Control
and Cybernetics
VOL. 4 (1975) No. 3–4

Maximum principle for differential open-loop games of pursuit in Banach spaces

by

CZESŁAW SIEMASZKO

J. Dąbrowski Military Technical Academy
Institute of Cybernetics
Warszawa, Poland

This paper considers the problem of open-loop differential games of pursuit in Banach spaces. The necessary conditions for optimal controls are obtained in the form of maximum principle. There are given some applications of this results to partial differential equations and functional differential equations.

Introduction

The object of this paper is to consider open-loop game of pursuit in a Banach space.

The game is described by differential equations in a Banach space. The aim of one player (called further pursuer) is to minimize a capture time of second player (called further evader). The aim of the evader is to maximize this time. Such a game was considered by Kalendzeridze [5]. He has obtained necessary condition for optimal controls in the form of maximum principle in finite dimensional space. For differential games with time lag, the maximum principle was proved by Oguztorelli in [9]. Similar result obtained Kirilova [6] using methods of functional analysis.

Acknowledgements

I wish to express my gratitude to Professor S. Rolewicz without whose valuable help and care this work could not have been carried out.

1. Preliminaries

This paragraph has an introductory character. We give here some definitions, notations and lemmas applicable in further part of the paper.

Let Z be a reflexive, separable Banach space.

Let C_z be a class of all convex, closed, bounded sets of the space Z.

DEFINITION 1. Under sum of sets $A, B \subset Z$ we understand the set of the form

$$A+B=\{x+y;\ x\in A \wedge y \in B\}, \tag{1.1}$$

and by multiplication of a set A by a real number α, we understand the set

$$\alpha A=\{\alpha x;\ x\in A\}. \tag{1.2}$$

From the Definition 1 follows that the sum of two sets from the class C_z belongs to that class, since the sum of two convex, weakly compact sets is convex, weakly compact. For the same reason to that class belongs also product of a set from that class by a real number.

Let X be an arbitrary Banach space.

DEFINITION 2. Function ρ defined for all sets $A, B \subset X$, by formula

$$\rho(A, B)=\sup_{a\in A}\rho(a, B)=\sup_{a\in A}\inf_{b\in B}\|a-b\| \tag{1.3}$$

will be called Hausdorff semimetric.

DEFINITION 3. Function $d(\cdot, \cdot)$ of the form

$$d(A, B)=\max\{\rho(A, B), \rho(B, A)\} \tag{1.4}$$

we shall call Hausdorff metric.

The Hausdorff semimetric has following properties:

PROPERTIES 1 [4]:

(a) $\rho(\alpha A, \alpha B)=\alpha\rho(A, B)$;

(b) $|\rho(A_1, B_1)-\rho(A_2, B_2)|\leqslant d(A_1, A_2)+d(B_1, B_2)$;

(c) $\rho(A_1+A_2, B_1+B_2)\leqslant\rho(A_1, B_1)+\rho(A_2, B_2)$.

It is possible to show that the class C_z is a complete metric space with Hausdorff distance $d(\cdot, \cdot)$ [4]. Let

$$K=\{x;\ \|x\|\leqslant 1\},$$

$$\Sigma=\{x;\ \|x\|=1\}.$$

K is the closed unit ball of X and Σ is the unit sphere of X.

DEFINITION 4. The ball K of Banach space X will be called uniformly strictly convex, if for each ε, $2\geqslant\varepsilon>0$ there exists $\delta(\varepsilon)>0$ such that for any $x_1, x_2\in K$, $\|x_1-x_2\|\geqslant\varepsilon$ implies $\frac{1}{2}\|x_1+x_2\|\leqslant 1-\delta(\varepsilon)$.

DEFINITION 5. Banach space X will be called uniformly strictly convex if its unit ball is uniformly strictly convex.

THEOREM 1 (see [3]). If a Banach space X is uniformly strictly convex, then:

(a) X is reflexive;

(b) $x_n, x_o \in X$, $x_n \to x_o$ weakly and $\|x_n\| \to \|x_o\|$ then $\|x_n - x_o\| \to 0$;

(c) if $x' \in X\, \|x'\| = 1$ and $A(\delta) = \{x; \langle x', x\rangle \geqslant 1-\delta\} \cap K$ then $\delta \to 0$ implies diam $A(\delta) \to 0$.

LEMMA 1. Let Z be uniformly strictly convex Banach space. Let Φ_t be a continuous multifunction with values $\Phi_t \in C_z$. If $x(t) \in \Phi_t$ and $\rho(0, \Phi_t) = \|x(t)\|$ then $x(t)$ is continuous.

Proof. We consider two cases

(1) $\rho(0, \Phi_t) > 0$, $t \in [0, T]$ and $T < \infty$.

(2) $\rho(0, \Phi_t) > 0$, $t \in [0, T]$, $\rho(0, \Phi_T) = 0$.

Case 1. Without loss of generality we can assume $\rho(0, \Phi_t) = 1$, since in another case from property (1, a) and continuity of $\rho(0, \Phi_t)$ on a compact set it follows that

$$\min_{0 \leqslant t \leqslant T} \rho(0, \Phi_t) = \rho(0, \Phi_{t_o}) > 0 \tag{1.5}$$

and putting

$$\Psi_t = \Phi_t / \rho(0, \Phi_t)$$

from property (1, 1) we have

$$\rho(0, \Psi_t) = \rho(0, \Phi_t / \rho(0, \Phi_t)) = 1\,.$$

Let $x(\mathrm{t})$ satisfy the assumptions of our Lemma. By the above considerations we have

$$\|x(t)\| = 1. \tag{1.6}$$

Let $t_n \to t_o$, then $\|x(t_n)\| = 1$.

For any $\varepsilon > 0$ there exists $N(\varepsilon)$ such that for each $n \geqslant N(\varepsilon)$ by continuity of Φ_t we have

$$x(t_n) \in \Phi_{t_n} \subset (\Phi_{t_o} + \varepsilon K)\,. \tag{1.7}$$

Let

$$B(\varepsilon) \overset{\mathrm{df}}{=\!=} (\Phi_{t_o} + \varepsilon K) \cap K\,. \tag{1.8}$$

$B(\varepsilon)$ is a convex set containing $x(t_o)$ and $x(t_n)$ for $n \geqslant N(\varepsilon)$. Let a linear continuous functional x_o' separate the sets Φ_{t_o} and K. Functional x_o' is also a supporting functional of $B(\varepsilon)$ at x_o, that means

$$\sup_{x \in B(\varepsilon)} \langle x_o', x\rangle = 1\,, \tag{1.9}$$

$$\inf_{x \in B(\varepsilon)} \langle x_o', x\rangle = \inf_{x \in \Phi_{t_o} + \varepsilon K} \langle x_o', x\rangle = 1 - \varepsilon\,. \tag{1.10}$$

It implies that

$$B(\varepsilon)\subset\{x;\ \langle x_o', x\rangle\geqslant 1-\varepsilon_{\wedge}\, x\in K\}\overset{\mathrm{df}}{=\!=}A(\varepsilon). \tag{1.11}$$

Hence by Theorem 1 diam $A(\varepsilon)\to 0$ when $\varepsilon\to 0$, so

$$\|x(t_n)-x(t_o)\|\leqslant \operatorname{diam} A(\varepsilon)\to 0 \text{ when } \varepsilon\to 0.$$

Therefore $x(t)$ is continuous.

Case 2. In the second case it follows from the proof of the case 1 that $x(t)$ is continuous on $[0, T)$. But $\|x(t)\|\to 0$ and $\|x(T)\|=0$, that finishes the proof. Q.E.D.

DEFINITION 6. We say the ball K of a Banach space X is smooth if at each point x of the sphere Σ there exists exactly one supporting hyperplane.

LEMMA 2. Let X be a uniformly strictly convex Banach space with a smooth ball, x_o, $x_n\in\Sigma$; x_n' — a supporting functional of K at x_n and $\|x_n'\|=1$. If $\|x_n-x_o\|\to 0$ then there exist subsequence $\{x_{n_k}'\}$ of $\{x_n'\}$ and $\tilde{x}'$ such that $x_{n_k}'\to\tilde{x}'$ in norm and $\langle\tilde{x}', x_o\rangle=1$, it means $\tilde{x}'$ is supporting functional of K at x_o.

Proof. Sequence $\{x_n'\}$ is conditionally weakly compact, since X is reflexive (by Theorem 1). Thus there exists subsequence $\{x_{n_k}'\}$ of $\{x_n'\}$ weakly convergent to some $\tilde{x}'$. We have to show that $\tilde{x}'$ is supporting functional of K at x_o.

$$|\langle\tilde{x}', x_o\rangle-1|=|\langle\tilde{x}', x_o\rangle-\langle x_{n_k}', x_{n_k}\rangle|\leqslant|\langle\tilde{x}', x_o\rangle-\langle x_{n_k}', x_o\rangle|+$$
$$+|\langle x_{n_k}', x_o\rangle-\langle x_{n_k}', x_{n_k}\rangle|\leqslant\varepsilon/2+\varepsilon/2=\varepsilon. \tag{1.12}$$

First estimation follows from weak convergence $x_{n_k}'\to\tilde{x}'$ and the second follows from norm convergence of x_n to x_o. Q.E.D.

In the following by measurability and integrability we shall mean Lebesque measurability and integrability although majority of theorems is true for arbitrary measure.

Let $U(\cdot)$ denote measurable multifunction

$$U(\cdot):[0,\infty)\to C_z. \tag{1.13}$$

The existence of measurable selectors follows from Theorem 1 of [10].

DEFINITION 7. We shall say that $U(\cdot)$ is locally integrable with p-power $1\leqslant p<\infty$ if

$$\int_A d^p(U(t), 0)\, dt<\infty, \tag{1.14}$$

where $A\subset[0,\infty)$ arbitrary compact set

$0=\{0\}$ set consisting of $0\in X$,

$d(\cdot,\cdot)$ Hausdorff metric.

DEFINITION 8. By the integral of $U(\cdot)$ over A we shall mean the set

$$\int_A U(t)\,dt=\left\{z\in Z;\ z=\int_A z(t)\,dt_\wedge z(t)\in U(t)\ \text{a.e.}\right\}. \tag{1.15}$$

The integral $\int_A z(t)\,dt$ is understood in the sense of Bochner. By $L_p[(0, T), Z]$ we denote the space of all functions integrable with p-power in the sense of Bochner.

LEMMA 3. If multifunction $U(\cdot)$ is integrable with p-power on $[0, T]$ then the set of all selectors U_s of $U(\cdot)$ is weakly compact in $L_p[(0, T), Z]$ $1\leqslant p<\infty$.

Proof. For $1<p<\infty$ it follows from the following facts:

(a) U_s is bounded and convex;

(b) $U(t)$ is closed for $t\in[0, T]$ a.e.

Thus U_s is strongly closed, convex what implies boundedness in the norm of L_p and weak closedness. Reflexivity of L_p, $1<p<\infty$ implies weak compactness. Definition 7 of $U(\cdot)$ for $p=1$ implies that for all measurable selectors φ of $U(\cdot)$ we have

$$\|\varphi(t)\|\leqslant f(t)\ \text{a.e.}\qquad \text{and}\qquad f(\cdot)\in L_1$$

that is enough for weak compactness in $L^1[(0, T), Z]$ by paper of Castaing [2]. Q.E.D.

LEMMA 4. Let $F_t: L_p[(0, T), Z]\to X$ be a linear continuous operator for each fixed $t\in[0, T]$. Let a multifunction $U(\cdot)$ be integrable with p-power on $[0, T]$ and satisfy

$$\sup_{u\in U_s}\|F_t(u)-F_{t+\delta}(u)\|\to 0\qquad \text{when}\qquad \delta\to 0 \tag{1.16}$$

then

(a) $\bigwedge_{t\in[0, T]} F_t(U_s)$ is convex weakly compact set,

(b) multifunction $F_t(U_s)$ of t is continuous in Hausdorff topology.

Proof. From Lemma 3 it follows that U_s is weakly compact in $L_p[(0, T), Z]$ so by continuity of F_t we obtain convexity and weak compactness of $F_t(U_s)$, what implies point (a). Point (b) follows from the estimation

$$\rho\left(F_{t_0}(U_s), F_{t_0+\delta}(U_s)\right)=\sup_{u\in U_s}\inf_{v\in U_s}\|F_{t_0}(u)-F_{t_0}(v)+F_{t_0}(v)-F_{t_0+\delta}(v)\|\leqslant$$
$$\leqslant\sup_{v\in U_s}\|F_{t_0}(v)-F_{t_0+\delta}(v)\|\to 0\qquad \text{when}\qquad \delta\to 0$$

because of (1.16)

Similarly we can show

$$\rho\left(F_{t_0+\delta}(U_s), F_{t_0}(U_s)\right)\to 0\qquad \text{when}\qquad \delta\to 0$$

hence we get continuity of $F_t(U_s)$. Q.E.D.

LEMMA 5. Let X be a linear locally convex space. Let $A_i \subset X$ for $i=1, \ldots, n$ and X' dual of X. Then the following equality holds

$$\sup_{x \in A_1 + \ldots + A_n} \langle x', x \rangle = \sum_{i=1}^{n} \sup_{a_i \in A_i} \langle x', a_i \rangle . \tag{1.17}$$

Proof. Let $D \stackrel{\mathrm{df}}{=} \mathop{\mathsf{X}}\limits_{i=1}^{n} A_i$ means Cartesian product, $D \subset \mathop{\mathsf{X}}\limits_{i=1}^{n} X$. We define $\varphi : D \to X$ in the manner $\varphi(x_1, \ldots, x_n) = \sum_{i=1}^{n} x_i \quad \sup_{x \in A_1 + \ldots + A_n} \langle x', x \rangle = \sup_{z \in D} \langle x', \varphi(z) \rangle$ what finishes the proof. Q.E.D.

Let us consider nonlinear integral equation of the form

$$y(t) = \int_0^t S(t, \tau) f(\tau, y(\tau), u(\tau))\, d\tau . \tag{1.18}$$

Let us assume that

$$S(t, \tau) : X \to X \tag{1.19}$$

linear bounded operator from Banach space X into X, for each t,

$$S(t, \tau)\, x \tag{1.20}$$

continuous function of t with fixed τ and continuous function of τ with fixed t, both in the norm topology

$$f(\cdot, \cdot, \cdot) : (0, T) \times X \times Z \to X \tag{1.21}$$

continuous function of all arguments in the norm topology.

Moreover we assume that $f(\cdot)$ has strong derivatives with respect to 2-nd and 3-rd arguments which are continuous in operator norm.

Let

$$\xi(u)(\cdot) \stackrel{\mathrm{df}}{=} y(\cdot) . \tag{1.22}$$

LEMMA 6 [13]. Under assumption (1.19), (1.20), (1.21) about $S(\cdot)$ and $f(\cdot)$, function $\xi(u)$ given by (1.22) has strong derivatives with respect to u at u_o and

$$\delta y(t) \stackrel{\mathrm{df}}{=} \xi'(u_o)(\Delta u)(t) = \int_0^t S(t, \tau)\, [f_y'\, \xi'(u_o)(\Delta u)(\tau) + f_u'(\Delta u)(\tau)]\, d\tau . \tag{1.23}$$

Remark 1. Since $\delta y(t)$ depends linearly on Δu. We shall note this by

$$E_t(\Delta u) \stackrel{\mathrm{df}}{=} \delta y(t) = (\xi'(u_o) \cdot \Delta u)(t) . \tag{1.24}$$

2. Maximum principle for differential open-loop games

2.1. Problem Formulation

Let x and y be two points moving in a Banach space X. A trajectories of this system of points can be given as solutions of some differential, integral or difference equations. In this paragraph we shall deal with trajectories described by differential equation. Equation describing position of a point $x(y)$ depends on some control $u \in U_s$ $(v \in V_s)$. The player using a control u (v) we call pursuer (evader) and note them player $U(V)$ respectively. Moreover let some convex closed set $\Omega \subset X$ be given. The aim of player U is such a choice of control u that at some moment t_1 should be

$$x(t_1) - y(t_1) \in \Omega$$

and the moment t_1 should be minimal. The aim of player V is to maximize this time t_1. In general case we have two differential equations in Banach space X describing trajectories of the pursuer and evader of the form

$$\dot{x}(t) = f_1(x, t, u), \qquad x(t_o) = x_1, \tag{2.1}$$

$$\dot{y}(t) = f_2(y, t, v), \qquad y(t_o) = y_1. \tag{2.2}$$

We assume that the player V chooses his control v at the beginning of the play and cannot change it in the course of the game. Next chooses his control player knowing already the control v.

Definition 9. By the time $T(u, v)$ we denote the smallest time t for which we have inclusion $x(t) - y(t) \in \Omega$ when players U and V apply controls u and v respectively.

Definition 10. We shall call T_o the optimal capture time if for each control v there exists control u such that $T(u, v) \leqslant T_o$ and for any $\varepsilon > 0$ there exists v_o such that for arbitrary u we have inequality

$$T(u, v_o) \geqslant T_o - \varepsilon. \tag{2.3}$$

If for the pair of controls $[u, v]$, $x(t) - y(t) \notin \Omega$ for all t then we put $T(u, v) = \infty$. In the following we shall assume that $T_o < \infty$

Definition 11. We shall call the pair of controls u_o, v_o optimal if

$$\sup_{v \in V_s} \inf_{u \in U_s} T(u, v) = \inf_{u \in U_s} T(u, v_o) = T(u_o, v_o) \overset{\mathrm{df}}{=} T_o. \tag{2.4}$$

2.2. Maximum principle for linear games

In this paragraph we consider differential game, described by the pair of equations of the form

$$x(t) = F_t(u) + x_1(t), \tag{2.5}$$

$$y(t) = E_t(v) + y_1(t), \tag{2.6}$$

where

(a) Z, Q denote reflexive separable Banach spaces;

(b) X denotes an arbitrary Banach space;

(c) $x_1(t)\, y_1(t), x(t), y(t) \in X$;

(d) $x_1(\cdot)$, $y_1(\cdot)$ — given continuous functions;

(e) $U(\cdot)$, $V(\cdot)$ — measurable, integrable with p-power, $1 \leqslant p < \infty$, multifunctions with values in$\to C_z$, C_Q respectively;

(f) $U_s(V_s)$ — the set of all measurable selectors of $U(\cdot)$ $V(\cdot)$ and $u = u(\cdot) \in U_s$, $v = v(\cdot) \in V_s$.

About operators F_t, E_t we assume:

(z_1) For fixed t, $F_t(E_t)$ is linear continuous operator from $L_p[(0, t), Z]$ $\big(L_p[(0, t)\, Q]\big)$ into the Banach space x

$$F_t : L_p[(0\ t), Z] \to X,$$

$$E_t : L_p[(0\ t), Q] \to X.$$

(z_2) Assume that $T < \infty$. Let for $t \leqslant T$ $F_t(E_t)$ satisfy assumptions of Lemma 4.

2.2.1. We consider here the case when Banach space X is uniformly strictly convex and unit ball of this space smooth. Such properties have for example: $L_p(R^n)$ — spaces with Lebesgue measure $1 < p < \infty$,
l_p — spaces and Hilbert spaces (see [7]).

We assume moreover that

$$\Omega = K, \tag{2.7}$$

where K — a unit ball of X.

For simplicity we introduce following notation

$$z(t) = y(t) - x(t) \tag{2.8}$$

where $x(t)$ and $y(t)$ are given by (2.5) (2.6),

$$z_1(t) = y_1(t) - x_1(t), \tag{2.9}$$

$$\Phi_t(u) = F_t(u) + z_1(t). \tag{2.10}$$

The aim of the player U is to reach K in shortest time and the aim of V is to be out of K for the longest time. By optimality of v_o we have

$$[\Phi_t(U_s) - E_t(v_o)] \cap K \quad \text{for} \quad t < T_o. \tag{2.11}$$

We shall need following lemma

Lemma 7. If v_o is optimal control of evader then

$$\bigwedge_{t_1 < T_o} \bigvee_{\varepsilon > 0} \bigwedge_{t \leqslant t_1} K \cap \big(\Phi_t(U_s) - (E_t(v_o) + \varepsilon K)\big) = \emptyset. \tag{2.12}$$

Proof. Let $t_1 < T_o$. We shall show, that there exist $\varepsilon > 0$ such that

$$\rho\big(0, \Phi_t(U_s) - E_t(v_o)\big) > 1 + \varepsilon \quad \text{for} \quad 0 \leqslant t \leqslant t_1, \tag{2.13}$$

where ρ Hausdorff semidistance.

It follows from Lemma 4 that $\Phi_t(U_s)$ is a continuous multifunction. $E_t(v_o)$ is continuous by definition. Moreover both functions are continuous on compact set $[0, t_1]$. Function ρ (continuous with respect to Hausdorff metric $d(\cdot)$) attains its minimum. So there exist a constant a and $t_o \in [0, t_1]$ such that

$$a = \min_{0 \leqslant t \leqslant t_1} \rho\left(0, \Phi_t(U_s) - E_t(v_o)\right) = \rho\left(0, \Phi_{t_o}(U_s) - E_{t_o}(v_o)\right) > 1. \tag{2.14}$$

Hence putting

$$\varepsilon = \frac{a-1}{2} \text{ we get (2.13)}$$

what means

$$(1+\varepsilon) K \cap \left(\Phi_t(U_s) - E_t(v_o)\right) = \emptyset. \tag{2.15}$$

Finally

$$K \cap \left(\Phi_t(U_s) - E_t(v_o) + \varepsilon K\right) = \emptyset \tag{2.16}$$

what finishes the proof. Q.E.D.

Let $\varDelta V_{t_1}$ denote the following set

$$\varDelta V_{t_1} = \{\varDelta v(\cdot) : \varDelta v(\tau) = v(\tau) - v_o(\tau) \wedge v(\tau) \in V(\tau)\ 0 \leqslant \tau \leqslant t_1\} \tag{2.17}$$

where v_o as before optimal control. $\varDelta V_{t_1}$ is the set of admissible variations of optimal control v_o with support contained in $[0, t_1]$. Of course the set $\varDelta V_{t_1}$ is convex by definition and convexity of $V(t)$.

LEMMA 8. If $t_1 < T_o$ and $\varDelta V_{t_1}$ given by (2.17) then

$$\bigwedge_{\varDelta v \in \varDelta V_{t_1}} \bigvee_{\beta_o > 0} \bigwedge_{\beta \leqslant \beta_o} K \cap \left(\Phi_t(U_s) - E_t(v_o + \beta \varDelta v)\right) = \emptyset \tag{2.18}$$

for $t \in [0, t_1]$.

Proof

$$E_t(v_o + \varDelta v) = E_t(v_o) + E_t(\varDelta v), \tag{2.19}$$

$$\sup_{0 \leqslant t \leqslant t_1} \|E_t(\varDelta v)\| = b > 0. \tag{2.20}$$

The case $b = 0$ follows from Lemma 7. Putting $\beta_o = \frac{\varepsilon}{b}$ where ε is the same as in Lemma 7 we get the proof immediately from Lemma 7. Q.E.D.

LEMMA 9. Let X be a uniformly strictly convex Banach space and K smooth unit ball of X. Moreover $x(\cdot)$ continuous function with value in X and $\|x(t)\| > 0$ for $t \in [0, t_1]$, $x'(t) \in X'$ such that $\langle x'(t), x(t)\rangle = \|x(t)\|$ $\|x'(t)\| = 1$ then $x'(\cdot)$ is continuous.

Proof follows immediately from Lemma 2. Q.E.D.

Under those assumptions the following theorem is true.

THEOREM 2. If there exists optimal capture time T_o at the game described by equations (2.5) and (2.6) and corresponding pair of optimal controls u_o, v_o, then there exists linear continuous functional $x'_o \in X'$ such that the optimal control u_o fulfils the equation.

$$\max_{u \in U_s} \langle x'_o, F_{T_o}(u) \rangle = \langle x'_o, F_{T_o}(u_o) \rangle \tag{2.21}$$

and each x'_o fulfiling (2.21) satisfies the equation

$$\max_{v \in V_s} \langle x'_o, E_{T_o}(v) \rangle = \langle x'_o, E_{T_o}(v_o) \rangle \tag{2.22}$$

Proof. Let $z(t)$ be given by (2.8) so it has the form

$$z(t) = E_t(v) - \Phi_t(u). \tag{2.23}$$

For $v = v_o$ multifunction

$$\Psi_t \stackrel{\mathrm{df}}{=} E_t(v_o) - \Phi_t(U_s) \tag{2.24}$$

has not common points with K for $t < T_o$ so

$$\Psi_t \cap K = \emptyset \quad \text{for} \quad t < T_o. \tag{2.25}$$

From the form (2.10) of function $\Phi_t(u)$ and definition of $E_t(v)$ it follows that Ψ_t is a continuous multifunction with respect to Hausdorff metric. From properties of semimetric ρ it follows that the function

$$\rho(0, \Psi_t) = \min_{x \in \psi_t} \|x\| \tag{2.26}$$

is continuous with respect to t.

Reflexivity of X implies the existence of $x(t)$ such that

$$\rho(0, \Psi_t) = \|x(t)\|. \tag{2.27}$$

Lemma 1 implies continuity of $x(t)$. Let $x'(t) \in X'$ be such that

$$\langle x'(t), x(t) \rangle = \|x(t)\| \quad \text{and} \quad \|x'(t)\| = 1. \tag{2.28}$$

From the assumptions on X and Lemma 9 it follows that $x'(t)$ is continuous. Equations (2.25) and (2.27) give us inequality.

$$\rho(0, \Psi_t) = \inf_{x \in \Psi_t} \langle x'(t), x \rangle = \langle x'(t), x(t) > \max_{x \in K} \langle x'(t), x \rangle = 1 \quad \text{for} \quad t < T_o. \tag{2.29}$$

Optimality of time T_o and control v_o implies

$$\min_{x \in \Psi_{T_o}} \langle x'(T_o), x \rangle = 1 \tag{2.30}$$

hence

$$\min_{x \in \Psi_{T_o}} \langle x'(T_o), x \rangle = \langle x'(T_o), E_{T_o}(v_o) \rangle + \min_{u \in U_s} \{-\langle x'(T_o), \Phi_{T_o}(u) \rangle\} \tag{2.31}$$

but

$$\min_{u \in U_s} \{-\langle x'(T_o), \Phi_{T_o}(u) \rangle\} = -\max_{u \in U_s} \langle x'(T_o), \Phi_{T_o}(u) \rangle$$

$$\max_{u \in U_s} \langle x'(T_o), \Phi_{T_o}(u) \rangle = \langle x'(T_o), z_1(T_o) \rangle + \max_{u \in U_s} \langle x'(T_o), F_{T_o}(u) \rangle$$

so

$$\max_{u \in U_s} \langle x'(T_o), F_{T_o}(u)\rangle = \langle x'(T_o), F_{T_o}(u_o)\rangle \tag{2.32}$$

that implies the first part of our theorem. Let us pass to the second part.

We put

$$\rho(0, \Psi_t) = \eta_o(t) \tag{2.33}$$

and

$$\eta(t) \overset{\mathrm{df}}{=} \eta_o(t) + \langle x'(t), E_t(\Delta v)\rangle \tag{2.34}$$

where $\Delta v \in \Delta V_{t_1}$ for some $t_1 \leqslant T_o$.

Let us suppose that the maximum principle does not hold.

That means that there exist $t_1 \leqslant T_o$ and $\Delta v_1 \in \Delta V_{t_1}$ such that

$$\langle x'(T_o), E_{T_o}(\Delta v_1)\rangle = c > 0. \tag{2.35}$$

For fixed Δv_1 left hand side of (2.35) is a continuous function of t, so there exist $h > 0$ such that for $t \in [T_o - h, T_o]$

$$\langle x'(t), E_t(\Delta v_1)\rangle > \frac{c}{2}. \tag{2.36}$$

Form Lemma 8 it follows that for $t \in [0, T_o - h]$ there exist $\beta_o > 0$ which fulfils equality

$$(-\Phi_t(U_s) + E_t(v_o) + \beta_o E_t(\Delta v_1)) \cap K = \emptyset \tag{2.37}$$

hence choosing a control $v_1 = v_o + \beta_o \Delta v_1$ we get

$$(E_t(v_1) - \Phi_t(U_s)) \cap K = \emptyset \quad \text{for} \quad t \in [0, T_o - h]. \tag{2.38}$$

Moreover

$$\eta(t) = \eta_o(t) + \beta_o \langle x'(t), E_t(\Delta v_1)\rangle$$

hence

$$\eta_o(t) + \beta_o \langle x'(t), E_t(\Delta v_1)\rangle > 1 + \beta_o \frac{c}{2} > 1 \tag{2.39}$$

for $t \in [T_o - h, T_o]$, what follows from (2.29) and (2.36). Inequality (2.39) and equality (2.38) imply

$$(E_t(v_1) - \Phi_t(U_s)) \cap K = \emptyset, \; t \in [0, T_o] \tag{2.40}$$

but it contradicts the optimality of time T_o. So for $t = T_o$ and $\bigwedge_{t_1 \leqslant T_o} \bigwedge_{\Delta v \in \Delta V_{t_1}}$ should be fulfilled inequality

$$\begin{gathered} \langle x'(T_o), E_{T_o}(\Delta v)\rangle \leqslant 0, \\ \langle x'(T_o), E_{T_o}(v - v_o)\rangle \leqslant 0, \end{gathered} \tag{2.41}$$

hence

$$\max_{v \in V_s} \langle x'(T_o), E_{T_o}(v)\rangle = \langle x'(T_o), E_{T_o}(v_o)\rangle \tag{2.42}$$

and putting $x'(T_o) = x_o$ we finish the proof. Q.E.D.

2.2.2. In this paragraph we shall prove maximum principle under different assumptions. Here X is an arbitrary Banach space, Ω — closed convex set with interior, U_s and F_t satisfy previous conditions and moreover

$$F_{t_1}(U_s) \subset F_{t_2}(U_s) \quad \text{for} \quad t_1 \leqslant t_2 . \tag{2.43}$$

The assumptions on E_t and V_s are not changed.

THEOREM 3. If there exist optimal capture time T_o and corresponding pair of optimal controls u_o, v_o then there exists linear continuous functional $x'_o \in X'$ such that

$$\max_{u \in U_s} \langle x'_o, F_{T_o}(u)\rangle = \langle x'_o, F_{T_o}(u_o)\rangle . \tag{2.44}$$

Proof. Let us put as before

$$x(t) = F_t(u) + x_1(t), \tag{2.45}$$

$$y(t) = E_t(v) + y_1(t) \tag{2.46}$$

trajectories of pursuer and evader. Without loss of generality we may assume that $x_1(t) \equiv 0$.

Let

$$y_o(t) = E_t(v_o) + y_1(t) \tag{2.47}$$

denote trajectory of evader. Control v_o and time T_o give

$$y_o(T_o) \in \partial (F_{T_o}(U_s) + \Omega)$$

where ∂ denotes boundary of the set. Since Ω has interior there exists continuous linear functional x'_o with $\|x'_o\| = 1$ such that

$$\sup_{x \in F_{T_o}(U_s)+\Omega} \langle x'_o, x\rangle = \langle x'_o, y_o(T_o)\rangle \tag{2.48}$$

Lemma 5 gives us equality

$$\sup_{x \in F_{T_o}(U_s)+\Omega} \langle x'_o, x\rangle = \sup_{x \in F_{T_o}(U_s)} \langle x'_o, x\rangle + \sup_{\omega \in \Omega} \langle x'_o, \omega\rangle \tag{2.49}$$

which implies

$$\sup_{x \in F_{T_o}(U_s)} \langle x'_o, x\rangle = \sup_{u \in U_s} \langle x'_o, F_{T_o}(u)\rangle = \langle x'_o, F_{T_o}(u_o)\rangle \quad \text{Q.E.D.} \tag{2.50}$$

In a similar way as in Lemmas 7 and 8 we can prove following lemmas.

LEMMA 10. If v_o is optimal control of evader than

$$\bigwedge_{t_1 < T_o} \bigvee_{\varepsilon > 0} \bigwedge_{0 \leqslant t \leqslant t_1} (y_0(t) + \varepsilon K) \cap F_t(U_s) = \emptyset . \tag{2.51}$$

LEMMA 11. Let $t_1 < T_o$ and ΔV_{t_1} given by (2.17) then

$$\bigwedge_{\Delta v \in \Delta V t_1} \bigvee_{\beta_o > 0} \bigwedge_{\beta \leqslant \beta_o} (y_1(t) + E_t(v_o + \beta \Delta v)) \notin F_t(U_s) \tag{2.52}$$

for $t \leqslant t_1$.

Now we can pass to the proof of the maximum principle for evader.

THEOREM 4. If there exists optimal capture time T_o and corresponding pair of controls u_o, v_o then each linear bounded functional $x'_o \in X'$ fulfilling (2.44) satisfies also

$$\max_{v \in V_s} \langle x'_o, E_{T_o}(v)\rangle = \langle x'_o, E_{T_o}(v_o)\rangle . \tag{2.53}$$

Proof. For arbitrary trajectory $y(t)$ of evader we have

$$y(t) = y_o(t) + E_t(\Delta v) . \tag{2.54}$$

Let us consider unequality

$$\langle x', E_{T_o}(\Delta v)\rangle \leqslant 0 \quad \text{for} \quad \Delta v \in \Delta V_{t_1} . \tag{2.55}$$

This inequality implies maximum principle for $t \leqslant t_1$. Let us suppose that maximum principle is not true, so there exist $t_1 \leqslant T_o$ and $\Delta v_1 \in \Delta V_{t_1}$ such that

$$\langle x'_o, E_{T_o}(\Delta v_1)\rangle = c > 0 . \tag{2.56}$$

From continuity of (2.56) with respect to t with fixed Δv_1, follows existence of $h > 0$ such that for $t \in [T_o - h, T_o]$

$$\langle x'_o, E_t(\Delta v_1)\rangle > \frac{c}{2} . \tag{2.57}$$

Let us consider equation of the form

$$y_p(t) \overset{\text{df}}{=} y_o(t) + E_t(\Delta v_1) .$$

Lemma 11 implies the existence of $\beta_o > 0$ such that

$$y_{\beta_o}(t) = y_o(t) + \beta_o E_t(\Delta v_1) \notin F_t(U_s) . \tag{2.58}$$

for $\quad t \leqslant T_o - h.$

Inequality (2.57) gives us

$$\langle x'_o, y_{\beta_o}(t) - y_o(t)\rangle = \langle x'_o, \beta_o E_t(\Delta v_1)\rangle \tag{2.59}$$

for $t \in [T_o - h, T_o]$ what means

$$y_{\beta_o}(t) \notin F_t(U_s) \quad \text{for} \quad t \in [T_o - h, T_o] \tag{2.60}$$

by assumption $F_t(U_s) \subset F_{T_o}(U_s)$, $t \leqslant T_o$, so (2.58) and (2.60) show us that

$$y_{\beta_o}(t) \notin F_t(U_s) \quad \text{for} \quad t \leqslant T_o \tag{2.61}$$

that contradicts the optimality of T_o so we have

$$\langle x'_o, E_{T_o}(\Delta v)\rangle \leqslant 0 \quad \text{for all} \quad t_1 \leqslant T_o \tag{2.62}$$

and

$$\Delta v \in \Delta V_{t_1}$$

hence

$$\max_{v \in V_s} \langle x'_o, E_{T_o}(v)\rangle = \langle x'_o, E_{T_o}(v_o)\rangle . \quad \text{Q.E.D.} \tag{2.63}$$

3. Nonlinear case

Similarly as before we can prove the maximum principle in nonlinear case in two variants.

3.1. Here Banach space X, operator F_t and the set Ω have properties as in point 2.1. The game is described by equations:

$$x(t)=F_t(u)+x_1(t), \tag{3.1}$$

$$y(t)=\int_0^t S(t,\tau)\, f\big(\tau, y(\tau), v(\tau)\big)\, d\tau. \tag{3.2}$$

The operators $S(\cdot,\cdot)$ and $f(\cdot,\cdot,\cdot)$ fulfil assumptions of Lemma 6.

THEOREM 5. If there exists an optimal capture time T_o at the game described by (3.1) and (3.2) and corresponding it pair of optimal controls u_o, v_o then there exists continuous bounded functional $x_o' \in X'$ such that for this u_o we have

$$\max_{u\in U_s} \langle x_o', F_{T_o}(u)\rangle = \langle x_o', F_{T_o}(u_o)\rangle \tag{3.3}$$

and any x_o', satisfying (3.3) fulfils also equation

$$\max_{v\in V_s} \langle x_o', E_{T_o}(v)\rangle = \langle x_o', E_{T_o}(v_o)\rangle \tag{3.4.}$$

where $E_T(v)$ is given by (1.24),

Proof. Let us define

$$z(t)=x(t)-y(t) \tag{3.5}$$

where $x(t)$ and $y(t)$ are given by (3.1) (3.2).

Let

$$x_1(t)+E_t(u) \overset{\text{df}}{=} \Phi_t(u) \text{ then we get} \tag{3.6}$$

$$z(t)=\Phi_t(u)-y(t). \tag{3.7}$$

Let $y_o(t)$ be a trajectory of evader corresponding to optimal control v_o. From lemma (6) we get

$$y(t)=y_o(t)+\varepsilon\delta y(t)+O_t(\varepsilon) \tag{3.8}$$

where $\delta y(t)$ corresponds to variation Δv of v_o

$$z(t)=\Phi_t(u)-y_o(t)-\varepsilon\delta y(t)-O_t(\varepsilon). \tag{3.9}$$

For $\varepsilon=0$ similarly as in linear case there exist $x'(t)\in X'$ such that $x'(t)$ is continuous and

$$\eta_o(t) \overset{\text{df}}{=} \min_{u\in U_s} \langle x'(t), \Phi_t(u)\rangle - \langle x'(t), y_o(t)\rangle > 1 \tag{3.10}$$

for $t<T_o$, and $\eta_o(T_o)=1$.

For a trajectory $y(t)$ we define

$$\eta_\varepsilon(t) \overset{\mathrm{df}}{=} \min_{u \in U_s} \langle x'(t), \Phi_t(u)\rangle - \langle x'(t), y_o(t)\rangle - \\ - \langle x'(t), \varepsilon\delta y(t)\rangle - \langle x'(t), O_t(\varepsilon)\rangle . \quad (3.11)$$

Let us notice that $\eta_\varepsilon(T_o) > 1$ implies strict separation of sets $\Phi_{T_o}(U_s) - y(T_o)$ and K. We get maximum principle if

$$\langle x'(T_o), \delta y(T_o)\rangle \geqslant 0 \quad \text{for any} \quad t_1 \leqslant T_o \text{ and } \bigwedge_{\Delta v \in \Delta V_{t_1}} \quad (3.12)$$

where Δv corresponds $\delta y(t)$. Let us suppose that the maximum principle does not hold. Then there exists $\delta y_1(t)$ such that

$$\langle x'(T_o), \delta y_1(T_o)\rangle = -c < 0 . \quad (3.13)$$

By continuity of $x'(t)$ and $\delta y_1(t)$ there exists $h > 0$ such that

$$\langle x'(t), \delta y_1(t)\rangle < -\frac{c}{2} \; t \in [T_o - h, T_o] \quad (3.14)$$

so for sufficiently small $\varepsilon_o > 0$ we get

$$\eta_{\varepsilon_o}(t) > 1 + \varepsilon_o \frac{c}{2} - \langle x'(t), O_t(\varepsilon_o)\rangle > 1 \quad (3.15)$$

for $t \in [T_o - h, T_o]$.

This inequality means that for the function

$$y_{\varepsilon_o}(t) = y_o(t) + \varepsilon_o \, \delta y_1(t) + O_t(\varepsilon_o)$$

the value of the multifunction $\Phi_t(U_s) - y_{\varepsilon_o}(t)$ can be strictly separated from K for any t. On the other hand

$$\|\varepsilon \delta y_1(t) + O_t(\varepsilon)\| \quad (3.16)$$

can be arbitrarily small for suitable $\varepsilon = \varepsilon_1 < \varepsilon_o$ for $t \in [0, T_o]$. Using Lemma 11 we get

$$(\Phi_t(U_s) - y_{\varepsilon_1}(t)) \cap K = \varnothing, \; t \in [0, T_o - h] . \quad (3.17)$$

Hence and from (3.15) follows

$$(\Phi_t(U_s) - y_{\varepsilon_1}(t)) \cap K = \varnothing, \; t \in [0, T_o] . \quad (3.18)$$

That contradicts the optimality of T_o and v_o so there should be

$$\langle x'(T_o), \delta y(T_o)\rangle \leqslant 0 \quad \text{for all} \quad t_1 \leqslant T_o \text{ and } \Delta v \in \Delta V_{t_1} . \quad (3.19)$$

As follows from (1.24) $\delta y(t)$ depends on Δv linearly so we can write

$$\delta y(t) \overset{\mathrm{df}}{=} E_t(\Delta v) . \quad (3.20)$$

Hence we have

$$\langle x'(T_o), E_{T_o}(\Delta v)\rangle \geqslant 0$$

$$\min_{v \in V_s} \langle x'(T_o), E_{T_o}(v)\rangle = \langle x'(T_o), E_{T_o}(v_o)\rangle \quad (3.21)$$

and putting $x'(T_o) = -x'$ we get second part of the theorem. To prove the first part we use (3.10) for $t = T_o$. We get then (3.22)

$$\min_{u \in U_s} \langle x'(T_o), \Phi_{T_o}(u) \rangle = \min_{u \in U_s} \langle x'(T_o), F_{T_o}(u) \rangle + \langle x'(T_o), x_1(T_o) \rangle \tag{3.22}$$

or

$$\max_{u \in U_s} \langle x'_o, F_{T_o}(u) \rangle = -\min_{u \in U_s} \langle x'(T_o), F_{T_o}(u) \rangle \tag{3.23}$$

so by optimality of u_o we finally get

$$\max_{u \in U_s} \langle x'_o, F_{T_o}(u) \rangle = \langle x'_o, F_{T_o}(u_o) \rangle . \quad \text{Q.E.D.} \tag{3.24}$$

Remark 2. In nonlinear case maximum principle under assumption of paragraph 2.2. can be proved in a similar way.

Corollary 1. If operators $E_t\,(F_t)$ satisfing (z_1) and (z_2) are represented by equation

$$E_t(v) = \int_0^t S(t, \tau)\, v(\tau)\, d\tau , \tag{3.25}$$

$$F_t(u) = \int_0^t W(t, \tau)\, u(\tau)\, d\tau , \tag{3.26}$$

then we can give Theorem 2 the following form.

THEOREM 5′. If there exists optimal capture time T_o at the game described by (3.25) and (3.26) and corresponding it pair of optimal controls u_o, v_o then there exists $x'_o \in X'$ such that for u_o we have

$$\max_{u \in U_s} \langle W^*(T_o, \tau)\, x'_o, u(\tau) \rangle = \langle W^*(T_o, \tau)\, x'_o, u_o(\tau) \rangle \tag{3.27}$$

and each x' fulfiling (3.27) fulfils also

$$\max_{v \in V_s} \langle S^*(T_o, \tau)\, x'_o, v(\tau) \rangle = \langle S^*(T_o, \tau)\, x'_o, v_o(\tau) \rangle . \quad \text{Q.E.D.} \tag{3.28}$$

4. Applications

In this paragraph we shall give some examples of operators E_t and F_t fulfiling assumptions of the theorems of previous paragraphs.

4.1. Abstract parabolic equation in Banach space

X — Banach space. Let us consider an equation of the form

$$\frac{dx(t)}{dt} = A(t)\, x(t) \tag{4.1}$$

where $x(t) \in X$

$$x(0) = x_o. \tag{4.2}$$

Following Sobolewski we have:

THEOREM 6 [12]. Let $A(t)$ be a linear unbounded operator with dense in X domain D independent of t for $t \in [0, T]$. Let for arbitrary $t, \tau, s \in [0, T]$

$$\|[A(t) - A(\tau)]\, A^{-1}(s)\| \leqslant c|t-\tau|^{\varepsilon} \tag{4.3}$$

for some $\varepsilon \in (0, 1]$ and moreover for arbitrary λ with $\operatorname{Re} \lambda \geqslant 0$, the operator $A(t) + \lambda I$ has bounded inverse with

$$\|(A(t) + \lambda I)^{-1}\| \leqslant c\, [|\lambda| + 1]^{-1} \tag{4.4}$$

then there exists an operator-function $S(t, \tau)$ defined and strongly continuous for $\tau \leqslant t$; $\tau, t \in [0, T]$ fulfilling the following conditions: $S(t, \tau)$ is uniformly differentiable with respect to t, $t > \tau$

$$\frac{\partial S(t, \tau)}{\partial t} + A(t)\, S(t, \tau) = 0, \tag{4.5}$$

$$S(t, \tau) = S(t, l)\, S(l, \tau) \quad \text{for} \quad T \geqslant t \geqslant l \geqslant \tau \tag{4.6}$$

$$S(t, t) = I. \tag{4.7}$$

Formula

$$x(t) = S(t, 0)\, x_o \tag{4.8}$$

gives the unique solution of (4.1) continuous for $t \in [0, T]$ and differentiable for $t > 0$. Q.E.D.

THEOREM 7 [12]. Let $A(t)$ fulfil assumption of the theorem 6 and moreover

$$\|A^{-1}(s)\, [A(t) - A(\tau)]\| \leqslant c\, |t-\tau|^{\eta} \tag{4.9}$$

for some $\eta \in (0, 1]$. The bar means closedness of operator in X. Then the operator-function $S(t, \tau)$ is uniformly continuously differentiable with respect to τ, $\tau < t$ and two-time in t and τ together and

$$\frac{\partial S(t, \tau)}{\partial \tau} - \overline{S(t, \tau)\, A(\tau)} = 0, \tag{4.10}$$

$$\frac{\partial^2 S(t, \tau)}{\partial t\, \partial \tau} - \overline{A(t)\, S(t, \tau)\, A(\tau)} = 0. \tag{4.11}$$

The following estimation is also true

$$\|A^{-\alpha}(t), S(t, \tau)\, A^{\beta}(\tau)\| \leqslant c\, |t-\tau|^{\alpha-\beta}, \qquad 0 \leqslant \alpha \leqslant \beta < 1 + \eta \tag{4.12}$$

$$\|A^{\alpha}(t)\, S(t, \tau)\, A^{\beta}(\tau)\| \leqslant c\, |t-\tau|^{-\alpha-\beta}, \qquad 0 \leqslant \alpha \leqslant 1 + \varepsilon, \quad 0 \leqslant \beta \leqslant 1 - +\eta. \tag{4.13}$$

We have the following estimation for $S(t, \tau)$. Q.E.D

LEMMA 12 [12]. If assumption of the Theorem 7 are satisfied then

$$\left\| A^{\alpha}(\xi) \int_{\tau}^{t+\Delta t} S(t+\Delta t, s) f(s)\, ds - \int_{\tau}^{t} S(t, s) f(s)\, ds \right\| \leqslant$$

$$\leqslant c(\alpha)\, (\Delta t)^{(p-1)/p} \left[\int_{\tau}^{t+\Delta t} \|f(s)\|^{p}\, ds \right]^{1/p} \tag{4.14}$$

for $f \in L_p[0, T]$ $0 \leqslant \alpha < \dfrac{p-1}{p}$. Q.E.D.

LEMMA 13. Under the same assumption as in the Theorem 7 the following estimation is true

$$\int_{t_o}^{t} \|S(t, \tau) - S(t, \tau - \Delta)\|\, d\tau \leqslant |\Delta \ln \Delta| +$$

$$+ |(t - t_o + \Delta) \ln (t - t_o + \Delta) - (t - t_o) \ln (t - t_o)|. \tag{4.15}$$

Proof. From Theorem 7 we have

$$S(t, \tau) - S(t, \tau - \Delta) = \int_{\tau-\Delta}^{\tau} S(t, s)\, A(s)\, ds. \tag{4.16}$$

From the same theorem we have the estimation

$$\|\overline{S(t, s)\, A(s)}\| \leqslant \frac{c}{t-s} \quad \text{for} \quad t > s \tag{4.17}$$

hence we get

$$\|S(t, \tau) - S(t, \tau - \Delta)\| \leqslant \int_{\tau-\Delta}^{\tau} \|\overline{S(t, s)\, A(s)}\|\, ds \leqslant$$

$$\leqslant \left| \int_{\tau-\Delta}^{\tau} \frac{c}{t-s}\, ds = c \ln\left(1 + \frac{\Delta}{t-\tau}\right) \right| \quad \text{for} \quad t > \tau. \tag{4.18}$$

We have

$$c \int_{t_o}^{t} \ln\left(1 + \frac{\Delta}{t-\tau}\right) d\tau = c \left[\int_{t_o}^{t} \ln (t - \tau + \Delta) - \ln (t - \tau)\, d\tau \right] =$$

$$= c \left\{ -(t - \tau + \Delta) \ln (t - \tau + \Delta) \Big|_{t_o}^{t} + (t - \tau) [\ln (t - \tau) - 1] \Big|_{t_o}^{t} \right\} =$$

$$= c [-\Delta \ln \Delta + (t - t_o + \Delta) \ln (t - t_o + \Delta) - (t - t_o) \ln (t - t_o)]. \tag{4.19}$$

So we get

$$\int_{t_o}^{t} \|S(t, \tau) - S(t, \tau - \Delta)\|\, d\tau \leqslant c\, | -\Delta \ln \Delta +$$

$$+ (t - t_o + \Delta) \ln (t - t_o + \Delta) - (t - t_o) \ln (t - t_o)|. \quad \text{Q.E.D.}$$

Remark 3. From the Lemma 13 it follows that the operator F_t defined by

$$F_t(u)=\int_{t_0}^{t} S(t,\tau)\,u(\tau)\,d\tau \tag{4.20}$$

fulfils assumptions of the Lemma 4 and therefore also assumptions of Theorem 2. In the next paragraph we shall apply our results to a partial differential equation. For this purpose the following definition will be useful.

Definition 12. We shall say that control $u(t)\in U(t)$ is of bang-bang type if $u(t)\in \in \operatorname{Extr} U(t)$ where $\operatorname{Extr} U(t)$ — the set of extremal points of $U(t)$

4.2. Bang-bang principle for the differential game of pursuit described by some partial differential equation of parabolic type

Let the game be described by partial differential equation of the form

$$\frac{\partial x(t,p)}{\partial t}=A\left(p,\frac{\partial}{\partial p}\right)x(t,p) \tag{4.21}$$

where $t\in(0,T]\ p\in D\subset R^n$ with initial condition

$$x(0,p)=0 \tag{4.21}$$

and boundary condition

$$\lim_{p\to y} B\left(y,\frac{\partial}{\partial p}\right)x(t,p)=u(t,y)-v(t,y) \tag{4.22}$$

where $u(t,y)$, $v(t,y)$ are measurable function such that

$$|u(t,y)|\leqslant M(t,y), \tag{4.23}$$

$$|v(t,y)|\leqslant N(t,y), \tag{4.24}$$

$M(t,y)\,N(t,y)$ — are measurable, bounded, positive functions. Linear operators A and B have the form

$$A\left(p,\frac{\partial}{\partial p}\right)=\sum_{j,i=1}^{n}\frac{\partial}{\partial p_i}\left(a_{ij}(p)\frac{\partial}{\partial p_j}\right)+a(p), \tag{4.25}$$

$$B\left(y,\frac{\partial}{\partial_p}\right)=\sum_{j,i=1}^{n}a_{ij}(y)\,v_j(y)\frac{\partial}{\partial p_i}+b(y), \tag{4.26}$$

and the operator $\left[A\left(p,\frac{\partial}{\partial p}\right),B\left(y,\frac{\partial}{\partial p}\right)\right]$ is selfadjoint. Domain D has Liapunov boundary $S\in C^3$, $v(y)=[v_1(y),\,\dots,\,v_n(y)]$ is the unit vector of the inward normal to the surface S. The operator $A\left(p,\frac{\partial}{\partial p}\right)$ is of eliptic type in D.

Let $\varphi_k(p)$ and λ_k denote eigenfunctions and eigenvalues of the operator (A, B). The system $\{\varphi_k\}$ is complete in the space $L^2(D)$ (see [8]). The time optimal control problem for this equation was considered by Lawruk, Rolewicz [8]. We assume that the remaining conditions from [8] are fulfilled. Let $\Omega \subset L^2(D)$ has the form

$$\Omega = \Omega_o + \varepsilon K, \tag{4.27}$$

where $\varepsilon > 0$ and K unit ball of $L^2(D)$. Ω_o — arbitrary closed convex bounded set. As before the aim of the player U is to bring the trajectory of equation (2.21) in a shortest time T_o to Ω and the aim of V is to maximize this T_o.

Let $x(t, p)$ be the solution of (4.21), (4.21′) then according to (10) in [8] we get

$$\int_D x(t, p)\, \varphi_k(p)\, dp = \int_0^t \int_S \{e^{\lambda_k(\tau - t)} [u(\tau, y) - v(\tau, y)]\, \varphi_k(y)\}\, d\tau\, dy. \tag{4.28}$$

We shall assume that $0 \notin \Omega$.

THEOREM 8. If there exists an optimal capture time T_o and corresponding pair of optimal controls u_o, v_o then these controls have to be bang-bang.

Proof. Let us consider the Banach space Z_t of measurable functions $u(t, y)$ with the norm

$$\|u\|_{t_1} = \operatorname{ess} \sup_{0 \leq t \leq t_1} \sup_y \frac{|u(t, y)|}{M(t, y)}. \tag{4.29}$$

Define U_s by

$$U_s = \{u(t, y);\ |u(t, y)| \leq M(t, y)\}. \tag{4.30}$$

Hence the set U_s is the unit ball in this space. Similarly as in [8] we can show that the operator

$$\mathscr{A}_{t_1}(u) = \left\{\int_0^t e^{\lambda_k(\tau - t_1)} \int_S u(\tau, y)\, \varphi_k(y)\, dy\, d\tau\right\} \tag{4.31}$$

is a weakly * continuous operator from Z_{t_1} into l^2. Hence the image of the unit ball of Z_{t_1} is weakly compact in l^2. Since the system $\{\varphi_K\}$ is complete in $L^2(D)$, $L^2(D)$ is isomorphic to l^2. The image of Ω under this isomorphism I is a closed convex set, containing a nonempty interior. The operator $\mathscr{A}_{t_1}(u)$ as a function of t_1 is continuous. Introducing the space Q_{t_1} of measurable functions $v(t, y)$ with the norm

$$\|v\|_{t_1} = \operatorname{ess} \sup_{0 \leq t \leq t_1} \sup_y \frac{|v(t, y)|}{N(z, y)} \tag{4.32}$$

we get a Banach space with properties similar to Z_{t_1}. The game will be over at time T_o when

$$\mathscr{A}_{T_o}(v_o) \in \mathscr{A}_{T_o}(U_s) + I(\Omega). \tag{4.33}$$

So we see that there exists an $x' \in l^2$ — the supporting functional to $\mathscr{A}_{T_0}(U_s)+$ $+I(\Omega)$ at $\mathscr{A}_{T_0}(v_0)$. Hence for optimal contols u_0, v_0 we have according to theorem (2) the maximum principle

$$\sup_{\|u\|_{T_0}\leq 1} \langle x_0', \mathscr{A}_{T_0}(u)\rangle = \langle x_0', \mathscr{A}_{T_0}(u_0)\rangle \tag{4.34}$$

$$\sup_{\|v\|_{T_0}\leq 1} \langle x_0', \mathscr{A}_{T_0}(v)\rangle = \langle x_0', \mathscr{A}_{T_0}(v_0)\rangle . \tag{4.35}$$

Since $x_0' = \{a_K\} \in l^2$ we can represent it on $\mathscr{A}_{T_0}(u)$ in the following way

$$\langle x_0', \mathscr{A}_{T_0}(u)\rangle = \int_0^{T_0} \int_S \Gamma(\tau, y)\, u(\tau, y)\, ds\, d\tau , \tag{4.36}$$

where

$$\Gamma(\tau, y) = \sum_{k=1}^{\infty} d_k\, e^{\lambda_k(\tau - T_0)}\, \varphi_k(y) . \tag{4.37}$$

Similarly as in [8] we can show that the supremums in (4.34) and (4.35) are attained for functions u_0, v_0 uniquely defined by the equations

$$|u_0(\tau, y)| = M(\tau, y) , \tag{4.38}$$

$$|v_0(\tau, y)| = N(\tau, y) . \tag{4.39}$$

Formulas (4.38) and (4.39) show that the optimal controls have to be bang-bang.

5. Games described by functional equation of neutral type

5.1. Differential equation of neutral type

We shall deal with linear differential equation of neutral type of the form

$$\begin{gathered} \frac{d}{dt} D(x(\cdot), t) = \int_0^t d_s \eta(t, s)\, x(s) + f(t), \\ x_{t_0} = \kappa , \quad \kappa \in C[-h, 0], \quad t \in [t_0, t_1] . \end{gathered} \tag{5.1}$$

All results and notation of this paragraph belong to Banks, Kent [1]. Following paper [1] we introduce notation; $C[-h, 0]$ space of continuous functions with values in R^n on an interval $[-h, 0]$ with norm sup.

$$x_{t_0}(\tau) \overset{\mathrm{df}}{=} x(t_0 + \tau) \in R^n , \quad \tau \in [-h, 0] .$$

Function $D(x(\cdot), t)$ has the form

$$D(x(\cdot), t) \overset{\mathrm{df}}{=} x(t) - \int_{t_0 - h}^{t} d_s\, \mu(t, s)\, x(s) . \tag{5.2}$$

Functions $\mu(\cdot)$ and $\eta(\cdot)$ are $n \times n$ matrices. By a solution of differential equation (5.1) with initial value κ we shall mean an $x \in C[t_0 - h, t_1]$ such that $t \to D(x(\cdot), t)$

is absolutely continuous on $[t_o, t_1]$ with (5.1) being satisfied a.e. Now we shall give an assumption under which there exists a solution of (5.1).

Assumption 1. $\mu(\sigma, \tau)=0$ for $\tau \geqslant \sigma$; $\mu(\sigma, \tau)=\mu(\sigma, t_o-h)$ for $\tau<t_o-h$. μ is Borel measurable, continuous from the right in its first argument and continuous from the left in its second argument; $\tau \to \mu(\sigma, \tau)$ is of bounded variation on every finite τ interval, uniformly in σ and the mapping $t \to \Gamma(y, t) \stackrel{\text{df}}{=} \int_{t_o-h}^{t} d_s \mu(t, s)\, y(s)$ is continuous on $[t_o, t_1]$ for each fixed $y \in C[t_o-h, t_1]$ which obviously implies that $(y, t) \to \Gamma(y, t)$ is continuous.

Assumption 2. There is a continuous nondecreasing function δ with $\delta(0)=0$ such that for each $t \in R^1$ and $\varepsilon>0$ we have

$$\operatorname{Var}\big([t-\varepsilon, t];\ \mu(t, \cdot)\big) \leqslant \delta(\varepsilon).$$

Assumption 3. $\eta(\sigma, \tau)=0$ for $\tau \geqslant \sigma$, $\eta(\sigma, \tau)=\eta(\sigma, t_o-h)$ for $\tau<t_o-h$, η is measurable, continuous from the left in its second argument on $(-\infty, \sigma)$: $\tau \to \eta(\sigma, \tau)$ is of bounded variation on every finite τ interval and there is an $m \in L_1^{\text{Loc}}$ such that

$$\operatorname{Var}\big([t_o-h, \sigma];\ \eta(\sigma, \cdot)\big) \leqslant m(\sigma).$$

THEOREM 9 [1]. Under assumption 1, 2, 3 for each fixed $t \in [t_o, t_1]$ the system

$$\begin{gathered} Y(s, t)=E_n+\int_s^t d_\alpha Y(\alpha, t)\, \mu(\alpha, s)-\int_s^t Y(\alpha, t)\, \eta(\alpha, s)\, ds \\ s \in [t_o, t],\ Y(t, t)=E,\ Y(s, t)=0,\ s>t \end{gathered} \tag{5.3}$$

has a unique solution on $[t_o, t_1]$. This solution $Y(s, t)$ is left continuous in its first argument and

$$|Y(s, t)| \leqslant B,\ \operatorname{Var}\big([t_o, t_1];\ Y(\cdot, t)\big) \leqslant B \quad \text{for} \quad (s, t) \in [t_o, t_1] \times [t_o, t_1].$$

The following theorem allows us to represent the solution of (5.1).

THEOREM 10 [1]. Let $x(\cdot)$ be the solution of (5.1) under assumptions 1, 2, 3. Then for $t \in [t_o, t_1]$

$$x(t)=Y(t_o, t)\, D(\kappa, t_o)+\int_{t_o-h}^{t_o-} d_\beta\, \gamma(t, \beta)\, \kappa(\beta)+\int_{t_o}^{t} Y(\beta, t) f(\beta)\, d\beta \tag{5.4}$$

where Y is given by (5.3) and

$$\gamma(t, \beta) \stackrel{\text{df}}{=} -\int_{t_o}^{t+} dY(\alpha, t)\, \mu(\alpha, \beta)+\int_{t_o}^{t} Y(\alpha, t)\, \eta(\alpha, \beta)\, d\alpha. \quad \text{Q.E.D.} \tag{5.5}$$

Assumption 4. μ has the following property; there exist $l>0$, $L>0$ such that

$$\int_{t-l}^{t} d_\tau \{\mu(t,\tau)-\mu(s,\tau)\}| \leqslant L\,|t-s| \tag{5.6}$$

for $s \leqslant t$.

Let $U(\cdot)$ — measurable multifunction and for fixed t, $U(t) \in C_R\, q$; U_s — the set of selectors of $U(\cdot)$.

LEMMA 14 [1]. Under assumptions 1, 2, 3, 4, Φ being compact in $C[-h, 0]$ implies

$$\mathscr{A} \overset{\mathrm{df}}{=} \{x(\kappa, u)(\cdot);\ (\kappa, u) \in \Phi \times U_s\} \tag{5.7}$$

is an equicontinuous subset of $C[t_o-h, t_1]$. Q.E.D.

Let us define an attainable set in $C[-h, 0]$ by

$$\mathscr{A}_t = \{z \in C[-h, 0];\ z = x_t(\kappa, u);\ (\kappa, u) \in \Phi \times U_s\}. \tag{5.8}$$

Let us fix $\kappa \in \Phi$, then we have by representation theorem

$$X_t(u)(\tau) = Y(t_o, t_o+\tau)\, D(\kappa, t_o) + \int_{t_o-h}^{t_o^-} d_\beta\, \gamma(t+\tau, \beta)\, \kappa(\beta) +$$

$$+ \int_{t_o}^{t+\tau} Y(\beta, t+\tau)\, u(\beta)\, d\beta, \qquad \tau \in [-h, 0]. \tag{5.9}$$

We can write this in another way, namely

$$x_t = x_t^1 + F_t(u), \tag{5.10}$$

where

$$F_t(u) \overset{\mathrm{df}}{=} \int_{t_o}^{t_o+\tau} Y(\beta, t+\tau)\, u(\beta)\, d\beta, \qquad \tau \in [-h, 0]. \tag{5.11}$$

$F_t(u) \in C[-h, 0]$ for fixed t, and is continuous function of t, so

$$F(u) \overset{\mathrm{df}}{=} F(\cdot)(u) \in C\,[[t_o, t_1],\ C[-h, 0]]. \tag{5.12}$$

It is easy to see that F is a continuous operator

THEOREM 11 [1]. Under assumption 1, 2, 3, 4, Φ being compact in $C[-h, 0]$ implies $\mathscr{A}_t$ given by (5.8) is campact in $C[-h, 0]$, $t \in [t_o, t_1]$. Furthermore the mapping $t \to \mathscr{A}_t$ is continuous in Hausdorff metric. Q.E.D.

5.2. Linear differential game of pursuit

We consider here the game described by the pair of equations

$$\frac{d}{dt} D_1(x(\cdot), t) = \int_{t_o-h}^{t} d_s\, \eta_1(t, s)\, x(s) + u(t), \tag{5.13}$$

$$\frac{d}{dt} D_2(y(\cdot), t) = \int_{t_o-h}^{t} d_s\, \eta_2(t, s)\, y(s) + v(t) \tag{5.14}$$

with initial values

$$x_{t_o}=\kappa_0, \quad y_{t_o}=\kappa_1, \quad \kappa_0, \kappa_1 \in C[-h, 0]. \tag{5.15}$$

Let $U(\cdot)$ — multifunction integrable with p-power, $1 \leqslant p < \infty$, $V(\cdot)$ – multifunction as above, $U_s(V_s)$ — the set of measurable selectors of $U(\cdot)(V(\cdot))$, $u(\cdot) \in U_s$, $v(\cdot) \in V_s$ — controls of players.

The operators D_1, D_2 are given by (5.2). We assume that all the assumptions of previous paragraph are fulfiled. Let I: $C[-h, 0] \to L^2[-h, 0]$ be an embedding of $C[-h, 0]$ into $L^2[-h, 0]$. I is continuous linear operator, so it mapps compact sets of $C[-h, 0]$ into compact sets of $L^2[-h, 0]$. The game will be considered in $L^2[-h, 0]$. The problem of pursuit is stated as at the beginning of paragraph 2. For this problem the following theorem is true.

THEOREM 12. If there exists an optimal capture time T_o and corresponding it pair of optimal controls u_o, v_o then there exists a linear continuous functional $x_o' \in L^2[-h, 0]$ such that the maximum principle holds:

$$\max_{u \in U_s} \langle x_o', Y_1(\beta, T_o) u(\beta)\rangle = \langle x_o', Y(\beta, T_o) u_o(\beta)\rangle, \tag{5.16}$$

$$\max_{v \in V_s} \langle x_o', Y_2(\beta, T_o) v(\beta)\rangle = \langle x_o', Y(\beta, T_o) v_o(\beta)\rangle. \tag{5.17}$$

Remark 2. The maximum principle proved in Theorem 2 is true as well for finite as for infinite dimensional spaces. Assumptions in Theorem 2 and further demand from Ω to contain a non-empty interior. This follows from application of the theorem on separation of two convex sets (where it is enough if one of them contains an interior). In the case when, the attainable set of $F_t(U_s)$ has interior, the assumption about interior of Ω can be omitted. For details connected with this remark regarding to maximum principle for optimal control in Banach spaces see [11] and [14].

References

1. BANKS H. T., KENT G. A., Control of functional differential equation of retarded and neutral type to target sets in function space. *SIAM J. Contr.* (1972) 567—593.
2. CASTAING Ch., Quelques resultats de compacité liés à 'integration. *C. R. Acad. Sci.* Paris **268** (1969) 327—329.
3. DAY M. M., Normed linear spaces. 1973.
4. DEBREU W. G., Integration of correspondences. V Berkeley Symp. on Mathematical Statistical Problems 1965, VII p. 351—372.
5. PONTRIAGIN L. S., BOLŁTIANSKI W. G., GRAMKRELIDZE R. W., MISHCHENKO, Matiematiczeskaja tieorija optimalnych processow. Moskwa 1969.
6. KIRIŁOWA F. M., The application of functional analysis to problem of pursuit. In: Mathematical Theory of Optimal Control 1967.
7. KÖTHE G., Topological linear spaces. 1969.
8. ŁAWRUK B., ROLEWICZ S., The minimum time control problem for linear parabolic equation controlled by boundary condition. *Bull. Acad. Pol. Sc.* **16**, 6 (1968) 489—493.

9. Oguztorelli M. N., Time lag control systems. New York 1966.
10. Rockafellar T., Convex integral functionals and duality. In: Contribution to nonlinear functional analysis. New York 1972.
11. Rolewicz S., On general theory of linear systems. In: Beitrage zur Analysis (to appear).
12. Sobolewski P. E., On equation of parabolic type in Banach space. *Tr. Mosk. Mat. Obsc.* **10** (1961) 297—350.
13. Toporowska G., The minimal norm problem and Pontriagin max principle for Banach spaces. II. *Studia Math.* **35** (1970) 105—110.
14. Wojtaszczyk P., A theorem on convex sets relate to the abstract Pontriagin's maximum principle. *Bull. Acad. Pol. Sc.* **21** (1973) 43—95.

Zasada maksimum dla różniczkowych gier pościgu w pętli otwartej

Rozważono zagadnienie różniczkowych gier pościgu w pętli otwartej zdefiniowanych dla przestrzeni Banacha. Wyznaczono warunki konieczne optymalności sterowań w postaci zasady maksimum. Podano przykłady zastosowań otrzymanych wyników dla równań różniczkowych cząstkowych i równań różniczkowych funkcyjnych.

Принцип максимума для дифференциальных игр преследования в разомкнутом контуре

Рассматривается вопрос дифференциальных игр преследования в разомкнутом контуре, определенных в банаховом пространстве. Приводятся необходимые условия оптимальности управлений в виде принципа максимума. Даны примеры применений полученных результатов для дифференциальных уравнений с частными производными и функциональных дифференциальных уравнений.