_k(0)=0$ for some $0\neq z_k(0)\in \mathscr{M}_k$. In both cases the rank of the matrix $[E_{1k};\,(I-J_k\, e^{\bar{A}_k})\, M_k]$ is less then rank $E_{1k}+\operatorname{rank} M_k=n+\operatorname{rank} M_k$. To see this it is enough to recall that the rank of a matrix is equal to the dimension of the subspace spanned by its columns. The contradiction to condition (ii) proves the sufficiency.

For (not strong) observability we get a slight modification of Theorem 3.3.

THEOREM 3.4. The system (2.1) is observable on $[0, k]$ if and only if both condition (i) of Theorem 3.3. and the following

$$\text{(ii)}' \qquad \operatorname{rank}\left(\begin{bmatrix} B & 0 \\ 0 & I \end{bmatrix} - (J_k + B_k D_k)\, e^{\bar{A}_k}\right) M_k = \operatorname{rank} M_k$$

are satisfied. Here D_k is defined by (2.9) and M_k in previous theorem.

Proof. By Theorem 3.2. condition $\mathscr{P}_k=0$ is necessary for observability. Therefore it remains to prove that under assumption $\mathscr{P}_k=0$ the system is not observable iff (ii)′ is not satisfied. In view of Lemma 2.7, 2.9 and Corollary 2.10 system (2.1) is not observable iff $\exists 0 \neq z_k(0) \in \mathscr{M}_k$ such that condition (2.3) holds (which may be written in the form (3.2)) and initial function f is continuous on [0, 1]. By uniqueness of the trajectory corresponding to $z_k(0)$ the function f is in general of the form $f(t)=D_k z_k(t)+f'(t)$, where $f' \in C[0, 1]$ is arbitrary with $f(t) \in \ker B$. So an unobservable state $(x(0), Bf)$ can be chosen with continuous f iff the difference $D_k z_k(1) - x(0) \in \ker B$, that is $BD_k z_k(1)=Bx(0)$ or using z_k^o and (2.8)

$$B_k D_k e^{\bar{A}_k} z_k(0) = \begin{bmatrix} B & 0 \\ 0 & I \end{bmatrix} z_k^o. \tag{3.3}$$

Summarizing, an equivalent characterization of unobservability is as follows: There exists $0 \neq z_k(0) \in \mathscr{M}_k$ such that

$$\begin{bmatrix} B & 0 \\ 0 & I \end{bmatrix} (I - J_k e^{\bar{A}_k})\, z_k(0) = B_k D_k e^{\bar{A}_k} z_k(0). \tag{3.4}$$

It is easy to check that (3.4) is equivalent to (3.3) and (3.2).

Observing that $\begin{bmatrix} B & 0 \\ 0 & I \end{bmatrix} J_k = J_k$ and that, by definition, the columns of M_k are linearly independent we get that condition (3.4) is the contrary to (ii)′ what was to be proven.

Now consider the case $T=k-1+\tau$, $\tau \in (0, 1)$. The proofs are quite similar so we pay attention to changes which are to made only.

THEOREM 3.5. The system (2.1) is strongly observable on $[0, T]$, $k-1<T<k$ iff

(i) $\operatorname{rank} B + \operatorname{rank} M_{k-1} = \operatorname{rank}[B_{k-1};\, M_{k-1}]$

and

(ii) $n + \operatorname{rank} M_k = \operatorname{rank}[E_{1k};\, (I - J_k e^{\bar{A}_k}) M_k]$.

Proof. Necessity of (ii) is clear in view of Theorem 3.3. and Remark 2.5. Necessity of (i) can be proven similarly as necessity of condition (i), in Theorem 3.3 by showing that if $\mathscr{P}_{k-1} \neq 0$ then there exists nonzero solution to (2.2), (2.3) $z_k(t) \in \mathscr{M}_{k-1} \times \{0\} \subset \mathscr{M}_k$ (see Lemma 2.11) and in particular $z_k(t)=0$ for $t \in [0, \tau]$, $t=1$. For sufficiency suppose (i) and (ii) are valid and the system is not strongly observable on $[0, T]$. Then it is not strongly observable on $[0, k-1]$ since $k-1<T$. From condition (i) (that is $\mathscr{P}_{k-1}=0$) and Corollary 2.10 the trajectory $z_{k-1}(t)$ corresponding to an unobservable state is analytic on [0, 1] and hence, by (2.4), $x(t)$ is analytic

on $[k-2, k-1]$. This implies by (2.1) that $x(t)$ is analytic on $[k-1, k]$ since the nonhomogenous term $g(t)=Bx(t-1)$ is analytic. By unobservability assumption $Cx(t)\equiv 0$ on $[0, T]$. Analytic extension of this identity gives $Cx(t)\equiv 0$ on $[0, k]$for a nonzero trajectory $x(t)$, which means that the system (2.1) is not strongly observable on $[0, k]$. Hence by Theorem 3.3 either $\mathscr{P}_k\neq 0$ or condition (ii) fails to hold. But $0\neq(z, x)\in \mathscr{M}_k\cap \operatorname{im} B_k\subset\mathscr{P}_k$, $x\in R^n$, implies that $z\neq 0$ (see the form (2.5) of B_k) and, by Lemma 2.11, $0\neq z\in \mathscr{M}_{k-1}\cap \operatorname{im} B_{k-1}\subset\mathscr{P}_{k-1}$ contradicting to condition (i). In any case a contradiciton is obtained and thus the proof is complete.

This proof can be immediately adapted in order to show the validity of the next theorem.

THEOREM 3.6. The system (2.1) is observable on $[0, T]$, $k-1<T<k$ if and only if

(i) $\operatorname{rank} B+\operatorname{rank} M_{k-1}=\operatorname{rank}[B_{k-1}; M_{k-1}]$,

and

(ii)′ $\operatorname{rank} M_k=\operatorname{rank}\left(\begin{bmatrix} B & 0\\ 0 & I\end{bmatrix}-(J_k+B_k D_k e^{\bar{A}_k}) M_k\right)$.

4. On Algorithm for checking observability

Although some of the matrices involved in observability criteria are not uniquely defined (namely the matrices D_k, $\bar{A}_k$) this is however immaterial in practice since they may be used in the form multiplied by M_k, e.g. $B_k D_k e^{\bar{A}_k} M_k$, and such form is unique. It is clarified in details below. Let us formulate.

Algorithm for checking observability on $[0, k]$.

Step 1. Given matrices A, B, C of system (2.1) construct matrices A_k, B_k, J_k according to (2.5). Compute the matrix N the columns of which form a basis for ker C and then construct $nk\times nr$ block diagonal matrix $N_k=\text{block diag}[N, \ldots, N]$ as a basis matrix for $\mathscr{N}_k$. Here $r=\dim\ker C$.

Step 2. Compute M_k, a basis matrix for the subspace $\mathscr{M}_k$ using Algorithm I or II given in section 2. If $M_k=0$ then STOP, the system is strongly observable on $[0, k]$ (and therefore observable). If $M_k\neq 0$ then go to step 3.

Step. 3. Check condition (i) in Theorem 3.3. If it is not fullfiled then STOP, the system is not observable on $[0, k]$ (and therefore not strongly observable). If conditon (i) holds go to step 4.

Step 4. Check condition (ii) of Theorem 3.4. In order to do this compute first the matrices Q_k and $\tilde{B}_k$ following from the representation $A_k M_k=M_k Q_k+\tilde{B}_k$. This representation is unique by condition (i) ($\mathscr{M}_k\cap \operatorname{im} B_k=0$) and by Lemma 2.8. Set by definition

$$\bar{A}_k M_k=M_k Q_k \text{ nad } B_k D_k M_k=-\tilde{B}_k. \tag{4.1}$$

Due to formula (4.1) we avoid nonuniqueness of $\bar{A}_k$ and D_k.

Compute then $e^{\bar{A}_k} M_k=M_k e^{Q_k}$ (4.2) and

$$B_k D_k e^{\bar{A}_k} M_k = B_k D_k M_k e^{Q_k} = -\tilde{B}_k e^{Q_k}. \tag{4.3}$$

Check condition (ii) of Theorem 3.4 substituting first formulas (4.2) and (4.3). If condition (ii) is satisfied the system (2.1) is observable on $[0, k]$, if not the system is not observable and therefore not strongly observable. END.

Remark 4.1. In a similar way every observability condition of section 3 can be verified. For strong observability there is no need in computing the matirx $\tilde{B}_k$ and the formula (4.3). The practical effectiveness of the algorithm above depends on technical details in realization of an algorithm computing the matirx M_k (e.g. the method of computing the basis for intersection of two subspaces).

Let introduce an example illustrating the algorithm presented.

Example 4.1. Let the matrices A, B, C of system (2.1) be as follows

$$A = \begin{bmatrix} 3 & 2 \\ 0 & 1 \end{bmatrix}; \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \quad C = [1, 1].$$

This system is not observable (and not strongly observable) on $[0, T]$, $T<2$. Indeed, here

$$N_1 = N = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \text{ and } M_1 = N$$

which can be checked directly by substituting $\mathcal{M}_1 = \mathcal{N}$ into defining formula (2.6) where $\operatorname{im} B_1 = \operatorname{im} B$ is all the space R^2.

Check observability on $[0, 2]$. Using an algorithm of Lemma 2.8 yields

$$M_2 = \begin{bmatrix} 0 \\ N \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

so we cannot use Theorem 3.1.

Condition (i) in Theorem 3.3. is satisfied:

$$\operatorname{rank} B + \operatorname{rank} M_2 = 2 + 1 = \operatorname{rank} [B_2; M_2].$$

Compute

$$A_2 M_2 = \begin{bmatrix} 0 \\ AN \end{bmatrix} = \begin{bmatrix} 0 \\ N \end{bmatrix} = M_2$$

Hence $Q_2 = 1$, $\tilde{B}_2 = 0$ in representation $A_2 M_2 = M_2 Q_2 + \tilde{B}_2$.

Check, for strong observability, condition (ii) of Theorem 3.3:

$$2 + \operatorname{rank} M_2 = 2 + 1 = \operatorname{rank} \left[\begin{matrix} I \\ 0 \end{matrix}; M_2 - \begin{bmatrix} 0 & 0 \\ I & 0 \end{bmatrix} M_2 \exp(1) \right] = \operatorname{rank} \begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix} = 3.$$

By Theorem 3.3 the system is strongly observable on $[0, 2]$ and by Remark 2.5 it is observable on $[0, 2]$.

5. Conclusions

Two observability problems has been considered for stationary delay system of the form (2.1); strong observability with integrable initial condition and observability with continuous initial function f. For both cases appropriate criteria for checking observability has been derived. The criteria depends on the length of the interval considered and the difficulties in checking observability arises as the length of the interval increases. So it would be useful if one has at least partial conditions for checking observability and strong observability of the system (2.1) (that is observability on some interval) not depending on a particular interval. This problem has not been solved in this paper but on the basis on examples the following conjecture can be stated.

Conjecture 5.1. The system (2.1) is observable (strongly observable) if and only if it is observable (strongly observable) on $[0, q+1]$, where $q=\dim \ker C$.

Since, cleary, the system (2.1) is observable if rank $C=n$ then, if the Conjecture 5.1 is valid , the system is observable if and only if it is observable on $[0, n]$ (the same for strong observability).

All the considerations of the previous sections can be repeated for the case when the lag $h\neq 1$ (it is an equivalent case after transformation of time) or for many commensurable delays. The criteria will obtain the same form, only the form of A_k, B_k will be modified as in [2]. Also the case of coefficients depending analytically on time might be probably treated similarly.

An important unsolved problem is in what topologies and under what conditions the map $y(\cdot)\mapsto(x(0), Bf)$ which is defined properly for observable systems only) is continuous.

The other problem connected with observability theory is the construction of observers. This problem is very important from practical point of view and for the case of time-lag systems its solution is not at a satisfactory level yet.

References

1. Basile G., Marro G., Controlled and conditioned invariant subspacesin linear system theory. *J. Optimiz. Theory Appl.* **3**, 5, (1969) 306—315.
2. Charrier P., Haugazean Y., On the degeneracy of linear time-invariant delay — differential systems (to appear in J. Math. Analysis Appl.).
3. Delfour M. C., Mitter S. K., Controllability and observability for infinite dimensional systems. *SIAM J. Contr.* **10**, 2 (1972) 329—334.
4. Delfour M. C., Mitter S. K., Controllability, observability and optimal feedback control of affine hereditary differential systems. *SIAM J. Contr.* **10**, 2 (1972) 298—328.
5. Dolecki S., Duality of F-obseravbility and im F-controllability (to appear).
6. Gabasov R. F., Zhevniak R. M., Kirillova F. M., Kopeikina T. B., Conditional observability of linear systems. *Probl. Control Inform. Theory* **1**, 3—4 (1972) 217—238.
7. Kopeikina T. B., Mularthik V. V., On observability and recognizability of systems with delays. *Differen. Uravnenia.* **10**, 8 (1974) 933—936.

8. Krasovskii N. N., Kurzhanskii A. B., On the problem of time-lag systems observability. *Differen. Uravnenia* **2**, 3 (1966) 299—309.
9. Manitius A., Optimal control of processes with delays — a survey and some new results. *Arch. Autom. i Telemech.* **15**, 2 (1970) 205—221.
10. Olbrot A.W., On degeneracy and related problems for linear constant time-lag systems. *Ricerche di Automatica* **3**, 3 (1972) 203—220.
11. Olbrot A. W., Algebraic criteria of controllability to zero function for linear constant time-lag systems. *Control a. Cybernetics* **2**, 1—2 (1973).
12. Triggiani R., Extensions of rank conditions for controllability and observability to Banach spaces and unbounded operators Unive. of Warwick, Dec. 1973, Rep. 28.
13. Wonham W. M., Morse A. S., Decoupling and pole assignment in linear multivariable systems — a geometric approach. *SIAM J. Contr.* **8**, 1, (1970) 1—18.
14. Zhevniak R. M., On observability theory in Banach spaces. *Differen. Uravnenia* **6**, 8 (1970) 1516—1519.
15. Hale J., Functional differential equations. Appl. Math. Sci. Vol. 3, Springer Verlag, Berlin 1971.

Kryteria obserwalności dla stacjonarnych układów z opóźnieniami

Rozważono problem obserwowalności dla układu, którego trajektoria jest rozwiązaniem równania różniczkowego z opóźnieniem

$$\dot{x}(t)=Ax(t)+Bx(t-1),\quad t\geqslant 0$$

z wyjściem

$$y(t)=Cx(t),$$

gdzie $x(t)\in R^n$, $y(t)\in R^p$; A,B,C — macierze odpowiednich rozmiarów o stałych w czasie współczynnikach. Przyjęto definicję stanu początkowego jako pary $(x(0), Bf)$, gdzie $f(t)=x(t-1)$ dla $t\in[0,1]$ jest funckją początkową. Określono obserwowalność i mocną obserwowalność układu jako rozróżnialność stanów początkowych na bazie informacji zawartej w przebiegu wyjścia przy ciągłej i (dla mocnej obserwowalności) nieciągłej funckji początkowej f. Przedstawiono równoważny układ bez opóźnień (wzór (2.2)) z dodatkowymi więzami dwugranicznymi typu (2.3), co pozwoliło sprowadzić problem obserwowalności do pewnego problemu inwariantnej sterowalności układu zastępczego (lemat 2.7). Podano algorytmy obliczenia odpowiednich podprzestrzeni niezmienniczej sterowalności oraz ich zastosowanie do sprawdzenia obserwowalności (twierdzenie 3.1 — prosty warunek dostateczny, twierdzenie 3.2 — prosty warunek konieczny, twierdzenia 3.3—3.6 — pełne kryteria dla odpowiednich przypadków). W pkcie 4 pracy podano przykładowy algorytm sprawdzania obserwowalności na przedziale $[0,k]$, k — liczba całkowita, odpowiadający kryterium zawartym w Twierdzeniu 3.3 oraz podano przykład liczbowy ilustrujący kolejne etapy algorytmu. Dla pozostałych przypadków algorytm buduje się analogicznie.

We wnioskach podkreślono pełną stosowalność metody dla ogólniejszych układów z wieloma opóźnieniami współmiernymi, silną zależność uzyskanych rezultatów od długości przedziału obserwacji oraz wskazane pewne nie rozwiązane problemy będące kontynuacją tematyki niniejszej pracy.

Проверка наблюдаемости в стационарных системах с запаздыванием

Исследуется проблема наблюдаемости для системы, которой траектория является решением дифференциального уравнения с запаздыванием

$$\dot{x}(t)\,Ax(t)+Bx(t-1),\quad t\geqslant 0$$

с выходом

$$y(t)=Cx(t),$$

где: $x(t) \in R^n$, $y(t) \in R^p$; A, B, C — матрицы соответствующей размерности с постоянными коэффициентами. В начестве определения начального состояния принимается пара $(x(0), Bf)$, где $f(t)=x(t-1)$ для $t \in [0, 1]$ является начальной функцией. Определены наблюдаемость и сильная наблюдаемость в виде различаемости началных состояний, исходя из информации о выходе $y(t)$ на некотором интервале $[0, T]$. При этом рассматриваются непрерывные функции f для задачи наблюдаемости. Указана эквивалентная система без запаздываний (2.2) с дополнительными ограничениями типа равенств (2.3). Это подволило дать новую формулировку проблемы наблюдаемости в виде некоторой проблемы инвариантной управляемости эквивалентнойист семы (Лемма 2,7). В работе даны алгоритмы для вычисления соответствующих подпространств инвариантной управляемости и их приложение для проверки системы на наблюдаемость (Теорема 3.1 — простое достаточное условие, Теорема 3.2 — простое необходимое условие, Теоремы 3.3, 3.4, 3.5, 3.6 — полные критерии наблюдаемости для соответствующих случаев). В п. 4 описан пример алгоритма проверки наблюдаемости на интервале $[0, k]$, k — целое число который соответствует Теореме 3.3. Дан также численный пример, поясняющий поочередные этапы алгоритма. Для остальных случаев можно легко построить аналогичный алгоритм.

В заключение отмечена возможность применения описанного метода в случае более сложных систем с многими, соизмеримыми запаздываниями, замечена сильная зависимость полученных результатов от длины интервала наблюдения, а также указаны некоторые открытые вопросы, связанные с темой этой работы.

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