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Sensitivity analysis of optimal control system with small time delay

by

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This paper presents the real sensitivity analysis of optimal control systems in various structures with small time delay. A computational method for a linear-quadratic case is described. An example illustrates the application of the method.

1. Introduction

Consider a time delay process

$$\dot{x}(t)=A_1 x(t)+A_2 x(t-h)+Bu(t) \tag{1}$$

with given initial state

$$x(t)=\varphi(t), \qquad t\in[-h_m, 0] \tag{2}$$

where $x(t)\in R^n$ denotes a state vector; $u(t)\in R^r$ is a control vector; A_1, A_2, B are real metrices and $h\in[0, h_m]$ is a small[1]) delay parameter. The optimal control problem consists of minimizing the functional.

$$I(x,u)=0.5\, x'(T)\, Fx(T)+0.5\int_0^T \left(x'(t)\, Qx(t)+u'(t)\, Ru(t)\right) dt \tag{3}$$

where prime denotes the transpose; F, Q, R are real matrices; T is a fixed final time.

Time delay optimization problems have been studied extensively in recent years because of numerous applications in physical, biological and social systems etc. Even in simple examples of the foregoing type, the solution of such problems leads to a large computational effort. On the other hand, small time delays (which appear in almost every application) are often neglected. The sensitivity analysis of optimal control systems will help to determine the performance loss in (3) due to neglecting the time delay, that is, by defining a more simple model:

$$\dot{x}(t)=(A_1+A_2)\, x(t)+Bu(t), \qquad x(0)=\varphi(0) \tag{4}$$

by setting $h=0$.

[1]) Actually, the "smallness" of the time delay can be determined on the basis of the sensitivity analysis — see further lines.

The optimal control computed from the model (4), can be applied to the real process (1) in various ways, i.e., in various optimal control structures which have, in general, different sensitivities. The sensitivity approach presented here also allows the comparison of various control structures and provides data for choosing the best one with respect to the performance index.

2. Basic results in the sensitivity analysis of optimal control systems

The problem, formulated above, corresponds to the general sensitivity problem which was investigated in [1]. We present now some basic results related to the time delay sensitivity.

Generally, the optimal control problem can be defined in the following way:

$$\min I(x, u, a) = \hat{I}(a) \tag{5a}$$

such that

$$P(x, u, a) = 0 \tag{5b}$$

where $x \in B_x$ represents the state, $u \in B$ is the control and $a \in B_a$ is a parameter. The constraining relation (5b), where $P_i, B_x \times B_i \times B_a \to B_x$, can be interpreted as a state equation and the functional I represents the costs of the process. Assume that for each a there exists an optimal solution $x = \hat{X}(a)$, $u = \hat{U}(a)$ for the problem (5a, b). The operators $\hat{X}(a)$, $\hat{U}(a)$ are called **basic state and control characteristics**[2]).

In practical applications the optimization problem is based on a model, e.g. (5a, b), which often differs from reality. Define another model which differs from the original one in the value of parameters (e.g. $P(x, u, \alpha) = 0$, $I(x, u, \alpha) = q$).

Suppose the optimal control law is represented by the operator equation:

$$u^i = R^i(x, a, \alpha) \tag{6}$$

where i denotes the i-th structure of the control system. Assume $R^i(\hat{X}(a), a, a) = \hat{U}(a)$. The state x^i of the process in the i-th control structure and the real performance functional are defined by the following relations:

$$P\left(x^i, R^i(x^i, a, \alpha), \alpha\right) = 0\,, \tag{7a}$$

$$I\left(x^i, R^i(x^i, a, \alpha), \alpha\right) \stackrel{\mathrm{df}}{=} I^i(a, \alpha)\,. \tag{7b}$$

We call the operators $X^i(a, \alpha) = x^i$, which is a solution of (7a) und $U^i(a, \alpha) = R^i\left(X^i(a, \alpha), a, \alpha\right)$ (if they exist) **structural state and control characteristics**. The functional

$$S^i(a, \alpha) \stackrel{\mathrm{df}}{=} I^i(a, \alpha) - \hat{I}(\alpha) \tag{8}$$

[2]) Analogously, one can define $\hat{\eta} = \hat{N}(a)$ the basic characteristic of the adjoint variable (which exists and is normal if P_x is onto; P_x denotes the Frechet derivative).

evaluates the performance loss due to imperfect knowledge of the process parameter α and is called the **sensitivity measure**[3]) of the optimal control problem. Since the optimality $S^i(a, \alpha)\geqslant 0$ and $S^i(a\ \alpha)=0$ for $a=\alpha$.

THEOREM 1. Suppose B_x, B_u are Banach spaces, B_x is reflexive; $B_a=R^1$, $a, \alpha \in [a_0, a_1]$. Suppose P, I are twice continuously differentiable with respect to x, u and $\hat{X}(a)$, $\hat{U}(a)$, $\hat{N}(a)$ are continuous in $[a_0, a_1]$. Suppose the structural characteristics $X^i(a, \alpha)$, $U^i(a, \alpha)$ are differentiable with respect to a in a neighborhood of $a=\alpha$, $\alpha \in [a_0, a_1]$ (with one-side derivatives at a_0, a_1) and their derivatives are: Lipshitz continuous with respect to a for $a=\alpha$ and continuous with respect to α for $a=\alpha$.

Then the sensitivity measure $S^i(a, \alpha)$ is twice differentiable with respect to a, α for $a=\alpha$ (with one-side derivatives in a_0, a_1) and for each $a=\alpha$, $\alpha \in [a_0, a_1]$

(i) $$S^i_a=S^i_\alpha=0; \tag{9a}$$

(ii) $$S^i_{aa}=-S^i_{a\alpha}=S^i_{\alpha\alpha}. \tag{9b}$$

Moreover, for each $a, \alpha \in [a_0, a_1]$ the sensitivity measure can be approximated by

$$S^i(a, \alpha)\approx S^i(\alpha, a)\approx 0.5\,(\langle X^{i*}_a(a), \hat{L}_{xx}(a)\, X^i_a(a)\rangle + $$
$$+2\,\langle X^{i*}_a(a), \hat{L}_{xu}(a)\, U^i_a(a)\rangle + \langle U^{i*}_a(a), \hat{L}_{uu}(a)\, U^i_a(a)\rangle) \tag{10}$$

where $\hat{L}_{xx}(a)=L_{xx}(\hat{X}(a), \hat{U}(a), N(a)), a$ etc.; $L(x, u, \eta, a)$ — the Lagrange functional, and

$$X^i_a(a)=\left.\frac{\partial X^i(a, \alpha)}{\partial a}\right|_{\alpha=a}, \qquad U^i_a(a)=\left.\frac{\partial U^i(a, \alpha)}{\partial a}\right|_{\alpha=a} \tag{11}$$

i.e. X^i_a, U^i_a are the derivatives of the structural characteristics with respects to a at $\alpha=a$. The accuracy of this approximation is in order $o(\Delta a^2)$, $\Delta a=\alpha-a$.

The proof of this theorem is given in the Appendix 1. The derivatives (11) are called **structural sensitivity functions.**

This result can be stated in a form of so-called **relative principle of the local sensitivity analysis** [1]. Namely, one can use the same approximation of the sensitivity measure no matter which of the parameters changes. Actually, the situation when the parameter of the process has value α and changes is of more practical importance. But the reverse assumption is more acceptable from the computational point of view.

For an effective calculation of the structural functions (11) so-called **basic sensitivity functions** $\hat{X}_a(a)$, $\hat{U}_a(a)$ are needed. They are the derivatives of the basic characteristics $\hat{X}(a)$, $\hat{U}(a)$ and can be found from the linearization of the optimality conditions (canonical equations) — see [1].

Suppose that there exist operators C, D such that

$$\hat{U}_a=C\hat{X}_a+D. \tag{12a}$$

[3]) This approach to the sensitivity was introduced by A. Wierzbicki in [2].

Let $R^i(x, a, \alpha)=R^1(x, a)$ be the optimal control law in the closed-loop structure, uniquely defined and differentiable with respect to (x, a). Then $\hat{U}_a$ can be also expressed as

$$\hat{U}_a = R_x^1 \hat{X}_a + R_a^1 \tag{12b}$$

which implies $R_x^1=C$, $R_a^1=D$.

The main purpose of this paper is to present a method for computing the second-order approximation (10) for time delay process. Therefore, the singular perturbation theorems for homogeneous equation and optimal control problem are involved.

3. The time delay optimal control problem

Consider the original problem (1), (2), (3). Suppose $\varphi(t)$ is an absolutely continuous function on $[-h_m, 0]$. Suppose R is symmetric positive-definite and Q, F are symmetric positive-semidefinite matrices. Assume $B_x=L^2(-h_m, 0; R^n)\times \times W_1^2(0, T; R^n)$, $B_u=L^2(0, T; R^r)$. The operator (5b) can be defined as follows[4])

$$P(x, u, h)=\left(\dot{x}-A_1 x-A_2 x(t-h)-Bu,\ x(t)-\varphi(t),\ x(0)-\varphi(0)\right). \tag{13}$$

The local maximum principle [3] implies, that along the optimal trajectory

$$u(t)=R^{-1} B'\eta(t) \tag{14}$$

while the state and adjoint variables $\eta \in L^2(0, T; R^n)\times L^2(T, T+h_m; R^n)\times R^n$ satisfy the following canonical equations

$$\dot{x}=A_1 x+A_2 x(t-h)+S\eta, \tag{15a}$$

$$\dot{\eta}=-A_1'\eta-A_2'\eta(t+h)+Qx, \tag{15b}$$

where $S=BR^{-1}B'$, with the boundary conditions

$$x(t)=\varphi(t),\ t\in[-h_m, 0];\ \eta(T)=-Fx(T),\ \eta(t)=0,\ t\in(T, T+h_m).$$

The optimal control can be also uniquely defined as a function of the state, i.e., in a form of closed-loop optimal control law — see [4]. Although broadly investigated, the synthesis of the optimal feedback controller is very complicated and difficult in practical applications.

The canonical difference-differential equations can be transformed into a system of ordinary diflerential equations [5]. Let $m=T/h$ be an integer. We set

$$\xi=[x_1, \ldots, x_m]' \qquad \psi=[\eta_1, \ldots, \eta_m]' \tag{16}$$

where

$$x_i=x(t), \qquad \eta_i=\eta(t) \qquad t\in[ih, (i+1)h] \qquad i=1, 2, \ldots, m \tag{17}$$

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[4]) This means that $x|_{[h_m, 0]}\in L^2(-h_m, 0; R^n)$, $x|_{[0, T]}\in W_1^2(0, T; R^n)$ while $P: B_x\times B_u\times R \to \to L^2(0, T; R^n\times L^2(-h_m, 0; R^n)\times R^n$ and (5b) is here equivalent to

$$\dot{x}(t)=A_1 x(t)+A_2 x(t-h)+Bu(t), \qquad t\in[0, T],$$
$$x(t)=\varphi(t), \qquad t\in[-h_m, 0]$$
$$x(0)=\varphi(0).$$

and define the matrices

$$
\mathscr{A}=\begin{bmatrix} A_1 & 0 & & & 0 \\ A_2 & A_1 & & & \\ 0 & \cdot & & & \\ \cdot & & \cdot & & \\ \cdot & & & \cdot & \\ \cdot & & & \cdot & \\ 0 & & & A_2 & A_1 \end{bmatrix}
\qquad
\mathscr{T}=\begin{bmatrix} 0 & & & & 0 \\ 1 & 0 & & & \\ & \cdot & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & \cdot & \\ 0 & & & 1 & 0 \end{bmatrix}
\qquad
\mathscr{E}=\begin{bmatrix} 0 & & & & \\ & 0 & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & \cdot & \\ & & & 0 & \\ 0 & & & 0 & -F \end{bmatrix}
$$

$$\mathscr{B}_1=\operatorname{diag}\{S\}_1^m, \qquad \mathscr{B}=\operatorname{diag}\{Q\}_1^m,$$

$$F_2=[\varphi(t)\,0\,\ldots\,0]', \qquad F_1=\operatorname{diag}\{A_2\}_1^m F_2 .$$

We obtain a $(2n\times m)$-dimensional system of ordinary differential equations with mixed boundary conditions:

$$\dot{\xi}=\mathscr{A}\xi+\mathscr{B}_1\psi+F_1, \qquad \xi(0)=F_2(0)+\mathscr{T}\xi(h), \tag{18a}$$

$$\dot{\psi}=-\mathscr{A}'\psi+\mathscr{B}_2\xi, \qquad \psi(h)=\mathscr{E}\xi(h)+\mathscr{T}\psi(0). \tag{18b}$$

This shows that the problem of determining analytically the optimal state and control becomes more and more difficult as h tends to zero, since dimensionality of the system (18a, b) tends to infinity.

Setting $h=0$ we obtain[5])

$$P(x,u,0)=\big(\dot{x}-(A_1+A_2)\,x-Bu,\; x(t)-\varphi(t),\; x(0)-\varphi(0)\big) \tag{19}$$

and the canonical equations, corresponding to the nondelayed model, are

$$\dot{x}_0=(A_1+A_2)\,x_0+S\eta_0, \qquad x(0)=\varphi(0), \tag{20a}$$

$$\dot{\eta}_0=-(A_1+A_2)'\,\eta_0+Qx_0, \qquad \eta(T)=-Fx_0(T). \tag{20b}$$

The optimal control

$$u_0(t)=R^{-1}B'\eta_0(t) \tag{21}$$

can be presented in a form of a feedback controller

$$u_0(t)=R^{-1}B'K(t)\,x_0(t) \tag{22}$$

where $K(t)$ is a symmetric, negative semidefinite matrix, which satisfies the Riccati equation

$$\dot{K}=-K(A_1+A_2)-(A_1+A_2)'K-KSK+Q, \qquad K(T)=-F, \tag{23}$$

and

$$\eta_0(t)=K(t)\,x_0(t). \tag{24}$$

[5]) We have assumed the state space B_x does not change with the parameter. Therefore the initial function appears in the nondelayed equation, in form.

4. Sensitivity analysis of the time delay problem

The asymptotic solution of differential equations with time delay has been presented in [6], [7], [8]. However, these results can not be directly applied in the sensitivity approximation. Therefore, the following theorems are proven in the Appendix 2.

Consider first the homogeneous equation

$$\dot{x}(t)=A_1 x(t)+A_2 x(t-h), \quad x(t)=\varphi(t),\ t\in[-h,0]. \tag{25}$$

THEOREM 2. The solution $x(t,h)$ of (25) has right-side derivative $X_h(t)$ with respect to h at $h=0$, which satisfies the equation

$$\dot{X}_h=(A_1+A_2)X_h-A_2\dot{x}_0 \qquad X_h(0)=0 \tag{26}$$

where $x_0(t)$ is the solution of the nondelayed equation

$$\dot{x}_0=(A_1+A_2)x_0, \qquad x_0(0)=\varphi(0). \tag{27}$$

Consider the canonical equations (15a, b)

THEOREM 3. The solution $(\hat{x}(t,h),\hat{\eta}(t,h))$ of (15a, b) has right-side derivative $(\hat{X}_h(t),\hat{N}_h(t))$ at $h=0$ which satisfies the equations

$$\dot{\hat{X}}_h=(A_1+A_2)\hat{X}_h+S\hat{N}_h-A_2\dot{\hat{x}}_0, \tag{28a}$$

$$\begin{gathered}\dot{\hat{N}}_h=-(A_1+A_2)'\hat{N}_h+Q\hat{X}_h-A_2'\dot{\hat{\eta}}_0,\\ \hat{X}_h(0)=0;\ \hat{N}_h(T)=-F\hat{X}_h(T)-A_2'\hat{\eta}_0(T),\end{gathered} \tag{28b}$$

where $\hat{x}_0$, $\hat{\psi}_0$ are solutions of the problem (20a, b).

Hence, the right-side derivative of the control

$$\hat{U}_h(t)=R^{-1}B'\hat{N}_h(t)=R^{-1}B'\left(K(t)\hat{X}_h(t)+M(t)\right) \tag{29}$$

where $M(t)$ satisfies the equation

$$\begin{gathered}\dot{M}=-(KS+A_1'+A_2')M+KA_2\dot{x}_0-A_2'\dot{\eta}_0,\\ M(T)=-A_2'\eta_0(T).\end{gathered} \tag{30}$$

In terms of sensitivity analysis $\hat{X}_h$, $\hat{U}_h$, $\hat{N}_h$ are the basic sensitivity functions. Having the existence of right-side derivatives, the Lipshitz-continuity of these derivatives for $h\geqslant 0$ can be easily proven.

According to Section 2, the relation (29) represents exactly the linear part of the approximation of the optimal feedback control law at $h=0$[6]).

In order to determine the structural variations, consider the following situation. Assume that delay parameter h is zero in the process and in the model the delay

[6]) The differentiability and Lipshitz-continuity of the operator $R^1(x,h)$ in $[0,h_m]$ can be directly proven on the basis of [4].

parameter changes from 0 to h. The structural variations are computed at the point $h=0$ and we obtain non-delayed equations for the structural variations. According to the Theorem 1 this approach is locally equivalent to the reverse situation which appears in practice.

Under this assumption we consider several well-known optimal structures. The structural sensitivity functions are obtained immediately from the general sensitivity models, presented in [1].

(0) Open-loop structure

The optimal control is applied in a control system in the same way as it is determined, hence $X^0(a, \alpha)=x(\alpha, \hat{u}(a))$; $U^0(a, \alpha)=\hat{U}(a)$ and $U_h^0(t)=\hat{U}_h(t)$. The structural state sensitivity function satisfies the equation

$$\dot{X}_h^0=(A_1+A_2)\,X_h^0+B\hat{U}_h, \qquad X_h(0)=0. \tag{31}$$

(1) Closed-loop structure

The optimal control law is synthesized on the basis of a model with delay and its linear approximation is given by (29). Hence, the structural sensitivity functions

$$\dot{X}_h^1=(A_1+A_2)\,X_h^1+BU_h^1, \tag{32a}$$

$$U_h^1=R^{-1}B'KX_h^1+R^{-1}B'M. \tag{32b}$$

(2) Optimal trajectory tracking

If the matrix B has an inverse and a nondelayed state is observable, the optimal trajectory tracking structure (or model-following structure) can be applied — see Fig. 1. The state of the real process is equal to the optimal state for the model; hence

$$X_h^2(t)=\hat{X}_h(t) \tag{33a}$$

and

$$U_h^2(t)=B^{-1}(\dot{X}_h^2-(A_1+A_2)\,X_h^2) \tag{33b}$$

or, a more useful formula is

$$U_h^2=\hat{U}_h+B^{-1}A_2\,\hat{x}_0. \tag{33c}$$

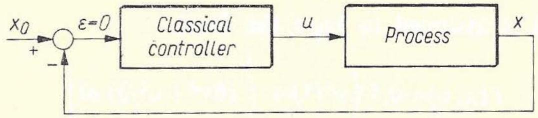

Fig. 1. Optimal trajectory tracking system

(3) Optimizing feedback

Assume that $\hat{\eta}(t)$ can be computed on the basis of the model. The optimal control can be determined by a peak-holding controller, performing the operation

$$\max_u \left(-0.5\, u' R u + \hat{\eta}' B u\right). \tag{34}$$

Such a structure — see [9] — is called the (open-loop) optimizing feedback. Because the function which is maximized at each instant of time does not depend on h explicitly, this structure has the same sensitivity as the classical open-loop structure.

The adjoint variable can be determined as well in the closed loop, as a function of the state, and then we obtain a closed-loop optimizing feedback structure. Analogously, this structure is also as sensitive as the classical closed-loop structure.

Because the process is linear, the second-order term of the sensitivity measure approximation has the form:

$$S^i(h) = 0.5 \left(X_h^{i\prime}(T)\, F X_h^i(T) + \int_0^T (X_h^{i\prime} Q X_h^i + U_h^{i\prime} R U_h^i)\, dt \right) h^2 + o(h^2). \tag{35}$$

The linear variation of the optimal value of the performance functional can also be easily computed

$$\delta \hat{I} = \hat{x}_0'(T)\, F\, \delta x + \int_0^T \hat{x}_0' Q\, \delta x\, dt. \tag{36a}$$

where

$$\delta \dot{x} = (A_1 + A_2)\, \delta x - A_2\, \hat{\dot{x}}_0\, h, \qquad \delta x(0) = 0. \tag{36b}$$

5. An example

To clarify the methodology of the sensitivity analysis and to illustrate the conclusions we take now a simple example.

Consider the process presented in Fig. 2:

$$\dot{x}(t) = -x(t) + x(t-h) + u(t), \qquad x(t) = 1, \; t \in [-h, 0].$$

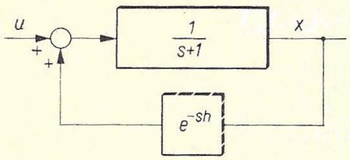

Fig. 2. Process considered in the example

The cost functional is assumed to have the form:

$$I(x, u) = 0.5 \left(x^2(1) + \int_0^1 (b x^2 + u^2/b)\, dt \right)$$

where b is a positive parameter.

Setting $h=0$ we obtain

$$\dot{x}_0=u, \qquad x_0(0)=1$$

and the optimal solution

$$\hat{x}_0(t)=\mathrm{e}^{-bt}, \qquad \hat{u}_0(t)=-b\,\mathrm{e}^{-bt},$$

where $K(t)=-1$.

The basic sensitivity functions satisfy the equations:

$$\dot{\hat{X}}_h=b\hat{N}_h+B\,\mathrm{e}^{-bt}, \qquad \hat{X}_h(0)=0,$$

$$\dot{\hat{N}}_h=b\hat{X}_h-b\,\mathrm{e}^{-bt}, \qquad \hat{N}_h(1)=-\hat{X}_h(1)+\mathrm{e}^{-b}.$$

Hence $M(t)=\mathrm{e}^{b(t-2)}$ and

$$\hat{X}_h(t)=0.5\,\mathrm{e}^{-2b}(\mathrm{e}^{bt}-\mathrm{e}^{-bt})+bt\,\mathrm{e}^{-bt},$$

$$\hat{U}_h(t)=b\left(-\hat{X}_h(t)+\mathrm{e}^{b(t-2)}\right).$$

The structural sensitivity functions of the considered structures are:

$$X_h^0=\hat{X}_h+\mathrm{e}^{-bt}-1,$$

$$U_h^1=-bX_h^1+b\,\mathrm{e}^{b(t-2)},$$

$$X_h^1=\hat{X}_h-bt\,\mathrm{e}^{-bt},$$

$$U_h^2=\hat{U}_h+b\,\mathrm{e}^{-bt}.$$

In Fig. 3 the behaviour of the second-order approximation of the sensitivity measure and the performance linear variation for $h=0.1$ are shown. If $b>1$ than the performance losses for the closed-loop structure are close to zero. The closed-loop is the least sensitive structure considered. Moreover, the feedback optimal controller, based on the nondelayed model, can be very easily constructed — see Fig. 4.

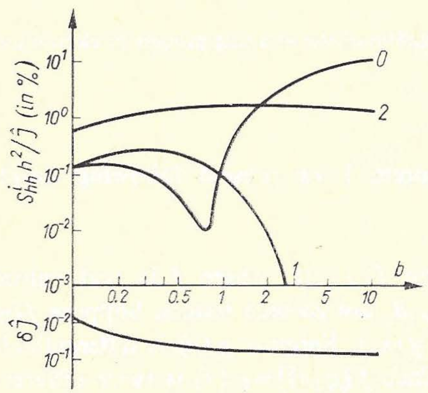

Fig. 3. Second-order sensitivity approximation and ideal performance variation for $h=0.1$ versus b for systems considered in the example

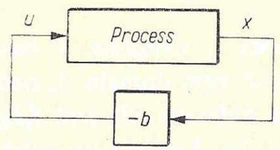

Fig. 4. Optimal feedback control system for the example

6. Conclusions

The method, presented in this paper results from the general theory of sensitivity. Solving the time delay optimization problem is usually extremely difficult but can be mitigated. The performance losses due to neglecting the time delay can be rather easily estimated. For instance, the local sensitivity analysis of the closed-loop systems requires less effort then the global synthesis. The example presented shows that the performance losses can be very small. Moreover, the sensitivity analysis allows the comparison of different optimal control structures.

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Appendix 1. In order to prove the Theorem 1 we present following simple lemmas.

Lemma 1. Consider a composite function $f(g(x))$, where f is real valued functional with domain B_g and $g: B_x \to B_g$; B_x, B_g are Banach spaces. Suppose $f(g)$ is twice differentiable and $f_g(g_0)=0$ for $g_0=g(x_0)$. Suppose $g(x)$ is differentiable and $g_x(x)$ is Lipshitz-continuous in $x=x_0$. Then $f(g(x))=\varphi(x)$ is twice differentiable with respect to x and

$$\varphi_{xx}(x_0)=g_x^*(x_0)\,f_{gg}(g_0)\,g_x(x_0). \tag{1.1}$$

Proof. Obviously $\varphi(x)$ is differentiable with respect to x.

$$\Delta\varphi_x = \varphi_x(x_0+\Delta_x) - \varphi_x(x_0) = f_g(g_0+\Delta g) \circ g_x(x_0+\Delta x) =$$

$$= f_g(g_0+\Delta g)\,[g_x(x_0+\Delta x) - g_x(x_0)] + f_g(g_0+\Delta g)\,g_x(x_0).$$

Since the differentiability of $f_g(g)$ and Lipshitz-continuity of $g_x(x)$ we have

$$\lim_{\|\Delta x\|\to 0} \|f_g(g_0+\Delta g)\,[g_x(x_0+\Delta x) - g_x(x_0)]\| / \|\Delta x\| = 0.$$

Because of the composite function theorem

$$\langle \Delta\varphi_x, \Delta x\rangle = \langle f_g(g_0+\Delta g), g_x\,\Delta x\rangle = \langle f_g(g_0)\,g_x(x_0)\,\Delta x, g_x(x_0)\,\Delta x\rangle + 0\,(\|\Delta x\|^2).$$

Hence

$$\Delta\varphi_x = g_x^* f_{gg}\,g_x\,\Delta x + 0\,(\|\Delta x\|)$$

and the proof is complete.

Lemma 2. Consider a function $f(x, y)$, $x, y \in [a_0, a_1]$. Let $f(x, y) = 0$ for each $x = y$ and $f(x, y)$ be differentiable with respect to x in a neighborhood of $x = y$, $y \in [a_0, a_1]$ (with one-side derivatives at a_0, a_1). Let $f_x(x, y)$ be continuous with respect to x, y for $x = y$. Then $f(x, y)$ is also differentiable with respect to y for $x = y$ and for each $x = y$ we have

$$f_x(x, y) = -f_y(x, y). \tag{1.2}$$

Proof.

$$\frac{f(y, y+\Delta y)}{\Delta y} = -\frac{f(y+\Delta y, y+\Delta y) - f(y, y+\Delta y)}{\Delta y} =$$

$$= -\frac{fx\,(y+\Theta\Delta y, y+\Delta y)\,\Delta y}{\Delta y} = -f_x(y, y) + 0\,(\Delta y)\ ^{7)}$$

because the uniform continuity of f_x. Hence (1.2) holds.

Let us now prove the Theorem 1. The sensitivity measure can be presented in a form

$$S^i(a, \alpha) = L\,(X^i(a, \alpha), U^i(a, \alpha), \hat{N}(\alpha), \alpha) - L\,(\hat{X}(\alpha), \hat{U}(\alpha), \hat{N}(\alpha), \alpha)$$

where $a, \alpha \in [a_0, a_1]$. Since the optimality

$$S_a^i(\alpha, \alpha) = \hat{L}_x(\alpha)\,X_a^i(\alpha) + \hat{L}_u\,U_a^i(\alpha) = 0.$$

Because of the lemma 2 there exists $S_\alpha^i = -S_a^i$ for $a = \alpha$ and the first part of the theorem is proven. Apparently, S_a^i can also denote the right-side derivative at $a = \alpha = a_0$.

Consider now $S_a^i(a, \alpha)$. Since the assumptions of the theorem and the Lemma 1 there exists S_{aa}^i for $a = \alpha$. But $S_a^i(a, \alpha) = 0$ for $a = \alpha$ and the structural derivatives

[7]) $0\,(\Delta y) \xrightarrow[\Delta y \to 0]{} 0$.

have continuous derivatives with respect to a, α for $a=\alpha$. Then the Lemma 2 can be applied and there exists $S^i_{a\alpha}=-S^i_{aa}$ for each $a=\alpha\in[a_0, a_1]$.

Let $a, \alpha\in[a_0, a_1]$ and $\delta a=\alpha-a$.

$$S^i(a, \alpha)=0.5\left(\langle X^{i*}_a(\alpha_1), \hat{L}_{xx}(\alpha_1)\, X^i_a(\alpha_1)\rangle+\right.$$
$$\left.+2\langle X^{i*}_a(\alpha_1), \hat{L}_{xu}(\alpha_1)\, U^i_a(\alpha_1)\rangle+\langle U^{i*}_0(\alpha_1), \hat{L}_{uu}(\alpha_1)\, U^i_a(\alpha_1)\rangle\right)\delta a^2$$

where $\alpha_1=a+\Theta\delta a$.

Since the continuity

$$X^i_a(\alpha_1)=X^i_a(a)+0\,(\delta a)=X^i_a(\alpha)+0\,(\delta a),$$
$$\hat{L}_{xx}(\alpha_1)=\hat{L}_{xx}(a)+0\,(\delta a)=\hat{L}_{xx}(\alpha)+0\,(\delta a)$$

and we have

$$S^i(a, \alpha)=D\,(a)\,\delta a^2+0\,(\delta a^2)=S^i_{aa}(\alpha)+0\,(\delta a^2).$$

Hence, there exists $S^i_{\alpha\alpha}=D$ for $a=\alpha$ and $S^i_{aa}=S^i_{\alpha\alpha}$. Since the continuity this equality holds also for the one-side derivatives and the approximation (10) can be performed.

Appendix 2. We prove first the Theorem 2. Consider the equations (25) and (27).

$$x\,(t, h)=e^{A_1 t}\,\varphi\,(0)+\int_{-h}^{0} e^{A_1(t-\tau-h)}A_2\,\varphi\,(\tau)\,d\tau+$$
$$+\int_{0}^{t} e^{A_1(t-\tau-h)}A_2\,x\,(\tau, h)\,d\tau-\int_{t-h}^{t} e^{A_1(t-\tau-h)}A_2\,x\,(\tau, h)\,d\tau,$$

$$x_0\,(t)=e^{A_1 t}\varphi\,(0)+\int_{0}^{t} e^{A_1(t-\tau)}A_2\,x_0\,(\tau)\,d\tau.$$

Since the continuity of $\varphi\,(x)$

$$\lim_{h\to 0+}\frac{1}{h}\int_{-h}^{0} e^{A_1(t-\tau-h)}A_2\,\varphi\,(\tau)\,d\tau=e^{A_1 t}A_2\,\varphi\,(0)$$

and the continuity of $x\,(t, h)$ — see [8]

$$\lim_{h\to 0+}\frac{1}{h}\int_{-h}^{0} e^{A_1(t-\tau-h)}A_2\,x\,(\tau, h)\,d\tau=A_2\,x_0\,(t). \tag{2.1}$$

Using the Taylor expansion

$$e^{-A_1 h}=1-A_1\,h+0\,(h) \tag{2.2}$$

and denoting $\Delta x\,(t, h)=(x\,(t, h)-x_0\,(t))\,/\,h$ we obtain

$$\Delta x\,(t, h)=e^{A_1 t}A_2\,\varphi\,(0)+\int_{0}^{t} e^{A_1(t-\tau)}A_2\,\Delta x\,(\tau, h)\,d\tau-A_2\,x_0\,(t)-$$
$$-A_1\int_{0}^{t} e^{A_1(t-\tau)}A_2\,x_0\,(\tau)\,d\tau+0\,(h).$$

But

$$A_1 \int_0^t e^{A_1(t-\tau)} A_2 x_0(\tau)\, d\tau = A_2 x_0(t) - e^{A_1 t} A_2 x_0(0) + \int_0^t e^{A_1(t-\tau)} A_2 \dot{x}_0(\tau)\, d\tau .$$

We have

$$\Delta x(t, h) = \int_0^t e^{A_1(t-\tau)} A_2 \Delta x(\tau, h) - \int_0^t e^{A_1(t-\tau)} A_2 \dot{x}_0(\tau)\, d\tau + 0(h). \tag{2.3}$$

Hence, for every $t \in [0, T]$

$$\lim_{h \to 0+} \Delta x(t, h) = X_n(t)$$

which is defined by (26).

In order to prove the Theorem 3 we use analogously (2.1), (2.2) for the equation (15b) and obtain

$$\eta(t, h) = -e^{-A_1'(T-t)} F x(T, h) - \int_T^t e^{-A_1'(\tau-t)} A_2' \hat{\eta}_0(\tau)\, d\tau -$$

$$- A_1 h \int_T^t e^{-A_1'(\tau-t)} A_2' \hat{\eta}_0(\tau)\, d\tau + \int_T^t e^{-A_1'(\tau-t)} Q x(\tau, h)\, d\tau -$$

$$- h A_2 \hat{\eta}_0(t) + 0(h).$$

Hence, denoting $\Delta\eta(t, h) = (\eta(t, h) - \hat{\eta}_0(t))/h$

$$\Delta\eta(t, h) = e^{-A_1'(T-t)} F \Delta x(T, h) - A_2 \hat{\eta}_0(t) + A_1' \int_T^t e^{-A_1'(\tau-t)} A_2 \hat{\eta}_0(\tau)\, d\tau +$$

$$+ \int_T^t e^{-A_1'(\tau-t)} Q \Delta x(\tau, h)\, d\tau + 0(h). \tag{2.4}$$

But

$$A_1' \int_T^t e^{-A_1'(\tau-t)} \Delta_2 \hat{\eta}_0(\tau)\, d\tau = A_2' \hat{\eta}_0 T)) - e^{-A_1'(T-t)} A_2 \hat{\eta}_0(T) -$$

$$- \int_T^t e^{-A_1'(\tau-t)} A_2 \hat{\dot{\eta}}_0(\tau)\, d\tau . \tag{2.5}$$

Combining (2.3—5) we have

$$\Delta x(t, h) = \int_0^t e^{A_1(t-\tau)} (\Delta_2 \Delta x(t, h) + S \Delta\eta(t, h) - A_2 \hat{\dot{x}}_0(\tau))\, d\tau + 0(h),$$

$$\Delta\eta(t, h) = -(F \Delta x(T, h) - A_2 \hat{\eta}_0(T))\, e^{-A_1'(T-t)} -$$

$$- \int_T^t e^{-A_1'(\tau-t)} (-Q \Delta x(\tau, h) + A_2 \hat{\eta}_0(\tau))\, d\tau .$$

This implies (28a, b) when $h \to 0$.

Analiza wrażliwości układów sterowania optymalnego z małym opóźnieniem czasowym

Przedstawiono analizę wrażliwości różnych struktur układów sterowania optymalnego z małym opóźnieniem czasowym. Zaproponowano metodę obliczeniową dla przypadku liniowo-kwadratowego. Zastosowanie metody zilustrowano przykładem.

Анализ чувствительности систем оптимального управления с небольшим временным запаздыванием

В статье представлено анализ чувствительности различных структур систем оптимального управления с небольшим временным запаздыванием. Предлагается расчетный метод для линейно-квадратного случая.

Применение метода проиллюстрировано на примере.