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On parametric optimal control for weak solutions of abstract linear parabolic equations

by

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Am optimal control problem for a system described by abstract linear parabolic equation (state equation) is investigated.
Coefficients of elliptic operator of this equation depend on control function.
We consider weak solutions of the state equation. These solutions belong to some functional space, called the space of observation. The cost functional is defined on the space of observation and depends on control by means of solution of state equation. It is to be minimize on a given set of admissible controls.
Two different spaces of observation are investigated. Sufficient conditions for existence of an optimal control and necessary conditions of optimality are formulated.
Using so called generalized adjoint state equation a simple formula for the gradient of the cost functional is obtained. Some examples of parametric optimization of concrete boundary value problems for second order partial differential equations of parabolic type are presented.

0. Introduction

Let us consider an abstract linear prabolic equation (the state equation) of the form

$$\left(\frac{dy}{dt}(t), z\right)_{V'V} + a_\theta(t; y(t), z) = (f(t), z)_{V'V} \qquad \forall z \in V$$
$$y(0) = y_0, \tag{0.1}$$

where V is a Hilbert space, V' denotes its dual and $a_\theta(t; y, z)$ is a family of bilinear forms defined on V, which depend on parameter θ.

Let us suppose that there are given:

(i) U_1 — Banach space of parameters θ;
(ii) U — Hilbert space of controls u;
(iii) set of admissible controls $U_{ad} \subset U$;
(iv) linear mapping

$$L: U \supset U_{ad} \ni u \mapsto \theta \in U_1. \tag{0.2}$$

The object of this paper is to study the following optimization problem:

$$\text{find } \hat{u} \in U_{ad} \text{ such that}$$
$$I(y_{\hat{\theta}}) \leqslant I(y_\theta) \tag{0.3}$$
$$\hat{\theta} = L\hat{u}, \quad \theta = Lu, \quad \forall u \in U_{ad},$$

where y_θ denotes the solution of the state equation (0.1) with $\theta = Lu$, $u \in U_{ad}$.

$I(\cdot)$ is a given continuous functional defined on Hilbert space X called the space of observation.

Problem (0.3) is well posed if the state trajectory has the following property

$$y_\theta \in X, \quad \theta = Lu, \quad \forall u \in U_{ad}.$$

In the paper we discuss two kinds of observation.

In section 1 there are presented some basic results from theory of abstract parabolic equations which will be used in next sections.

In section 2 we consider simpler case of observation of the state trajectory in the space $W(0, T)$. We give sufficient conditions for existence of an optimal parameter $\hat{\theta} = L\hat{u}$ and for differentiability of the cost functional $J(u)$. Then there are given necessary conditions of optimality for the problem (0.3).

In section 3 we consider the observation in the spece $W^1(0, T)$ (in particular $W^1(0, T) \subset C(0, T; V)$ with continuous injection) and we prove similar results as in section 2.

In this case a simple formula for gradient of cost functional $J(u)$ is given.

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1. Abstract parabolic equation

In this section we define an abstract parabolic problem [1] and then we shortly recall sufficient conditions for existence and uniqueness of the solutions of such a problems.

We recall also some regularity results [2] which we will need in the sequel.

Let V and H be two Hilbert spaces with

$$V \subset H, \quad V \text{ is dense in } H \text{ with continuous injection.} \tag{1.1}$$

Identifying H with its dual and denoting by V' the dual of V we have $V \subset H \subset V'$ each space being dense in the following one.

$L^2(0, T; V)$ denotes the Hilbert space of (classes of) functions

$$y(\cdot): [0, T] \mapsto V$$

strongly measurable with the norm

$$\|y\|^2_{L^2(0,T;V)} = \int_0^T \|y(t)\|^2_V \, dt.$$

In the same way are defined Hilbert spaces $L^2(0, T; H)$ and $L^2(0, T; V')$.

We recall [1] that we can identify function $y \in L^2(0, T; V)$ with distribution $y \in \mathscr{D}'(]0, T[; V) \subset \mathscr{D}'(]0, T[; V')$ and hence there is defined the derivative in the sense of distribution

$$\frac{dy}{dt} \in \mathscr{D}'(]0, T[; V').$$

We set

$$W(0, T) = \left\{ y \in L^2(0, T; V) \,\middle|\, \frac{dy}{dt} \in L^2(0, T; V') \right\}$$

provided with the norm

$$\|y\|_{W(0,T)} = \left(\|y\|^2_{L^2(0,T;V)} + \left\| \frac{dy}{dt} \right\|^2_{L^2(0,T;V')} \right)^{1/2}.$$

$W(0, T)$ is a Hilbert space [1].

If $y \in W(0, T)$ then [1] after a possible modification on a set of measure zero $y \in C(0, T; H)$ and inclusion $W(0, T) \subset C(0, T; H)$ is continuous.

As it was mentioned in Introduction by $\theta \in U_1$ we denote functional parameter which depends on control $u \in U_{ad}$ by linear mapping (0.2). We assume that there is given family of forms

$$\{a_\theta(t; y, v)\}_{\theta \in \mathscr{U}}, \qquad \mathscr{U} = LU_{ad} \subset U, \tag{1.2}$$

Such that:

$$\text{for all} \quad t \in [0, T] \quad \text{the form} \quad a_\theta(t; y, v) \tag{1.3}$$

is bilinear with respect to $y, v \in V$ and uniformly bounded, i.e.

$$|a_\theta(t; y, v)| \leqslant M \|y\|_V \|v\|_V, \qquad \forall y, v \in V,$$

for all $y, v \in V$ function

$$t \mapsto a_\theta(t, y, v) \tag{1.4}$$

is measurable there exists a number $\lambda \geqslant 0$ such that

$$a_\theta(t; y, y) + \lambda \|y\|_H \geqslant \alpha \|y\|^2_V, \qquad \alpha > 0, \quad \forall y \in V, \quad \forall t \in [0, T]. \tag{1.5}$$

If we can choose $\lambda = 0$ then family (1.2) is said to be coercive (V — eliptic).

Remark. In this section we denote family (1.2) by $a_\theta(t; y, v)$ instead of $a_{Lu}(t; y, v)$ $u \in U_{ad}$.

In the next sections we will write $a_u(t; y, z)$ to denote $a_{Lu}(t; y, z)$.

Example 1. Let Ω be an open subset of R with smooth boundary $\Gamma = \partial\Omega$. We set

$$V = H^1(\Omega), \qquad H = L^2(\Omega)$$

and bilinear form

$$a_\theta(t; y, z) = \sum_{i,j=1}^{n} \int_{\Omega} c_{ij}(x, t, \theta) \frac{\partial y}{\partial x_i} \frac{\partial z}{\partial x_j} d\Omega + \\ + \int_{\Gamma} b(x, t, \theta)\, yz\, d\Gamma, \qquad \forall y, z \in H^1(\Omega), \tag{1.6}$$

where $\theta = \theta(x, t)$ is a functional parameter. We assume:

(i) $\theta = L^\infty(Q)$ and there exists trace $\theta|_\Sigma \in L^\infty(\Sigma)$ where
$0 \leqslant \theta(x, t) \leqslant M$ a,e. in $Q = \Omega \times]0, T[$
$0 \leqslant \theta|_\Sigma(x, t) \leqslant M$ a,e. in $\Sigma = \Gamma \times]0, T[$;

(ii) $c_{ij}(\cdot, \cdot, r) \in L^\infty(Q)$, $\forall r \in [0, M]$
$b(\cdot, \cdot, r) \in L^\infty(\Sigma)$, $\forall r \in [0, M]$
$c_{ij}(x, t, \cdot) \in C(0, M)$ a, e. in Q
$b(x, t, \cdot) \in C(0, M)$ a, e. in Σ;

(iii) Furthermore there exist constants $M_1, M_2 < \infty$ such that for all $r, s \in [0, M]$

$$|c_{ij}(x, t, r) - a_{ij}(x, t, s)| \leqslant M_1 |r - s| \quad \text{a,e. in } Q,$$
$$|b(x, t, r) - b(x, t, s)| \leqslant M_2 |r - s| \quad \text{a.e. in } \Sigma.$$

Remark. (i) (ii) (iii) imply the following estimation which will be useful in the sequel:

$$|a_{\theta 1}(t; y, z) - a_{\theta 2}(t; y, z)| \leqslant M_1 \|\theta_1 - \theta_2\|_{L^\infty(Q)} \|y\|_{H^1(\Omega)} \|z\|_{H^1(\Omega)} + \\ + M_2 \|\theta_1|_\Sigma - \theta_2|_\Sigma\|_{L^\infty(\Sigma)} \|y|_\Gamma\|_{L^2(\Gamma)} \|z|_\Gamma\|_{L^2(\Gamma)} \quad \text{a.e. in } [0, T].$$

Moreover, if we assume

(iv) $M_3 \sum_{i=1}^{n} \xi_i^2 \geqslant \sum_{i,j=1}^{n} c_{ij}(x, t, r)\, \xi_i \xi_j \geqslant \alpha \sum_{i=1}^{n} \xi_i^2, \quad \alpha > 0$
$\forall (x, t) \in Q, \quad \forall r \in [0, M], \quad \forall \xi = (\xi_1, \ldots, \xi_n) \in R^n$;

(v) $0 \leqslant b(x, t, r) \leqslant M_3$
$\forall (x, t) \in \Sigma, \quad \forall r \in [0, M]$,

then family (1.6) satisfies [4] (1.2)—(1.5).

Let us consider an abstract parabolic equation (state equation) of the form

$$\left(\frac{dy}{dt}(t), v\right)_{V'V} + a_\theta(t, y, v) = (f(t), v)_{V'V}, \qquad \forall v \in V; \tag{1.7}$$

$$y(0) = y_0; \tag{1.8}$$

where $y_0 \in H, f \in L^2(0, T; V')$ are given.

We denote by $y = y_\theta(\cdot) \in W(0, T)$ the state trajectory, that is the solution (if it exists) of the problem (1.7), (1.8).

THEOREM 1. [1]. Let us assume (1.3)—(1.5). Then for each parameter $\theta \in \mathcal{U}$ there exists the unique solution y_θ of the problem (1.7), (1.8) and the following a priori estimation holds

$$\|y_\theta\|_{W(0,T)} \leqslant C\left(\|y_0\|_H + \|f\|_{L^2(0,T;V')}\right), \tag{1.9}$$

where constant C depends only on the constants M, α, λ from (1.3), (1.5) and on T.

Remark. From (1.9) we also have

$$\|y_\theta\|_{C(0,T;H)} \leqslant C\left(\|y_0\|_H + \|f\|_{L(0,T;V')}\right)$$

hence the value $y_\theta(T)$ is a well defined element of the space H, however in general $y_\theta(T) \notin V$.

For our purpose interesting is the case where state trajectory is more regular, that is

$$y_\theta \in C(0, T; V), \qquad \forall \theta \in \mathcal{U} \tag{1.10}$$

but from the theorem 1 we cannot deduce (1.10).

Now we investigate the problem of sufficient conditions under which (1.10) holds.

We assume

(i) family (1.2) is symetric

$$a_\theta(t; y\ v) = a_\theta(t; v\ y), \qquad \forall y, v \in V; \tag{1.11}$$

(ii) for all $y, v \in V$, and for all $\theta \in \mathcal{U}$ the function

$$t \mapsto a_\theta(t; y, v) \tag{1.12}$$

belongs to the space $C^1([0, T]) = C^1(0, T)$.

Remark. From (1.3) we can deduce that the family (1.2) generates the family of bounded operators

$$A_\theta(\cdot) \in L^\infty(0, T; \mathcal{L}(V, V')), \qquad \theta \in \mathcal{U}, \tag{1.13}$$

which is defined in the following manner:

$$a_\theta(t; y, v) = (A_\theta(t)\, y, v)_{V'V}, \qquad \forall y, v \in V. \tag{1.14}$$

If we assume (1.12) then the family (1.13) is more regular:

$$A_\theta(\cdot) \in C^1(0, T); \mathcal{L}(V, V')), \qquad \forall \theta \in \mathcal{U}.$$

We use the so called abstract Green formula [5], to obtain another representation of the form (1.2).

Let us assume, that there is given a Hilbert space S, and a linear bounded operator $\gamma \in \mathcal{L}(V, S')$. We denote $V_0 = \ker \gamma$, it is a Hilbert space with topology induced by V.

Furthermore we assume that the space V_0 is dense in the space H, and operator γ is onto S. Bilinear form $a_\theta(t; y, v)$ is continuous on $V \times V$ hence it is continuous on $V\times V_0$. By $A_\theta(t)$ we denote a linear, unbounded operator in H, with domain $\mathscr{D}(A_\theta(t))$, which is defined in the following way:

(i) $y\in\mathscr{D}(A_\theta(t))$ if the mapping $V_0\ni v\mapsto a_\theta(t;y,z)$ is continuous in topology of the space H.

(ii) $A_\theta(t)\,y=z\in H$ if $a_\theta(t;y,v)=(z,v)_H$, $\forall v\in V_0$.

We can represent family (1.2) in the following way

$$a_\theta(t;y,v)=\big(A_\theta(t)\,y,v\big)_H,\qquad \forall v\in V_0, \tag{1.15}$$

$\mathscr{D}(A_\theta(t))$ is a Hilbert space provided with the graph norm

$$\|y\|_{\mathscr{D}(A_\theta(t))}=\big(\|y\|_V^2+\|A_\theta(t)\,y\|_H^2\big)^{1/2}. \tag{1.16}$$

THEOREM 2. There exists the unique linear operator

$$\sigma_\theta(t)\in\mathscr{L}\big(\mathscr{D}(A_\theta(t),S')\big) \tag{1.17}$$

such that the following abstract Green's formula holds:

$$a_\theta(t,y,v)=\big(A_\theta(t)\,y,v\big)_H+\big(\sigma_\theta(t)\,y,\gamma v\big)_{S'S}\qquad \forall y\in\mathscr{D}\big(A_\theta(t)\big),\ \forall v\in V. \tag{1.18}$$

Proof is given in [5] (p.174).

Let us introduce the following:

Condition (A1). For any given θ Hilbert space $\mathscr{D}(A_\theta(t))$ does not depend on $t\in[0,T]$.

If the above Condition is satisfied then from (1.12) we have

$$A_\theta(\cdot)\in C^1\Big(0,T;\mathscr{L}\big(\mathscr{D}(A_\theta(t));H\big)\Big); \tag{1.19}$$

$$\sigma_\theta(\cdot)\in C^1\Big(0,T;\mathscr{L}\big(\mathscr{D}(A_\theta(t));S'\big)\Big). \tag{1.20}$$

Indeed, for any $v\in V_0$ we get

$$a_\theta(t;y,v)=\big(A_\theta(t)\,y,v\big)_H,\qquad \forall y\in\mathscr{D}(A_\theta),\qquad \forall v\in V_0.$$

Hence (1.12) and the fact that V_0 is dense in H imply (1.19). Then

$$[0,T]\ni t\mapsto\big(\sigma_\theta(\cdot)\,y,v\big)_{S'S}=a_\theta(\cdot)\,y,v\big)-\big(A(\cdot)\,y,v\big)_H\in C^1(0,T)$$

$$\forall y\in\mathscr{D}(A_\theta),\qquad v\in V,$$

whence (1.20) follows.

Let us introduce

$$X_\theta=\left\{y\in L^2(0,T;V)\;\middle|\;\frac{dy}{dt}\in L^2(0,T;H),\ A_\theta y\in L^2(0,T;H)\right\} \tag{1.21}$$

with the norm

$$\|y\|_{X_\theta}=\left(\|y\|^2_{L^2(0,T;V)}+\left\|\frac{dy}{dt}\right\|^2_{L^2(0,T;H)}+\|A_\theta y\|_{L^2(0,T;H)}\right)^{1/2}. \tag{1.22}$$

X_θ is a Hilbert space.

We introduce also a Hilbert space Y_θ defined as the set

$$Y_\theta=\{\varphi \in L^2(0, T; S') | \exists y \in X_\theta \text{ such that } \varphi=\sigma_\theta y\} \tag{1.23}$$

with the norm

$$\|\varphi\|_{Y_\theta}=\Big(\inf_{y \in X_\theta} \|y\|^2_{X_\theta}+\|\varphi\|^2_{L^2(0,T;S')}\Big)^{1/2} \tag{1.24}$$
$$\sigma_\theta y=\varphi .$$

Remark. For given $\varphi \in Y_\theta$ there exists $y_\varphi \in X_\theta$ such that $\sigma_\theta y_\varphi=\varphi$ and $\|\varphi\|_{Y_\theta} \geqslant \|y_\varphi\|_{X_\theta}$.

We define the restriction $\tilde{A}_\theta(t)$ of the operator $A_\theta(t)$ to the domain

$$\mathscr{D}(\tilde{A}_\theta(t))=\{y \in \mathscr{D}(A_\theta(t)) | \sigma_\theta(t) y=0\}. \tag{1.25}$$

$\mathscr{D}(\tilde{A}_\theta(t))$ is a Hilbert space provided with the graph norm.

We have the following regularity theorem

Theorem 3. [2]. Let us assume (A1), (1.11), (1.12), (1.13), (1.14) and (1.5) with $\lambda=0$. Then the problem

$$\frac{dy}{dt}+\tilde{A}_\theta(t) y=f, \tag{1.26}$$

$$y(0)=y_0 \tag{1.27}$$

with $y_0 \in v$ and $f \in L^2(0, T; H)$, has the unique solution y_θ such that

$$y_\theta \in L^2(0, T; \mathscr{D}(\tilde{A}_\theta(t))), \tag{1.28}$$

$$\frac{dy_\theta}{dt} \in L^2(0, T; H), \tag{1.29}$$

and the following a priori estimation holds

$$\|y_\theta\|_{X_\theta} \leqslant C(\|y_0\|_V+\|f\|_{L^2(0,T;H)}), \tag{1.30}$$

where C is a constant which does not depend on y_0, f.

Remark. Constant C in estimation (1.30) $\mathscr{L}(V, V')$ does not depend on $\theta \in \mathscr{U}$ if the family (1.13) is bounded in the space $C^1(0, T; \mathscr{L}(V, V'))$.

Corollary 1. Under the same assumptions as in Theorem 3, there exists the unique solution $y_\theta \in X_\theta$ of the following abstract parabolic problem:

$$\left(\frac{dy}{dt}, v\right)_{V'V}+a_\theta(t; y, v)=(f, v)_{V'V}+(\varphi, \gamma v)_{S'S}, \quad \forall v \in V, \quad y(0)=y_0, \tag{1.31}$$

where $y_0 \in V, f \in L^2(0, T; H)$, $\varphi \in Y_\theta$ are given.

Moreover the following a priori estimation holds

$$\|y_\theta\|_{X_\theta} \leqslant C(\|y_0\|_V+\|f\|_{L^2(0,T;H)}+\|\varphi\|_{Y_\theta}), \tag{1.32}$$

where C is a constant which does not depend on y_0, f and φ.

Remark. By (1.21), (1.22) and Condition (A1) for given θ we have [1]:

$$X_\theta \subset C(0, T: V)$$

with continuous injection.

In particular it means that the following estimation holds:

$$\|y|_{t=0}\|_V \leqslant C \|y\|_{X_\theta}, \quad \forall y \in X_\theta .$$

Proof. For each $\varphi \in Y_\theta$ there exists $y_\varphi \in X_\theta$ such that

$$\sigma_\theta y_\varphi = \varphi$$

and

$$\|y_\varphi\|_{X_\theta} \leqslant \|\varphi\|_{Y_\theta}$$

hence

$$\|y_\varphi|_{t=0}\|_V \leqslant C \|y_\varphi\|_{X_\theta} \leqslant C' \|\varphi\|_{Y_\theta} .$$

We define $\tilde{y}_\theta \in X_\theta$ as the solution of the following problem

$$\frac{d\tilde{y}}{dt} + \tilde{A}_\theta(t) \tilde{y} = f,$$

$$\tilde{y}(0) = \tilde{y}_0 ,$$

where

$$\tilde{y}_0 = y_0 - y_\varphi(0) \in V,$$

$$\tilde{f} = f - \frac{dy_\varphi}{dt} - A_\theta(t) y_\varphi \in L^2(0, T; H) .$$

But

$$\|\tilde{y}\|_{X_\theta} \leqslant C_1 \left(\|y_0\|_V + \|\tilde{f}\|_{L^2(0,T;H)}\right),$$

hence for $y_\theta = \tilde{y} + y_\varphi$ we obtain

$$\|y_\theta\|_{X_\theta} \leqslant \|\tilde{y}\|_{X_\theta} + \|y_\varphi\|_{X_\theta} \leqslant C_2 \left(\|y_0\|_V + \|f\|_{L^2(0,T;H)} + \|\varphi\|_{Y_\theta}\right).$$

Example 2. Let us put

$$V = H^1(\Omega), \; H = L^2(\Omega), \; V_0 = H_0^1(\Omega),$$

$$S = H^{1/2}(\Gamma), \; S = H^{-1/2}(\Gamma) .$$

Consider bilinear from which was introduced in Example 1

$$a_\theta(t; y, z) = \sum_{i,j=1}^{n} \int_\Omega c_{ij}(x, t, \theta) \frac{\partial y}{\partial x_i} \frac{\partial z}{\partial x_j} d\Omega + \int_\Gamma b(x, t, \theta) yz \, d\Gamma .$$

First let us formulate sufficient conditions under which

$$[0, T] \ni t \mapsto a_\theta(t; y, z) \in C^1(0, T), \forall y, z \in H^1(\Omega) .$$

We have

(i) $c_{ij}(x, \cdot, r) \in C^1(0, T), \quad \forall x \in \Omega, \quad \forall r \in [0, M],$

$c_{ij}(x, t, \cdot) \in C^1(0, M), \quad \forall x \in \Omega, \quad \forall t \in [0, T],$

$\theta(x, \cdot) \in C^1(0, T), \quad \forall x \in \Omega;$

(ii) $b(x, \cdot, r) \in C^1(0, T), \quad \forall x \in \Gamma, \quad \forall r \in [0, M],$

$b(x, t, \cdot) \in C^1(0, M), \quad \forall x \in \Gamma, \quad \forall t \in [0, T],$

$\theta(x, \cdot) \in C^1(0, T), \quad \forall x \in \Gamma.$

For $y \in H^1(\Omega)$ such that

$$\sum_{i,j=1}^{n} \frac{\partial}{\partial x_j}\left(c_{ij}(x, t, \theta(x, t)) \frac{\partial y}{\partial x_i}\right) \in L^2(\Omega)$$

we obtain [5] the following representation

$$a_\theta(t; y, z) = (A_\theta(t) y, z)_{L^2(\Omega)} + ((\sigma_\theta(t) y, \gamma z)),$$

where

$((\cdot, \cdot))$ denotes scalar product between $H^{-1/2}(\Gamma)$ and $H^{1/2}(\Gamma)$

$$A_\theta(t) y = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_j}\left(c_{ij}(x, t, \theta(x, t)) \frac{\partial y}{\partial x_i}\right),$$

$$\sigma_\theta(t) y = \left(\frac{\partial y}{\partial \eta_A} + b(x, t, \theta(x, t))\right) y|_\Gamma,$$

$$\gamma y = y|_\Gamma,$$

$$\frac{\partial y}{\partial \eta_A} = \sum_{i,j=1}^{n} c_{ij}(x, t, \theta(x, t)) \frac{\partial y}{\partial x_i} \cos(\vec{\eta}, x_j)$$

$\vec{\eta}$ — is a unit vector orthogonal to and directed outside of Ω.

2. Observation in $W(O, T)$

Let U be Banach space of control functions. We denote by $U_{ad} \subset U$ a given set of admissible controls. We assume that there is given a family of bilinear forms

$$a_u(t; y, z): U_{ad} \times [0, T] \times V \times V \mapsto L^\infty(0, T) \tag{2.1}$$

which are continuous

$$|a_u(t; y, z)| \leqslant M \|y\|_V \|z\|_V, \quad \forall y, z \in V, \quad \forall u \in U_{ad}, \quad \forall t \in [0, T] \tag{2.2}$$

and V — elliptic,

$$a_u(t; y, y) \geqslant \alpha \|y\|_V^2, \quad \alpha > 0, \quad \forall y \in V, \quad \forall t \in [0, T], \quad \forall u \in U_{ad}, \tag{2.3}$$

where α, M are constants which do not depend on $u \in U_{ad}$.

We denote by $y_u \in W(0, T)$ the state trajectory, i.e. the unique solution of the abstract parabolic equation:

$$\left(\frac{dy}{dt}, z\right)_{V'V} + a_u(t; y, z) = (f(t), z)_{V'V}, \qquad \forall z \in V, \quad y(0) = y_0, \tag{2.4}$$

where $y_0 \in H, f(\cdot) \in L^2(0, T; V')$ are given.

THEOREM 4. If for all $u, v \in U_{ad} \subset U$

$$|a_u(t; y, z) - a_v(t; y, z)| \leqslant M \, \|y\|_V \, \|z\|_V \, \|u - v\|_U \tag{2.5}$$

then the state equation (2.4) generates continuous mapping

$$U \supset U_{ad} \ni u \mapsto y_u \in W(0, T). \tag{2.6}$$

Proof. By $y_1 = y_{u1}, y_2 = y_{u2}$, we denote state trajectories for controls $u_1, u_2 \in U_{ad}$. Let

$$\tilde{y} = y_1 - y_2 = y_{u1} - y_{u2}$$

then

$$\left(\frac{d\tilde{y}}{dt}, z\right)_{V'V} + a_{u_1}(t; \tilde{y}, z) = a_{u_2}(t; y_2, z) - a_{u_1}(t; y_2, z), \qquad \forall z \in V \tag{2.7}$$

$$\tilde{y}(0) = 0. \tag{2.8}$$

The right hand side of (2.7) defines linear functional $\tilde{f}(\cdot) \in L^2(0, T; V')$ where

$$(\tilde{f}(t), z)_{V'V} = a_{u_2}(t; y_2, z) - a_{u_1}(t; y_2, z)$$

which is bounded by assumption (2.5)

$$\|\tilde{f}\|_{L^2(0,T; V)} \leqslant M \, \|y_2\|_{L^2(0,T; V)} \, \|u_1 - u_2\|_U.$$

Using (1.9) we obtain for the problem (2.5), (2.8)

$$\|\tilde{y}\|_{W(0,T)} \leqslant C \, \|\tilde{f}\|_{L^2(0,T; V')} \leqslant C(y_2) \, \|u_1 - u_2\|_U$$

hence the mapping (2.4) is continuous.

Let us assume, that there is given functional $I(\cdot)$ which is continuous in the topology of the space $W(0, T)$.

We define on the set U_{ad} cost functional

$$J(u) = I(y_u), \qquad u \in U_{ad}.$$

COROLLARY 2. Let us assume that the set $U_{ad} \subset U$ is compact. Then there exists an optimal control $\hat{u} \in U_{ad}$ such that

$$J(\hat{u}) \leqslant J(u), \qquad \forall u \in U_{ad}.$$

Proof. By Theorem 4 functional $J(u)$ is continuous on the set U_{ad}, which is a compact subset of the space U. Hence by Weierstrass theorem Corollary follows.

Let us consider problem of differentiability of cost functional $J(u)$.

We denote by A_u, $u \in U_{ad}$ a bounded linear operator

$$A_u \in \mathscr{L}(\mathscr{V}, \mathscr{V}'), \quad \text{where} \quad \mathscr{V} = L^2(0, T; V), \ \mathscr{V}' = L^2(0, T; V') \tag{2.9}$$

which is defined by bilinear form on $\mathscr{V}$

$$\mathscr{A}_u(y, z) = \int_0^T a_u(t; y(t), z(t))\, dt, \qquad \forall y, z \in \mathscr{V}. \tag{2.10}$$

Lemma. Let us assume that at every point $\bar{u} \in U_{ad}$ there exist:

(i) form $\mathscr{B}_{\bar{u}}(v; y, z)$ which is linear with respect to $v \in U$, $y, z \in \mathscr{V}$ and bounded

$$|\mathscr{B}_u(v; y, z)| \leqslant M(\bar{u}) \|v\|_U \|y\|_{\mathscr{V}} \|z\|_{\mathscr{V}} \tag{2.11}$$

(ii) bilinear form $\mathscr{R}_{\bar{u}}(v; y, z)$ such that

$$|\mathscr{R}_{\bar{u}}(v; y, z)| = o(\|v\|_U) \tag{2.12}$$

for any $y, z \in \mathscr{V}$.

Moreover

(iii)
$$\mathscr{A}_{\bar{u}+\delta u}(y, z) - \mathscr{A}_{\bar{u}}(y, z) = \mathscr{B}_{\bar{u}}(\delta u; y, z) + \mathscr{R}_{\bar{u}}(\delta u; y, z) \tag{2.13}$$

in some open neighbourhood of the point $\bar{u} \in U$ for any $y, z \in \mathscr{V}$.

Then the mapping

$$U_{ad} \ni u \mapsto A_u \in \mathscr{L}(\mathscr{V}, \mathscr{V}') \tag{2.14}$$

is differentiable in the strong sense and its differential $\left\langle \frac{\partial A\bar{u}}{\partial u}; \delta u \right\rangle$ is determined by the form $\mathscr{B}_{\bar{u}}(\delta u; y, z)$:

$$\mathscr{B}_{\bar{u}}(\delta u; y, z) = \left(\left\langle \frac{\partial A_{\bar{u}}}{\partial u}; \delta u \right\rangle y, z, \right)_{\mathscr{V}'\mathscr{V}}, \qquad \forall \delta u \in U, \quad \forall y, z \in \mathscr{V}. \tag{2.15}$$

Proof. By assumption (i) there exists a linear mapping

$$U \ni v \mapsto B(v) \in \mathscr{L}(\mathscr{V}, \mathscr{V}')$$

such that

$$\mathscr{B}_{\bar{u}}(v; y, z) = (B(v)\, y, z)_{\mathscr{V}'\mathscr{V}}, \qquad \forall v \in U, \quad \forall y, z \in \mathscr{V}.$$

Hence from (iii) we obtain

$$\big((A_{\bar{u}+\delta u} - A_{\bar{u}} - B(\delta u))\, y, z\big)_{\mathscr{V}'\mathscr{V}} = \mathscr{R}_{\bar{u}}(\delta u; y, z) \qquad \forall y, z \in \mathscr{V}$$

then by (ii)

$$\sup_{\substack{\|y\|_{\mathscr{V}} \leqslant 1 \\ \|z\|_{\mathscr{V}} \leqslant 1}} \left| \big((A_{\bar{u}+\delta u} - A_u - B(\delta u))\, y, z\big)_{\mathscr{V}'\mathscr{V}} \right| = o(\|\delta u\|_U)$$

whence

$$B(\delta u) = \left\langle \frac{\partial A_{\bar{u}}}{\partial u}; \delta u \right\rangle.$$

To determine the gradient

$$\left\langle \frac{dJ(\bar{u})}{du}; \delta u \right\rangle, \quad \bar{u} \in U_{ad}, \quad \delta u \in U \tag{2.16}$$

of the cost functional $J(u)$ we introduce Lagrangian of the form:

$$\mathscr{L}(y, u, p) = I(y) + \int_0^T \left(\frac{dy}{dt} + A_u(t)\, y - f, p \right)_{V'V} dt \tag{2.17}$$

which is well defined for any

$$y \in W(0, T), \quad u \in U_{ad}, \quad p \in L^2(0, T; V).$$

If $y = y_u$ is the state trajectory, then

$$\mathscr{L}(y_u, u, p) = I(y_u), \quad \forall p \in L^2(0, T, V). \tag{2.18}$$

Let us assume that functional $I(\cdot)$ is differentiable, that is for every $\bar{u} \in U_{ad}$ where exists the gradient

$$\frac{dI}{dy}(y_u) \in (W(0, T))'. \tag{2.19}$$

Example. For

$$I_1(y) = \frac{1}{2} \|y(T) - z\|_H^2, \quad y \in W(0, T), \quad z \in H$$

we obtain

$$\left\langle \frac{dI_1}{dy}(y_{\bar{u}}); \varphi \right\rangle = (y_{\bar{u}}(T) - z, \varphi(T))_H, \quad \forall \varphi \in W(0, T).$$

Since [1] we have $W(0, T) \subset C(0, T; H)$ and injection is continuous, then

$$\frac{dI_1}{dy}(y_{\bar{u}}) \in (W(0, T))'.$$

For given $\bar{u} \in U_{ad}$ we define the adjoint state $\bar{p} = p_{\bar{u}}$ as the solution of the adjoint equation

$$\int_0^T \left(\frac{d\varphi}{dt} + A_{\bar{u}}(t)\, \varphi, p \right)_{V'V} dt = -\left\langle \frac{dI}{dy}(y_{\bar{u}}); \varphi \right\rangle \quad \forall \varphi \in W_0(0, T) \tag{2.20}$$

where

$$W_0(0, T) = \{\varphi \in W(0, T) | \varphi|_{t=0} = 0\}.$$

Example. For functional $I_1(\cdot)$ we obtain adjoint state equation of the form:

$$-\left(\frac{d\bar{p}}{dt}, v \right)_{V'V} + a_{\bar{u}}(t; v, \bar{p}) = 0, \quad \forall v \in V, \quad \bar{p}(T) = -y_{\bar{u}}(T) + z \tag{2.21}$$

which has unique solution $\bar{p} \in W(0, T)$.

To obtain existence of the solutions to problem (2.20) we will use some general results given in [7], which can be formulated as follows:

Let W be a Hilbert space and Φ a normed linear subspace of W. We denote by Φ' the dual space to Φ.

Let be given a bilinear form

$$E: W \times \Phi \mapsto R \tag{2.22}$$

which is continuous with resepct to the first argument. We define a mapping

$$M: \Phi \mapsto M[\Phi] \subset W \tag{2.23}$$

by the equality

$$(w, M\varphi)_W = E(w, \varphi), \quad \forall w \in W, \quad \forall \varphi \in \Phi. \tag{2.24}$$

Let L be any given element of space Φ'.

THEOREM 5 [7]. Suppose that there exists a real constant $C>0$ such that

$$\|M\varphi\|_W \geqslant C\,\|\varphi\|_\Phi, \quad \forall \varphi \in \Phi \tag{2.25}$$

then there exists a solution w_L to the variational problem

$$E(w_L, \varphi) = (L, \varphi)_{\Phi'\Phi}, \quad \forall \varphi \in \Phi \tag{2.26}$$

such that

$$\|w_L\|_W \leqslant \frac{1}{C}\,\|L\|_{\Phi'}. \tag{2.27}$$

The solution w_L of (2.26) is uniquely determined if the set $M[\Phi]$ is dense in W. Proof is given in [7].

In the case of the problem (2.20) we define:

(i) $W = L^2(0, T; V')$,

(ii) $\Phi = W_0(0, T)$,

(iii) functional L is of the form

$$(L, \varphi)_{\Phi'\Phi} = \left\langle \frac{dI}{dy}(y_{\bar{u}});\ \varphi \right\rangle, \tag{2.28}$$

(iv) bilinear form E has the form

$$E(w, \varphi) = \left(\frac{d\varphi}{dt} + A_{\bar{u}}\,\varphi,\ w \right)_{L^2(0,T;V')}. \tag{2.29}$$

In the case Theorem 1 implies that norms

$$\|\varphi\|_{W_0(0,T)} \quad \text{and} \quad \left\| \frac{d\varphi}{dt} + A_{\bar{u}}\,\varphi \right\|_{L^2(0,T;V')}$$

are equivalent, hence from Theorem 5 it follows that there exists the unique solution w_L of the problem

$$E(w_L, \varphi) = \left(\frac{d\varphi}{dt} + A_{\bar{u}}\,\varphi,\, w_L\right)_{L^2(0,T;V')} = -\left\langle \frac{dI}{dy}(y_{\bar{u}});\, \varphi \right\rangle,$$

$$\forall \varphi \in W_0(0, T), \qquad w_L \in L^2(0, T; V').$$

We can determine $p = \Pi_{w_L}^{-1} \in L^2(0, T; V)$ where Π^{-1} denotes canonical isomorphism between spaces $L^2(0, T; V')$ and $L^2(0, T; V)$. If is easy to verify that

$$\frac{\partial \mathscr{L}}{\partial y}(y_{\bar{u}}, \bar{u}, \bar{p}) = 0.$$

Hence if we assume (2.11), (2.12), (2.13), then the mapping (2.14) is differentiable and we obtain

$$\left\langle \frac{dJ}{du}(\bar{u});\, \delta u \right\rangle = \left\langle \frac{\partial \mathscr{L}}{\partial u}(y_{\bar{u}}, \bar{u}, \bar{p});\, \delta u \right\rangle = (\delta A_{\bar{u}}\, y_{\bar{u}}, \bar{p})_{\mathscr{V}'\mathscr{V}}, \tag{2.30}$$

where

$$\delta A_{\bar{u}} = \left\langle \frac{\partial A_{\bar{u}}}{\partial u};\, \delta u \right\rangle.$$

THEOREM 6. Let $\hat{u} \in U_{ad}$ be an optimal control and let cost functional $J(u)$ be differentiable then

$$\left(\left\langle \frac{\partial A_{\hat{u}}}{\partial u};\, u - \hat{u} \right\rangle y_{\hat{u}},\, p_{\hat{u}} \right)_{\mathscr{V}'\mathscr{V}} \geq 0, \qquad \forall u \in U_{ad}. \tag{2.31}$$

Proof. Using (2.30) we get (2.31) as the usual form of necessary conditions of optimality [6].

Example. For the sake of simplicity we assume that parameter $\theta = \theta(t)$ does not depend on x. Furthermore, parameter θ depends on control u by means of the following ordinary differential equation:

$$\begin{cases} \dot{\theta}(t) = -\beta\theta(t) + u, & \beta > 0, \quad u \in U_{ad}, \\ \theta(0) = 0, \end{cases} \tag{2.32}$$

where

$$U = L^2(0, T),$$

$$U_{ad} = \{u \in L^2(0, T) | 0 \leq u(t) \leq 1, \text{ a.e. in } [0, T]\}.$$

Remark. Linear mapping

$$L^2(0, T) \supset U_{ad} \ni u \mapsto \theta(u) \in H^1(0, T)$$

which is defined by (2.32) is continuous.

Taking into consideration, that injection $H^1(0, T) \subset H^{1-\varepsilon}(0, T)$ is compact for any $\varepsilon > 0$ and $H^{1/2+\delta}(0, T) \subset C(0, T)$ with continuous injection for any $\delta > 0$ the mapping:

$$L^2(0, T) \supset U_{ad} \ni u \mapsto \theta(u) \in H^{1-\varepsilon}(0, T) \subset C(0, T) \subset L^\infty(0, T)$$

is compact for any $0 < \varepsilon < \frac{1}{2}$.

Let us consider bilinear form which was introduced in Example 1 (section 1)

$$a_u(t; y, z) = a_{\theta(u)}(t; y, z) =$$
$$= \sum_{i,j=1}^{n} \int_\Omega c_{ij}(x, t, \theta_u(t)) \frac{\partial z}{\partial x_j} \frac{\partial y}{\partial x_i} d\Omega + \int_\Gamma b(x, t, \theta_u(t)\, yz\, d\Gamma. \tag{2.33}$$

Remark. If we assume that $\frac{\partial c_{ij}}{\partial \theta} \in L^\infty(\theta)$, $\frac{\partial b}{\partial \theta} \in L^\infty(\Sigma)$ then for (2.33) the following estimation holds

$$|a_{\theta(u_1)}(t; y, z) - a_{\theta(u_2)}(t; y, z)| \leqslant C \|\theta(u_1) - \theta(u_2)\|_{L^\infty(0,T)} \|y\|_{H^1(\Omega)} \|z\|_{H^1(\Omega)} \leqslant$$
$$\leqslant C \|u_1 - u_2\|_{L^2(0,T)} \|y\|_{H^1(\Omega)} \|z\|_{H^1(\Omega)}, \tag{2.34}$$
$$\forall y, z \in H^1(\Omega), \qquad \forall t \in [0, T],$$

where constant C depend on

$$\max_{u \in U_{ad}} \left\{ \left\| \frac{\partial c_{ij}}{\partial \theta} \right\|_{L^\infty(\theta)}, \left\| \frac{\partial b}{\partial \theta} \right\|_{L^\infty(\Sigma)} \right\}$$

and β, T.

Let there be given $w \in L^2(\theta)$, $z \in L^2(\Omega)$. For $\varepsilon = (\varepsilon_1, \varepsilon_2)$, $\varepsilon_1 \geqslant 0$, $\varepsilon_2 \geqslant 0$ we introduce cost funtional $J_\varepsilon(u)$

$$J_\varepsilon(u) = \frac{1}{2} \int_\Omega (y_u(x, T) - z)^2 d\Omega + \frac{\varepsilon_1}{2} \int_Q (y_u(x, t) - w(x, t))^2 dQ +$$
$$+ \frac{\varepsilon_2}{2} \int_0^T u^2(t)\, dt, \tag{2.35}$$

where for any $u \in U_{ad}$, $y_u \in W(0, T)$ denotes the state trajectory which is defined by state equation of the form

$$\int_\Omega \frac{\partial y}{\partial t}(t)\, z\, d\Omega + \sum_{i,j=1}^{n} \int_\Omega c_{ij}(x, t, \theta_u(t)) \frac{\partial y}{\partial x_i} \frac{\partial z}{\partial x_j} \cdot d\Omega +$$
$$+ \int_\Gamma b(x, t, \theta_u(t))\, yz\, d\Gamma = \int_\Omega f(t)\, z\, d\Omega, \tag{2.36}$$
$$\forall z \in H^1(\Omega) \text{ a.e. in } [0, T]$$
$$y(x, 0) = y_0(x)$$

where $f \in L^2(Q)$, $y_0 \in L^2(\Omega)$ are given.

From Theorem 6 we get the following:

LEMMA. For any $\varepsilon=(\varepsilon_1, \varepsilon_2)$, $\varepsilon_1 \geqslant 0$, $\varepsilon_2 \geqslant 0$ there exists an optimal control $\hat{u}_\varepsilon \in U_{ad}$ such that

$$J_\varepsilon(\hat{u}_\varepsilon) \leqslant J_\varepsilon(u), \quad \forall u \in U_{ad}.$$

At a given point $\bar{u} \in U_{ad}$ the gradient of cost functional (2.5) has the form:

$$\left\langle \frac{dJ_\varepsilon}{du}(\bar{u}); \delta u \right\rangle = \sum_{i,j=1}^{n} \int_Q \left(\frac{\partial c_{ij}}{\partial \theta}(x, t; \theta_{\bar{u}}(t))\, \delta\theta \right) \frac{\partial y_{\bar{u}}}{\partial x_i} \frac{\partial \bar{p}}{\partial x_j}\, dQ +$$
$$+ \int_\Sigma \left(\frac{\partial b}{\partial \theta}(x, t, \theta_{\bar{u}}(t))\, \delta\theta \right) y_{\bar{u}}|_\Sigma\, \bar{p}|_\Sigma\, d\Sigma + \varepsilon_2 \int_0^T \bar{u} \cdot \delta u\, dt$$

where $\delta\theta$ is definied by ordinary differential equation

$$\begin{cases} \delta\theta = -\beta\delta\theta + \delta u, \quad \delta u \in U \\ \delta\theta(0) = 0 \end{cases}$$

and adjoint state $\bar{p}$ is the unique solution of the adjoint state equation:

$$\begin{cases} -\left(\dfrac{d\bar{p}}{dt}, z \right) + a_{\theta(\bar{u})}(t; z, \bar{p}) = -\varepsilon_1 (y_{\bar{u}} - w, z) \quad \forall z \in H^1(\Omega), \quad \text{a.e. in } [0, T] \\ p(x, T) = -y_{\bar{u}}(x, T) + z(x). \end{cases}$$

This equation is obtained using (2.20).

3. Observation in $W^1(O,T)$ [1]

As in the previous section we denote by U_{ad} a given set of admissible controls. Suppose that there is given a family of bilinear forms

$$\{a_u(t, y, z)\}_{u \in U_{ad}} \tag{3.1}$$

which satisfy (2.1)—(2.3).

Furthermore we assume

(i) $a_u(t; y, z) = a_u(t; z, y), \quad \forall u \in U_{ad}, \quad \forall t \in [0, T], \quad \forall y, z \in V;$ (3.2)

(ii) for any $u \in U_{ad}$

$$[0, T] \ni t \mapsto a_u(t; y, z) \in C^1(0, T);$$

moreover

$$\|a_u(\cdot\,; y, z)\|_{C^1(0,T)} \leqslant C\, \|y\|_V\, \|z\|_V \quad \forall u \in U_{ad}, \quad \forall y, z \in V,$$

where constant C does not depend on $u \in U_{ad}$ and $y, z \in V$. As in section 1 (1.21), (1.22) we introduce a family of Hilbert spaces X_u, $u \in U_{ad}$ as well as Y_u and $\mathscr{D}(A_u(t))$.

[1] Space $W^1(0, T)$ will be defined in the sequel (3.9).

We assume

(iii) $$y \in \mathscr{D}\left(A_{u_1}(t)\right) \Leftrightarrow y \in \mathscr{D}\left(A_{u_2}(t)\right), \quad \forall u_1, u_2 \in U_{ad}, \tag{3.3}$$

$$C_1 \|y\|_{\mathscr{D}(A_{u_1}(t))} \leqslant \|y\|_{\mathscr{D}(A_{u_2}(t))} \leqslant C_2 \|y\|_{\mathscr{D}(A_{u_1}(t))}, \tag{3.4}$$

where constants C_1, C_2 can be chosen independently of u_1, u_2, $u_1, u_2 \in U_{ad}$.

From (iii) and (1.21), (1.22) we can deduce

$$y \in X_{u_1} \Leftrightarrow y \in X_{u_2}, \quad \forall u_1, u_2 \in U_{ad}, \tag{3.5}$$

$$\tilde{C}_1 \|y\|_{X_{u_1}} \leqslant \|y\|_{X_{u_2}} \leqslant \tilde{C}_2 \|y\|_{X_{u_1}}. \tag{3.6}$$

Moreover we assume:

(iv) $$\varphi \in Y_{u_1} \Leftrightarrow \varphi \in Y_{u_2}, \quad \forall u_1, u_2 \in U_{ad}, \tag{3.7}$$

$$\bar{C}_1 \|\varphi\|_{Y_{u_1}} \leqslant \|\varphi\|_{Y_{u_2}} \leqslant \bar{C}_2 \|\varphi\|_{Y_{u_1}}. \tag{3.8}$$

It seems that in general (iii) does not imply (iv) but it is an open problem for the author.

We choose some $\tilde{u} \in U_{ad}$ and denote

$$W^1(0, T) = X_{\tilde{u}}, \tag{3.9}$$

$$Y = Y_{\tilde{u}}. \tag{3.10}$$

Furthermore we assume that $\mathscr{D}\left(A_{\tilde{u}}(t)\right)$ does not depend on $t \in [0, T]$. We denote

$$\mathscr{D} = \mathscr{D}\left(A_{\tilde{u}}(t)\right). \tag{3.11}$$

If we assume that family (3.1) satisfies (2.3) and above conditions (i) (ii) then by Corollary 1 state trajectory y_u, $u \in U_{ad}$, which is defined by state equation of the form (1.33), is an element of the space $W^1(0, T)$. Moreover under the above assumptions (iii), (iv) the following estimation holds:

$$\|y_u\|_{W^1(0,T)} \leqslant C\left(\|f\|_{L^2(0,T;H)} + \|y_0\|_V + \|\varphi\|_Y\right). \tag{3.12}$$

We define cost functional

$$J(u) = I(y_u), \quad u \in U_{ad}, \tag{3.13}$$

where $I(\cdot)$ is given continuous functional on the space $W^1(0, T)$.

THEOREM 7. If for any $u_1, u_2 \in U_{ad}$

$$|a_{u_1}(t; y, z) - a_{u_2}(t; y, z)| \leqslant C \|L(u_1 - u_2)\|_{U_1} \|y\|_{\mathscr{D}} \|z\|_H,$$

$$\forall t \in [0, T], \quad \forall y \in \mathscr{D}, \quad \forall z \in V,$$

and if operator $L \in \mathscr{L}(U, U_1)$ is compact then cost functional (3.13) is continuous on the set U_{ad} in the weak topology of the space U.

Proof. Taking advantage of (3.12) and using the same argument as in Theorem 4 we obtain for

$$\tilde{y}=y_{u_1}-y_{u_2}=y_1-y_2$$

the following estimation:

$$\|\tilde{y}\|_{W^1(0,T)}\leqslant C\,\|y_2\|_{W^1(0,T)}\,\|L\,(u_1-u_2)\|_{U_1}.$$

Theorem follows from the fact, that operator $L\in\mathscr{L}\,(U, U_1)$ is compact.

COROLLARY 3. If we assume that the set $U_{ad}\subset U$ is weakly compact then according to Theorem 7 under the above assumptions there exists an optimal control $\hat{u}\in U_{ad}$ such that

$$J\,(\hat{u})\leqslant J\,(u),\qquad \forall u\in U_{ad}.$$

The gradient of cost functional (3.13) is determined in the similar way as it was done in previous section.

Let us assume, that there is given a Hilbert space $\tilde{Y}$ such that

$$Y\subset\tilde{Y}\quad \text{with continuous injection, } Y \text{ is dense in } \tilde{Y}. \tag{3.14}$$

Hence, as usually

$$Y\subset\tilde{Y}=\tilde{Y}'\subset Y'.$$

We define Lagrangian

$$\mathscr{L}\,(y, u, p, r)=I\,(y)+\int_0^T\left(\frac{dy}{dt}+A_u(t)\,y-f, p\right)_H dt+ \\ +\int_0^T(\sigma_u(t)\,y-\varphi, r)_{YY'}\,dt. \tag{3.15}$$

Remark.

$$(\varphi, \psi)_{YY'}=(\varphi, \Pi_1^{-1}\psi)_Y,\qquad \forall\varphi\in Y,\qquad \forall\psi\in Y'. \tag{3.16}$$

Π_1^{-1} denotes canonical isomorphism between Hilbert spaces Y' and Y. For $y=y_u$ we have

$$\mathscr{L}\,(y_u, u, p, r)=J\,(u),\qquad \forall u\in U_{ad},\qquad \forall p\in L^2(0, T; H),\qquad \forall r\in Y'. \tag{3.17}$$

Let us assume that functional $I\,(\cdot)$ is differentiable. Its differential at the point $\bar{y}=y_{\bar{u}}\in W^1(0, T)$, $\bar{u}\in U_{ad}$ for an increment $\delta y\in W^1(0, T)$ we denote by

$$\left\langle\frac{dI}{dy}(\bar{y});\,\delta y\right\rangle. \tag{3.18}$$

Generalized adjoint state $(\bar{p}, \bar{r})\in L^2\,(0, T\colon H)\,xY'$ at the point $\bar{u}\in U_{ad}$ we define as a solution (if it exists) of the following variational problem:

$$\int_0^T\left(\frac{d\varphi}{dt}+A_{\bar{u}}(t)\,\varphi, p\right)_H dt+\int_0^T(\sigma_{\bar{u}}(t)\,\varphi, r)_{YY'}\,dt= \\ =-\left\langle\frac{dI}{dy}(y_{\bar{u}});\,\varphi\right\rangle,\qquad \forall\varphi\in W_0^1(0, T), \tag{3.19}$$

where

$$W_0^1(0, T)=\{\varphi \in W^1(0, T)|\varphi|_{t=0}=0\}.$$

LEMMA. There exists the unique solution of the variational problem (3.19).

Proof. In this case Theorem 5 can be applied. Indeed, it is enough to define

$$\begin{aligned} &W=L^2(0, T; H)\, xY, \\ &w=(p, \tilde{r}) \end{aligned} \tag{3.20}$$

where

$$\tilde{r}=\Pi_1^{-1} r, \quad r \in Y', \quad \tilde{r} \in Y,$$

$$E(w, \varphi)=\int_0^T \left(p, \frac{d\varphi}{dt}+A_{\bar{u}}(t)\,\varphi\right)_H dt+\int_0^T (\Pi_1^{-1} r, \sigma_u(t)\,\varphi)_Y\, dt, \tag{3.21}$$

$$(L, \varphi)_{\Phi'\Phi}=-\left\langle \frac{dI}{dy}(y_{\bar{u}}); \varphi \right\rangle, \quad \frac{dI}{dy}(y_{\bar{u}}) \in (W^1(0, T))'. \tag{3.22}$$

Using (3.5), (3.6), (3.9) and (3.12) we obtain from Theorem 3 and Corollary 1:

$$\begin{aligned} &\left(\left\| \frac{d\varphi}{dt}+A_{\bar{u}}(t)\,\Phi \right\|_{L^2(0,T;H)}+\|\sigma_{\bar{u}}(t)\,\varphi\|_Y\right) \geqslant C\,\|\varphi\|_{W^1(0,T)}, \\ &\qquad \forall \varphi \in W_0^1(0, T). \end{aligned} \tag{3.23}$$

Hence by Theorem 5 there exists a solution $w_L=(\bar{p}, \Pi_{-1}^{-1}\bar{r}) \in W$ of the problem

$$E(w_L, \varphi)=(L, \varphi)_{\Phi'\Phi}, \quad \forall \varphi \in W_0^1(0, T). \tag{3.24}$$

On the other hand for any $f \in L^2(0, T; H)$ and any $\psi \in Y$ there exists the unique solution $\varphi \in W_0^1(0, T)$ of the problem

$$\begin{cases} \dfrac{d\varphi}{dt}+A_{\bar{u}}(t)\,\varphi=f, \\ \sigma_{\bar{u}}\,\varphi=\psi, \\ \varphi|_{t=0}=0, \end{cases} \tag{3.25}$$

hence Theorem 5 implies that the generalized adjoint state $(\bar{p}, \bar{r})$ is determined uniquely.

In order to find the gradient of (3.13) we have to introduce the spaces

$$\begin{aligned} &\mathcal{S}=L^2(0, T; S), \quad (\mathcal{S})'=\mathcal{S}'=L^2(0, T; S'), \\ &\mathcal{H}=L^2(0, T; H), \quad \mathcal{V}=L^2(0, T; V), \quad \mathcal{V}'=L^2(0, T; V') \end{aligned}$$

and define linear operators

$$A_u \in \mathcal{L}(W^1(0, T); \mathcal{H}), \tag{3.26}$$

$$\sigma_u \in \mathcal{L}(W^1(0, T); Y), \tag{3.27}$$

in the following way:

$$(A_u y, z)_{\mathscr{H}} = \int_0^T (A_u(t) y(t), z(t))_H \, dt, \quad \forall y \in W^1(0, T), \quad \forall z \in \mathscr{V}; \tag{3.28}$$

$$(\sigma_u y, \tilde{\gamma} z)_{\mathscr{S}'\mathscr{S}} = \int_0^T (\sigma_u(t) y(t), \gamma z(t))_{S'S} \, dt, \quad \forall y \in W^1(0, T), \quad \forall z \in \mathscr{V}, \tag{3.29}$$

where

$$(\tilde{\gamma} z)(t) = \gamma z(t), \quad \forall z \in \mathscr{V}, \quad \tilde{\gamma} \in \mathscr{L}(\mathscr{V}, \mathscr{S}).$$

Remark 1. Operators (3.28), (3.29) are connected with bilinear form

$$\mathscr{A}_u(y, z) = \int_0^T a_u(t; y(t), z(t)) \, dt \quad \forall y, z \in \mathscr{V}, \tag{3.30}$$

namely

$$\mathscr{A}_u(y, z) = (A_u y, z)_{\mathscr{H}} + (\sigma_u y, \tilde{\gamma} z)_{\mathscr{S}'\mathscr{S}}, \quad \forall y \in W^1(0, T), \quad \forall z \in \mathscr{V}. \tag{3.31}$$

Let us consider given control $\bar{u} \in U_{ad}$ and assume that nonlinear mappings

$$U_{ad} \ni u \mapsto A_u \in \mathscr{L}(W^1(0, T); \mathscr{H}), \tag{3.32}$$

$$U_{ad} \ni u \mapsto \sigma_u \in \mathscr{L}(W^1(0, T); Y), \tag{3.33}$$

are differentiable in strong sense at an arbitrary point $\bar{u} \in U_{ad}$. We denote derivatives of mappings (3.32), (3.33) at $\bar{u}$ by $B_A(\cdot)$ and $B\ (\cdot)$.

Conditions under which $B_A(\cdot)$ and $B_\sigma(\cdot)$ exist are discussed in Appendix.

Let $(\bar{p}, \bar{r}) \in L^2(0, T; H) \times Y = \mathscr{H} \times Y$ denotes generalized adjoint state at point $\bar{u} \in U_{ad}$.

Using the same argument as in Section 2 it can be shown that under the above assumptions the gradient of functional (3.13) is of the form

$$\left\langle \frac{dJ}{du}(\bar{u}); \delta u \right\rangle = (B_A(\delta u) y_{\bar{u}}, \bar{p})_{\mathscr{H}} + (B_\sigma(\delta u) y_{\bar{u}}, \bar{r})_{YY'}, \quad \forall \delta u \in U.$$

In exactly the same way as in the case of Theorem 6 we obtain the following:

THEOREM 8. If $\hat{u} \in U_{ad}$ is an optimal control which minimizes functional (3.13) over the set U_{ad} and functional (3.13) is differentiable at the point $\hat{u} \in U_{ad}$ then

$$(B_A(u - \hat{u}) y_{\hat{u}}, p_{\hat{u}})_{\mathscr{H}} + (B_\sigma(u - \hat{u}) y_{\hat{u}}, r_{\hat{u}})_{YY'} \geq 0, \quad \forall u \in U_{ad}.$$

Example. Let Ω be an open region in R^n with smooth boundary. Let us consider the following problem of optimization.

Find

$$\min_{u \in U_{ad}} \frac{1}{2} \|y_{\theta(u)}(T) - z\|^2_{H_1(\Omega)} = \min_{u \in U_{ad}} J_1(\theta(u)) = \min_{u \in U_{ad}} J(u).$$

Subject to the constraints (state equation):

$$(1)\begin{cases} \dfrac{\partial y}{\partial t} - F(\theta)\,\Delta y = f, & (x, t) \in Q, \\ \sigma_\theta(t)\, y = F(\theta)\dfrac{\partial y}{\partial \eta} + g(\theta)\, y = \varphi, & (x, t) \in \Sigma, \\ y(x, 0) = y_0(x), \quad x \in \Omega, \end{cases}$$

where

$$f \in L^2(Q), \quad \varphi \in H^{1/2, 1/4}(\Sigma), \quad y_0 \in H^1(\Omega), \tag{3.35}$$

are given functions. We assume:

$$\begin{aligned} &\theta(u) \in C^1(0, T), \quad \forall u \in U_{ad}, \\ &0 \leqslant \theta(u)(t) \leqslant M, \quad \forall t \in [0, T], \\ &F(\cdot),\ g(\cdot) \in C^1(0, M). \end{aligned} \tag{3.36}$$

Moreover

$$\begin{aligned} &F(r) \geqslant \alpha > 0, \quad \forall r \in [0, M], \\ &g(r) \geqslant 0, \quad \forall r \in [0, M]. \end{aligned} \tag{3.37}$$

Under the above assumptions for each $\theta = \theta(u)$ there exists [4] the unique solution $y = y_\theta \in H^{2,1}(Q)$ such that

$$\|y_\theta\|_{H^{2,1}(Q)} \leqslant C\left(\|y_0\|_{H^1(\Omega)} + \|f\|_{L^2(Q)} + \|\varphi\|_{H^{1/2,1/4}(\Sigma)}\right). \tag{3.38}$$

In this case we have

$$W^1(0, T) = H^{2,1}(Q), \quad Y = H^{1/2, 1/4}(\Sigma), \quad \mathscr{D} = H^1(\Omega; \Delta),$$

where

$$H^1(\Omega; \Delta) = \{y \in H^1(\Omega) \mid \Delta y \in L^2(\Omega)\}.$$

Let us assume that parameter θ depends on control $u \in L^2(0, T)$ in the following way

$$\theta = Lu$$

where L is given by the system of ordinary, linear, differential equations

$$\frac{d\theta}{dt} = -a_1\,\theta + u_1, \quad a_1 > 0,$$

$$\frac{du_1}{dt} = -a_2\,u_1 + u, \quad a_2 > 0,$$

$$\theta(0) = 0, \quad u_1(0) = 0, \quad u \in U_{ad},$$

and let

$$U_{ad} = \{u \in L^2(0, T) \mid 0 \leqslant u(t) \leqslant 1, \quad a.e. \text{ in } [0, T]\}.$$

From Theorem 6 we get the following

LEMMA 1. There exists an optimal control $\hat{u} \in U_{ad}$ such that

$$J(\hat{u}) \leqslant J(u), \qquad \forall u \in U_{ad}.$$

Remark. In the case of state equation (1) appropriate bilinear forms are defined as follows:

$$a_u(t; y, z) = F((Lu)(t)) \sum_{i=1}^{n} \int_{\Omega} \frac{\partial y}{\partial x_i} \frac{\partial z}{\partial x_i} d\Omega + \\ + g((Lu)(t)) \int_{\Gamma} yz \, d\Gamma, \qquad \forall y, z \in H^1(\Omega),$$

$$\mathscr{A}_u(y, z) = \int_0^T \left\{ F(Lu) \sum_{i=1}^{n} \int_{\Omega} \frac{\partial y}{\partial x_i} \frac{\partial z}{\partial x_i} d\Omega \right\} dt + \\ + \int_0^T \left\{ g(Lu) \int_{\Gamma} yz \, d\Gamma \right\} dt, \qquad \forall y, z \in L^2(0, T; H^1(\Omega)),$$

$$\mathscr{B}_u(v; y, z) = \int_0^T \left\{ \left[\frac{dF}{d\theta}(Lu) \sum_{i=1}^{n} \int_{\Omega} \frac{\partial y}{\partial x_i} \frac{\partial z}{\partial x_i} d\Omega \right] v \right\} dt + \\ + \int_0^T \left\{ \left[\frac{dg}{d\theta}(Lu) \int_{\Gamma} yz \, d\Gamma \right] v \right\} dt,$$

$$\forall u \in U_{ad}, \quad \forall v \in U = L^2(0, T), \quad \forall y, z \in L^2((0, T; H^1(\Omega)).$$

From Theorem 8 we obtain the following

LEMMA 2. If $\hat{\theta} = L\hat{u}$ is an optimal parameter (control) then

$$\int_Q \left\{ \left[\frac{dF}{d\theta}(L\hat{u}) \left(L(u - \hat{u})\right) \right] \cdot \Delta y_{\hat{u}} \cdot p_{\hat{u}} \right\} dQ + \\ + \int_{\Sigma} \left[\frac{dF}{d\theta}(Lu) \frac{\partial y_{\hat{u}}}{\partial \eta} + \frac{dg}{d\theta}(L\hat{u}\, y_{\hat{u}}) \right] \left(L(u - \hat{u})\right) r_{\hat{u}} \, d\Sigma \geqslant 0, \qquad \forall u \in U_{ad},$$

where

$$y_{\hat{u}} = y_{\theta(\hat{u})}, \qquad \hat{\theta} = L\hat{u},$$

$L^2(Q) \times H^{-1/2, -1/4}(\Sigma) \ni (p_{\hat{\theta}}, r_{\hat{\theta}})$ is the unique solution of generalized adjoint equation

$$\int_0^T \left(\frac{d\varphi}{dt} - F(\hat{\theta}) \Delta\varphi, p \right)_{L^2(\Omega)} dt + \int_0^T \left(F(\hat{\theta}) \frac{\partial \varphi}{\partial n} + g(\hat{\theta}) \varphi, r \right)_{L^2(\Gamma)} dt = \\ = -\left(y_{\hat{\theta}}(T) - z, \varphi(T)\right)_{H^1(\Omega)}, \qquad \forall \varphi \in W_0^1(0, T),$$

where

$$W_0^1(0, T) = \{\varphi \in H^{2,1}(Q) | \varphi|_{t=0} = 0\}.$$

4. Appendix

In Appendix we consider the problem of existence of Frechet derivatives $B_\Lambda(\cdot)$, $B_\sigma(\cdot)$ of nonlinear mappings

$$U_{ad} \ni u \mapsto \Lambda_u \in \mathscr{L}\left(W^1(0, T): \mathscr{H}\right),$$
$$U_{ad} \ni u \mapsto \sigma_u \in \mathscr{L}\left(W^1(0, T); \mathscr{S}'\right).$$

Let us start with technical remark:

Remark. In section 1 we introduced linear operator $\gamma \in \mathscr{L}(V, S)$ such that

(i) γ maps V onto S,

(ii) $V_0 = \operatorname{Ker} \gamma$, is dense in H.

Let us denote by $\gamma' \in \mathscr{L}(S', V')$ the adjoint of γ, i.e.

$$(\varphi, \gamma y)_{S'S} = (\gamma' \varphi, y)_{V'V}, \qquad \forall \varphi \in S', \qquad \forall y \in V.$$

Operator γ' has the closed range in V'

$$\text{Range } \gamma' = V_0^\perp$$

where $V_0^\perp$ denotes the orthogonal of subspace V_0 in V'. Hence there exists bounded inverse $(\gamma')^{-1} \in \mathscr{L}(V_0^\perp, S')$ which maps $V_0^\perp$ onto S'. Similarly for operator $\tilde{\gamma}' \in \mathscr{L}(\mathscr{S}', \mathscr{V}')$ there exists its inverse $(\tilde{\gamma}')^{-1} \in \mathscr{L}(\mathscr{V}_0^\perp, \mathscr{S}')$ where $\mathscr{V}_0^\perp$ denotes orthogonal of subspace $\mathscr{V}_0 = L^2(0, T; V_0)$ in $\mathscr{V}'$.

Let $\bar{u}$ be a given element of the set U_{ad}.

Lemma. Let there exist:

(i) form $\mathscr{B}_{\bar{u}}(v; y, z)$ which is linear with respect to $v \in U$ and $y, z \in \mathscr{V}$ and bounded:

$$|\mathscr{B}_{\bar{u}}(v; y, z)| \leq M(\bar{u}) \|v\|_U \|y\|_{\mathscr{V}} \|z\|_{\mathscr{V}}. \tag{4.1}$$

Moreover, for each $z \in \mathscr{V}_0 = L^2(0, T; V_0)$.

$$|\mathscr{B}_{\bar{u}}(v; y, z)| \leq M(\bar{u}) \|v\|_U \|y\|_{W^1(0,T)} \|z\|_{\mathscr{H}}; \tag{4.2}$$

(ii) bilinear form $\mathscr{R}_{\bar{u}}(v; y, z)$ for which the following estimations hold

$$\frac{|\mathscr{R}_{\bar{u}}(v; y, z)|}{\|y\|_{\mathscr{V}} \|z\|_{\mathscr{V}}} = o(\|v\|_U), \qquad \forall y, z \in \mathscr{V}, \quad y \neq 0, \quad z \neq 0; \tag{4.3}$$

$$\frac{|\mathscr{R}_{\bar{u}}(v; y, z_0)|}{\|y\|_{W^1(0,T)} \|z_0\|_{\mathscr{H}}} = o(\|v\|_U), \qquad \forall y \in W^1(0, T), \quad \forall z_0 \in \mathscr{V}_0, \quad y \neq 0, \quad z_0 \neq 0. \tag{4.4}$$

Such that

$$\mathscr{A}_{\bar{u}+\delta u}(y, z) - \mathscr{A}_{\bar{u}}(y, z) = \mathscr{B}_{\bar{u}}(\delta u; y, z) + \mathscr{R}_{\bar{u}}(\delta u; y, z), \qquad \forall y \in \mathscr{V}, \quad \forall z \in \mathscr{V}, \tag{4.5}$$

for any δu with small enough: $\|\delta u\|_U < \varepsilon$.

Then there exist linear operators

$$B_\Lambda(\cdot)\in\mathcal{L}\big(U;\mathcal{L}\left(W^1(0,T);\mathcal{H}\right)\big), \tag{4.6}$$

$$B_\sigma(\cdot)\in\mathcal{L}\big((U;\mathcal{L}\left(W^1(0,T);\mathcal{S}'\right)\big), \tag{4.7}$$

which are uniquely determined by the form $\mathcal{B}_{\bar{u}}(\cdot\,;y,z)$, such that

$$B_\Lambda(v)=\left\langle\frac{\partial\Lambda_{\bar{u}}}{\partial u};v\right\rangle\in\mathcal{L}\left(W^1(0,T);\mathcal{H}\right), \tag{4.8}$$

$$B_\sigma(v)=\left\langle\frac{\partial\sigma_{\bar{u}}}{\partial u};v\right\rangle\in\mathcal{L}\left(W^1(0,T);\mathcal{S}'\right), \tag{4.9}$$

where $\frac{\partial\Lambda_{\bar{u}}}{\partial u}$, $\frac{\partial\sigma_{\bar{u}}}{\partial u}$ denote derivatives of nonlinear mappings

$$U_{ad}\ni u\mapsto\Lambda_u\in\mathcal{L}\left(W^1(0,T);\mathcal{H}\right), \tag{4.10}$$

$$U_{ad}\ni u\mapsto\sigma_u\in\mathcal{L}\left(W^1(0,T);\mathcal{S}'\right), \tag{4.11}$$

at the point $\bar{u}\in U_{ad}$.

Remark. (4.11) makes sense because $Y\subset\mathcal{S}'$ with continuous injection, hence $\mathcal{L}\left(W^1(0,T);Y\right)\subset\mathcal{L}\left(W^1(0,T);\mathcal{S}'\right)$.

Proof. By (3.33) and (3.36) for every $z\in\mathcal{V}_0$ we have

$$\big((\Lambda_{\bar{u}+\delta u}-\Lambda_{\bar{u}}-B_\Lambda(\delta u))\,y,z\big)_{\mathcal{H}}=\mathcal{R}_{\bar{u}}(\delta u;y,z),\qquad\forall y\in W^1(0,T), \tag{4.12}$$

where $B_\Lambda(\cdot)$ is defined by equality

$$\mathcal{B}_{\bar{u}}(\delta u;y,z)=\left(B_\Lambda(\delta u)\,y,z\right)_{\mathcal{H}},\qquad\forall y\in W^1(0,T),\qquad\forall z\in\mathcal{V}_0. \tag{4.13}$$

But $\mathcal{V}_0$ is dense in $\mathcal{H}$ due to the fact, that V_0 is dense in H.

Then by assumption (4.3) we get

$$\sup_{\|y\|_{W^1(0,T)}\leqslant 1}\left|\big((\Lambda_{\bar{u}+\delta u}-\Lambda_{\bar{u}}-B_\Lambda(\delta u))\,y,z\big)_{\mathcal{H}}\right|=o\left(\|\delta u\|_U\right)$$
$$z\in V_0,\qquad\|z\|_{\mathcal{H}}\leqslant 1, \tag{4.14}$$

which shows that $B_\Lambda(\cdot)$ is Frechet derivative of the mapping (4.10).

To prove (4.11) we define linear operators $P_{\bar{u}}$ and $T(\cdot)$ of the form:

$$(P_{\bar{u}}y,z)_{\mathcal{V}'\mathcal{V}}=\mathcal{A}_{\bar{u}}(y,z)-(\Lambda_{\bar{u}}y,z)_{\mathcal{H}},\qquad\forall y\in W^1(0,T),\qquad\forall z\in\mathcal{V}, \tag{4.15}$$

$$\left(T(v)\,y,z\right)_{\mathcal{V}'\mathcal{V}}=\mathcal{B}_{\bar{u}}(v;y,z)-\left(B_\Lambda(v)\,y,z\right)_{\mathcal{H}},$$
$$\forall v\in U,\qquad\forall y\in W^1(0,T),\qquad\forall z\in\mathcal{V}. \tag{4.16}$$

It is easy to show (cf. (3.31)) that

(i)
$$P_{\bar{u}}\in\mathcal{L}\left(W^1(0,T);\mathcal{V}'\right),$$
$$P_{\bar{u}}\text{ maps }W^1(0,T)\text{ into }\mathcal{V}_0^{\perp}, \tag{4.17}$$
$$P_{\bar{u}}=\tilde{\gamma}'\,\sigma_{\bar{u}};$$

(ii)
$$T(\cdot)\in\mathscr{L}\big(U;\mathscr{L}\,(W^1(0,T);\mathscr{V}')\big),\qquad T(\cdot)=\tilde{\gamma}'B_\sigma(\cdot),\tag{4.18}$$

where $B_\sigma(\cdot)\in\mathscr{L}\,\big(U;\mathscr{L}\,(W^1(0,T);\mathscr{S}')\big)$ is defined by the following equality:

$$(B_\sigma(v)\,y,\tilde{\gamma}z)_{\mathscr{S}'\mathscr{S}}=\mathscr{B}_{\bar{u}}(v;y,z)-(B_A(v)\,y,z)_{\mathscr{H}},\qquad \forall v\in U,\quad \forall y\in W^1(0,T),\quad \forall z\in\mathscr{V}.\tag{4.19}$$

From (4.5) we deduce

$$\begin{aligned}\big((P_{\bar{u}+\delta u}-P_{\bar{u}}-T(\delta u))\,y,z\big)_{\mathscr{V}'\mathscr{V}}&=\mathscr{A}_{\bar{u}+\delta u}(y,z)-\mathscr{A}_{\bar{u}}(y,z)-\\&-\mathscr{B}_{\bar{u}}(\delta u;y,z)-\big((\Lambda_{\bar{u}+\delta u}-\Lambda_{\bar{u}}-B_A(\delta u))\,y,z\big)_{\mathscr{H}}=\\&=\mathscr{R}_{\bar{u}}(\delta u;y,z)-\big((\Lambda_{\bar{u}+\delta u}-\Lambda_{\bar{u}}-B_A(\delta u))\,y,z\big)_{\mathscr{H}}=\\&=\mathscr{R}_{\bar{u}}(\delta u;y,z)-G_{\bar{u}}(y,z),\qquad \forall y\in W^1(0,T),\quad \forall z\in\mathscr{V},\end{aligned}$$

where

$$G_{\bar{u}}(y,z)=\big((\Lambda_{\bar{u}+\delta u}-\Lambda_{\bar{u}}-B_A(\delta u))\,y,z\big)_{\mathscr{H}}$$

hence using inequalities

$$\frac{|\mathscr{R}_{\bar{u}}(\delta u;y,z)|}{\|y\|_{\mathscr{V}}\,\|z\|_{\mathscr{V}}}\geqslant\frac{|\mathscr{R}_{\bar{u}}(\delta u;y,z)|}{\|y\|_{W^1(0,T)}\,\|z\|_{\mathscr{V}}},\qquad \frac{|G_{\bar{u}}(y,z)|}{\|y\|_{W^1(0,T)}\,\|z\|_{\mathscr{H}}}\geqslant\frac{|G_{\bar{u}}(y,z)|}{\|y\|_{W^1(0,T)}\,\|z\|_{\mathscr{V}}},$$

which hold for every $y\in W^1(0,T)$, $y\neq 0$ and every $z\in\mathscr{V}$, $z\neq 0$ (in fact $\|y\|_{W^1(0,T)}\geqslant\|y\|_{\mathscr{V}}$, $\forall y\in W^1(0,T)$, also $\|z\|_{\mathscr{V}}\geqslant\|z\|_{\mathscr{H}}$, $\forall z\in\mathscr{V}$); from (4.3) and (4.14) we get

$$\|P_{\bar{u}+\delta u}-P_{\bar{u}}-T(\delta u)\|_{\mathscr{L}(W^1(0,T);\mathscr{V}')}=o\,(\|\delta u\|_U)$$

hence

$$\begin{aligned}&\|\sigma_{\bar{u}+\delta u}-\sigma_{\bar{u}}-B_\sigma(\delta u)\|_{\mathscr{L}(W^1(0,T);\mathscr{S}')}=\\&\qquad=\|(\tilde{\gamma}')^{-1}\,(P_{\bar{u}+\delta u}-P_{\bar{u}}-T(\delta u))\|_{\mathscr{L}(W^1(0,T);\mathscr{S}')}=o\,(\|\delta u\|_U).\end{aligned}$$ Q.E.D.

Remark. For existence of the gradient of functional (3.13) we need an additional assumption, namely

$$B_\sigma(\cdot)=\Big\langle\frac{\partial\sigma_{\bar{u}}}{\partial u};(\cdot)\Big\rangle\in\mathscr{L}\big(U;\mathscr{L}\,(W^1(0,T);Y)\big).$$

The above assumption is connected with the fact that the Lagrange multiplier r is an element of the space Y', which is defined uniquely.

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O optymalnym sterowaniu parametrycznym dla słabych rozwiązań abstrakcyjnych liniowych równań parabolicznych

Rozważono problem optymalizacyjny dla przypadku sterowania parametrycznego. Równaniem stanu, którego rozwiązania zależą od sterowania, jest tu liniowe abstrakcyjne równanie paraboliczne; rozważono jego tzw. słabe rozwiązania.

Sterowanie występuje we współczynnikach operatora eliptycznego. Postawiony problem optymalizacyjny polega na wyznaczaniu w zbiorze sterowań dopuszczalnych sterowania optymalnego minimalizującego funkcjonał jakości zależny od sterowania przez rozwiązanie równania stanu, przy czym rozwiązanie równania stanu jest traktowane jako element pewnej przestrzeni funkcyjnej, tzw. przestrzeni obserwacji. Rozważono dwa zasadniczo różniące się rodzaje obserwacji, dla których podano warunki wystarczające do istnienia rozwiązania optymalnego oraz różniczkowe warunki konieczne optymalności.

Dla uzyskania prostej postaci gradientu funkcjonału jakości wprowadzono tzw. uogólnione równanie sprzężone.

Uzyskane rezultaty wykorzystano w przykładach problemów optymalizacji parametrycznej dla konkretnych równań różniczkowych cząstkowych typu parabolicznego.

Об оптимальном параметрическом управлении для слабых абстрактных решений линейных параболических уравнений

В работе рассматривается оптимизационная проблема для случая параметрического управления. Уравнением состояния, которого решеня зависят от управления, здесь является мнейшое абстрактное параболическое уравнение; рассматриваются его так называемые слабые решения.

Управление имеет место в коэффициентах эллиптического оператора. Поставленная оптимизационная проблема состоит в определении на множестве допустимых управлений оптимального управления, минимизирующего функционал качества, зависящий от управления посредством решения уравнения состояния, причем решение уравнения состояния воспринимается в качестве элемента некоторого функционального пространства, так называемого пространства наблюдений. В работе рассматриваются два принципально различных вида наблюдений, для которых даны достаточные условия существования оптимального решения и необходимые дифференциальные условия оптимальности.

С целью получения простого вида градиента функционала качества вводится так называемое обобщенное сопряженное управление.

Полученные результаты используются в примерах задач параметрической оптимизации для конкретных дифференциальных уравнений с частными производными параболического типа.