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On discrete approximation for a class of sets of admissible control

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The problem of discrete approximation for closed and convex sets of admissible control is formulated. It is shown that for a broad class of such sets the discrete approximation exists and is determined uniquely.

Two types of discretization are considered. For each of them the method of constructing of approximations of sets of admissible control is presented. Such construction requires solving of a two-points boundary-value problem for an ordinary differential equation.
The obtained results are illustrated by examples.

1. Introduction

For majority of optimal control problems for continuous systems it is impossible to find optimal control analytically, mostly due to difficulties connected with solving of state equation.

Therefore instead of solving initial problem of this type, we approximate it by another finite dimensional problem, which can be solved using a computer.

The solutions of such approximating problems depend on some parameter of discretization τ (it can be a vector) which is destinated to tends to zero. It is said that the approximation is formulated properly if the solutions of approximating problems are convergent in some sense to the solution of initial one for $\tau \to 0$.

In order to construct an approximating problem for a given problem of optimal control, subject to constraints of control functions, it is necessary

1) to approximate the state equation

2) to approximate control functions and the set of admissible control.

First of this tasks can be performed using one of many known methods of approximate solving of state equation (it may be differential equation: ordinary or partial or functional equation) for example finite elements or finite differences methods (see e.g. [1], [6], [8]).

On the other hand the author do not know any paper concerning the problem of approximating sets of admissible control. This problem is trivial for simple cases of such sets and becomes important for more complex cases.

Generally speaking the problem of approximating of given set of admissible control U_{ad} can be formulated as follows: introducing the discretization of the state equation we introduce also some approximation of control functions. Often such approximating functions are less regular then admissible controls (for example continuous controls can be approximated by piece-wise constant ones).

We would like to construct the set $V_{\tau ad}$ approximating U_{ad} in such a way that

1) the approximation of each element of U_{ad} (i.e. of each admissible control) belongs to $V_{\tau ad}$
2) for each element of $V_{\tau ad}$ there exists an admissible control (an element of U_{ad}) close, in some sense, to this element.

More detailed discussion of these condition can be found in [5], [7]. Here we restrict ourself to heuristic statement that condition 1) assures that the set $V_{\tau ad}$ is rich enough, while condition 2) assures that having the solution of properly formulated approximating problem we can construct an admissible control, which converges to optimal one for $\tau \to 0$. For τ small enough such a control function can be treated as a good approximation of optimal control.

This paper is devoted to the problem of constructing of sets $V_{\tau ad}$ for a class of sets U_{ad} in the cases where discrete approximation is used.

2. Problem statement and method of solution

In the sequel we shall assume that the controls are scalar functions belonging to the space

$$U = L^2(0, T)$$

where $(0, T)$ is a fixed interval of time.

The set of admissible control is given by

$$U_{ad} = \Big\{ u \in H^r(0, T) : r \geq 1,\ u^{(m)}(0) = 0,\ 0 \leq m \leq p \leq r,\ J(u) = \int_0^T \boldsymbol{u}^T(t)\, A \boldsymbol{u}(t)\, dt \leq 1 \Big\} \quad (2.1)$$

where r — is a given integer, $H^r(0, T)$ — is the Sobolev space [10] of functions square summable together with all its derivative up to r-th (the derivatives are understood in the sense of distributions). Recall that

$$H^r(0, T) \subset C^{r-1}(0, T).$$

$\boldsymbol{u}^T(t) = [u(t), u^{(1)}(t) \ldots, u^{(r)}(t)]$ is a $(r+1)$ — dimensional vector of values at t of the function u and its r derivatives. $A = [a_{ij}]$, $i, j = 0, 1, \ldots r$ is $(r+1) \times (r+1)$ — dimensional symmetric matrix, non-negatively definite.

It is assumed that the quadratic form $J(u)$ satisfies condition

$$J(u) \geqslant \alpha \|u\|_r^2 \qquad \alpha > 0 \qquad \forall u \in H_0^r(0, T) \tag{2.2}$$

where $\|\cdot\|_r$ denotes the norm in $H^r(0, T)$ and closed subspace $H_0^r(0, T) \subset H^r(0, T)$ is given by

$$H_0^r(0, T) = \{u \in H^r(0, T) : u^{(m)}(0) = 0;\ 0 \leqslant m \leqslant p\}. \tag{2.2a}$$

The condition (2.2) implies

$$a_{rr} > 0. \tag{2.2b}$$

Note that the set U_{ad} is convex, bounded and closed both in $L^2(0, T)$ and $H^r(0, T)$ topologies.

Remark: if the initial conditions are not imposed on functions u then $H_0^r(0, T) = H^r(0, T)$.

The controls are approximated by piece-wise constant functions, i.e. as the space V_τ approximating U we choose M-dimensional space $E_\tau(0, T)$ of functions of the form

$$v(t) = \sum_{i=0}^{M-1} v(i\tau) W_i(t) \tag{2.3}$$

where $M\tau = T$ and $W_i(t)$ is the characteristic function of interval $[i\tau, (i+1)\tau)$.

On the space $H^r(0, T)$ there is defined a linear operator $\mathscr{R}_\tau$ mapping $H^r(0, T)$ onto V_τ. We shall consider two types of operators given respectively by

$$\mathscr{R}_{1\tau} u(t) = v_1(t) = \sum_{i=0}^{M-1} u(i, \tau) W_i(t) \tag{2.4a}$$

$$\mathscr{R}_{2\tau} u(t) = v_2(t) = \sum_{i=0}^{M-1} \frac{1}{\tau} \int_{i\tau}^{(i+1)\tau} u(t)\, dt\, W_i(t). \tag{2.4b}$$

We are looking for a closed convex and bounded set $V_{\tau\text{ad}} \subset E_\tau(0, T)$ which approximates U_{ad} in the following sense

$$\forall u \in U_{\text{ad}} : \mathscr{R}_\tau u \in V_{\tau\text{ad}} \tag{2.5a}$$

$$\forall v \in V_{\tau\text{ad}}\ \exists u \in U_{\text{ad}} : \mathscr{R}_\tau u = v. \tag{2.5b}$$

It is easy to see that conditions (2.5) imply in particular

$$\forall u \in U_{\text{ad}}\ \exists v \in V_{\tau\text{ad}} : \|u - v\|_0 = 0(\tau) \tag{2.6a}$$

$$\forall v \in V_{\tau\text{ad}}\ \exists u \in U_{\text{ad}} : \|u - v\|_0 = 0(\tau) \tag{2.6b}$$

where $\|\cdot\|_0$ denotes the norm in $L^2(0, T)$ and $0(\tau)$ denotes a function which tends to zero at least as fast as τ.

Note that conditions (2.6) are necessary to estimate the rate of convergence of approximation for a class of optimal control problems [5], [7].

Let us denote

$$E_\tau^0(0, T) = \mathscr{R}_\tau H_0^r(0, T). \tag{2.7}$$

It is obvious that $E_\tau^0(0, T)$ is a closed space. In the case where $\mathscr{R}_\tau = \mathscr{R}_{2\tau}$ we have $E_\tau^0(0, T) = E_\tau(0, T)$ and in the case $\mathscr{R}_\tau = \mathscr{R}_{\tau 1}$ the space $E_\tau(0, T)$ is the $(M-1)$ — dimensional subspace of $E_\tau(0, T)$ of functions v satisfying initial condition $v(0) = 0$.

For each element $v \in E_\tau^0(0, T)$ we define the element u_v satisfying the conditions

$$u_v \in \mathscr{R}_\tau^{-1}(v), \quad J(u_v) \leqslant J(u) \quad \forall u \in \mathscr{R}_\tau^{-1}(v) \tag{2.8}$$

where the set $\mathscr{R}_\tau^{-1}(v) \subset H^r(0, T)$ denotes the counterimage of the element v restricted to the subspace $H_0^r(O, T)$.

The element u_v satisfying (2.8) exists and is defined uniquely. Indeed $\mathscr{R}_\tau^{-1}(v)$ is the set of all functions $u \in H_0^r(0, T)$ satisfying the finite number of additional linear conditions: $M-1$ — conditions $u(i\tau) = v(i\tau)$, $i = 1, 2, \ldots, M-1$ in the case $\mathscr{R}_\tau = \mathscr{R}_{1\tau}$ or M — conditions $\frac{1}{\tau} \int\limits_{i\tau}^{(i+1)\tau} u(t)\, dt = v(i\tau)$, $i = 1, 2, \ldots, M$ in the case $\mathscr{R}_\tau = \mathscr{R}_{2\tau}$. Therefore $\mathscr{R}_\tau^{-1}(v)$ is a closed subspace with codimension $(M-1)$ or M.

By (2.2) the functional $J(u)$ is strictly convex and radially unbounded on $H_0^r(0, T)$ hence it assumes the unique minimum u_v on the closed subspace $\mathscr{R}_\tau^{-1}(v)$ [12].

Note that

$$U_\tau = \{u_v : v \in E_\tau^0(0, T)\} \tag{2.9}$$

is a subspace of $H_0^r(0, T)$ — $(M-1)$-dimensional in the case of operator $\mathscr{R}_{1\tau}$ and M-dimensional in the case of $\mathscr{R}_{2\tau}$.

Indeed an element u_v satisfies (2.8) if

$$J(u_v) \leqslant J(u_v + u) \quad \forall u \in \mathscr{R}^{-1}(\theta) \tag{2.10}$$

where θ is the zero element of $E_\tau^0(0, T)$.

The set $\mathscr{R}_\tau^{-1}(\theta) \subset H_0^r(0, T)$ is the subspace with codimension $(M-1)$ (for $\mathscr{R}_{1\tau}$) or M (for $\mathscr{R}_{2\tau}$) hence taking into consideration the form (1.1) of $J(u)$ we conclude that the condition (2.10) is equivalent to

$$\int\limits_0^T u^T(t)\, A u_v(t)\, dt = 0 \quad \forall u \in \mathscr{R}_\tau^{-1}(\theta). \tag{2.11}$$

It follows from (2.2) and from the form of $\mathscr{R}_\tau^{-1}(\theta)$ that the set U_τ characterized by (2.11) is really a $(M-1)$ or M dimensional subspace.

Let $\bar{\mathscr{R}}_\tau$ denotes the restriction of $\mathscr{R}_\tau$ to the subspace U_τ. It is a linear operator which maps U_τ onto the space $E_\tau^0(0, T)$ of the same dimension, hence it has the inverse $\bar{\mathscr{R}}_\tau^{-1}$ and

$$u_v = \bar{\mathscr{R}}^{-1}(v). \tag{2.12}$$

The above results enable us to construct the approximation of U_{ad}. Namely we put

$$V_{\tau ad} = \{v \in E_\tau^0(0, T) : u_v \in U_{ad}\} = \{v \in E_\tau^0(0, T) : J(u_v) \leqslant 1\}. \tag{2.13}$$

It is easy to see that the set (2.13) satisfies (2.5). Indeed, let $\bar{u}$ be any arbitrary element of the set U_{ad}, i.e. $J(\bar{u})\leqslant 1$. Let us put $\bar{v}=\mathscr{R}_\tau \bar{u}$. Taking into account (2.8) we get

$$J(u_{\bar{v}})\leqslant J(\bar{u})\leqslant 1,$$

hence $\bar{v}\in V_{\tau ad}$ and (2.5a) is satisfied. Condition (2.5b) is obviously satisfied for $u=u_v$.

Moreover the set (2.13) is the only set satisfying (2.5). Indeed it is easy to see that the set smaller then (2.13) can not satisfy (2.5a). We shall show that none set larger than (2.13) can satisfy (2.5b). Assume that $\hat{v}$ does not belong to the set (2.13), i.e. $J(u_{\hat{v}})>1$, then from (2.8) we have

$$1<J(u_{\hat{v}})\leqslant J(u) \qquad \forall u\in \mathscr{R}^{-1}(\hat{v})$$

and for $\hat{v}$ condition (2.5b) is not satisfied. This completes the proof of the uniqueness of $V_{\tau ad}$.

Taking advantage of (2.12) we can rewrite the definition (2.13) of the set $V_{\tau ad}$ in the form

$$V_{\tau ad}=\{v\in E_\tau(0, T): J(\bar{\mathscr{R}}_\tau^{-1}(v))\leqslant 1\}. \tag{2.14}$$

Note that it follows from (2.1) and (2.2) that $J(\bar{\mathscr{R}}^{-1}(v))$ is a quadratic form positive definite on $E_\tau^0(0, T)$, hence the set $V_{\tau ad}$ given by (2.14) is closed, convex and bounded. Therefore it is the required approximation of the set U_{ad}.

The above results are summarized in the following.

THEOREM 1. If the set of admissible control U_{ad} is given by (2.1) and the condition (2.2) is satisfied, then there exists the uniquely defined closed convex and bounded approximating set $V_{\tau ad}$, which satisfies conditions (2.5). This set is given by (2.14).

It follows from Theorem 1, that the construction of the set $V_{\tau ad}$ can be reduced to determination of the operator $\bar{\mathscr{R}}_\tau^{-1}$. It turns out that this last problem is not a simple one.

The next parts of the paper are devoted to determination of operator $\bar{\mathscr{R}}_\tau^{-1}$. The cases where $\mathscr{R}_\tau$ is given by (2.4a) and (2.4b) are considered successively.

3. Construction of the set $V_{\tau ad}$ in the case $\mathscr{R}_\tau-\mathscr{R}_{1\tau}$

To find the operator $\bar{\mathscr{R}}_\tau^{-1}$ we shall use the formula (2.11) characterizing elements u_v.

In our case the set $\mathscr{R}_\tau^{-1}(\theta)$ is given by

$$\mathscr{R}_{1\tau}^{-1}(\theta)=\{u\in H^r(0, T): u(i\tau)=0,\ i=0, 1, ..., M-1,$$
$$u^{(m)}(0)=0,\ m=1, 2, ,..., p\}. \tag{3.1}$$

Rewriting integral (2.11) in the form of the sum of integrals over the intervals $[i\tau, (i+1)\tau)$ and performing elementary transformations, we get

$$\sum_{i=0}^{M-1} \sum_{j=0}^{r} \int_{i\tau}^{(i+1)\tau} a_j^T u_v(t)\, u^{(j)}(t)\, dt = 0 \tag{3.2}$$

where a_j^T denotes the $(j+1)$-st verse of the matrix A.

Let us assume arbitrarily that the function $u_v \in H^r(0, T)$ is more regular, namely that on each interval $(i_\tau, (i+1)_\tau)$ it is of the class C^{2r}.

Each integral in (3.2) we integrate by parts j times. After some elementary transformation we get for the interval $(i\tau, (i+1)\tau)$

$$\sum_{j=0}^{r} \int_{i\tau}^{(i+1)\tau} a_j^T u_v(t)\, u^{(j)}(t)\, dt =$$

$$= \left[\sum_{k=0}^{r-1} \sum_{j=k+1}^{r} (-1)^{j-k-1} a_j^T u_v^{(j-k-1)}(t)\, u^{(k)}(t)\right]\Bigg|_{i\tau}^{(i+1)\tau} +$$

$$+ \int_{i\tau}^{(i+1)\tau} \left[\sum_{j=0}^{r} (-1)^j a_j^T u_v^{(j)}(t)\right] u(t)\, dt.$$

Hence the condition (3.2) takes on the form

$$\sum_{i=0}^{M-1} \left\{ \left[\sum_{k=0}^{r-1} \sum_{j=k+1}^{r} (-1)^{j-k-1} a_j^T u_v^{(j-k-1)}(t)\, u^{(k)}(t)\right]\Bigg|_{i\tau}^{(i+1)\tau} + \right.$$

$$\left. + \int_{i\tau}^{(i+1)\tau} \left[\sum_{j=0}^{r} (-1)^j a_j^T u_v^{(j)}(t)\right] u(t)\, dt \right\} = 0, \qquad \forall u \in \mathscr{R}_{1\tau}^{-1}(\theta). \tag{3.2a}$$

Taking into account definition (3.1) of the set $\mathscr{R}_{1\tau}^{-1}(\theta)$ as well as the fact that functions $u_v(t)$ are of the class $C^{r-1}(0, T)$ we find, that the condition (3.2a) is equivalent to the following system of equations

$$\sum_{j=0}^{r} (-1)^j a_j^T u_v^{(j)}(t) = 0, \qquad t \in (i\tau, (i+1)\tau), \qquad i = 0, 1, \ldots, M-1 \tag{3.3}$$

$$\sum_{j=k+1}^{r} (-1)^{j-k-1} a_j^T u_v^{(j-k-1)}(0) = 0 \qquad k = p+1, \ldots, r-1 \tag{3.3a}$$

$$\sum_{j=k+1}^{r} (-1)^{j-k-1} a_j^T \delta u_v^{(j-k-1)}(i\tau) = 0; \qquad i = 1, 2, \ldots, M-1, \quad k = 1, 2, \ldots, r-1 \tag{3.3b}$$

$$\sum_{j=k+1}^{r} (-1)^{j-k-1} a_j^T u_v^{(j-k-1)}(T) = 0 \qquad k = 0, 1, \ldots, r-1 \tag{3.3c}$$

where $\delta u_v(i\tau) = u_v(i\tau_+) - u_v(i\tau_-)$ denotes the difference between right and left limits of the function u_v at the point $i\tau$.

Using definitions of vectors a_j^T and $u_v^{(j)}(t)$, the initial conditions of u_v and taking advantage of symmetry of matrix A after tedious transformations we rewrite (3.3) in the form

$$\sum_{j=0}^{r} b_{2j}\, u_v^{(2j)}(t)=0; \qquad t\in(i\tau,(i+1)\tau); \qquad i=0,1,\ldots,M-1 \tag{3.4}$$

where

$$b_{2j}=\sum_{k+l=2j}(-1)^k a_{kl} \tag{3.4a}$$

$$D\tilde{u}_v(0)=0 \tag{3.5}$$

$$F\,\delta\tilde{u}_v(i\tau)=0, \qquad i=1,2,\ldots,M-1 \tag{3.6}$$

$$G\tilde{u}_v(T)=0 \tag{3.7}$$

where

$$\tilde{u}_v^T(t)=\left[u_v(t),\,u_v^{(1)}(t),\,\ldots,\,u_v^{(2r-2)}(t),\,u_v^{(2r-1)}(t)\right].$$

The components of matrices $D=[d_{km}]$, $G=[g_{km}]$, $k=0,1,2,\ldots,r-1$; $m=0,1,\ldots,2r-1$ and of $F=[f_{km}]$, $k=1,2,\ldots,r-1$; $m=0,1,\ldots,2r-1$ are given respectively by

$$g_{km}=\begin{cases}\sum\limits_{j=\beta_{km}}^{\gamma_{km}}(-1)^{j-k-1}a_{j,m-j+k+1} & \text{for} \quad k+m<2r\\ & \text{for} \quad k+m\geq 2r\end{cases} \tag{3.8a}$$

where

$$\beta_{km}=\max\{k+1,\,m-r+k+1\}$$

$$\gamma_{km}=\min\{k+m+1,\,r\} \tag{3.8b}$$

$$f_{km}=g_{km} \qquad 1\leq k\leq r-1$$

$$d_{km}=\begin{cases}\delta_{km} & k\leq p\\ g_{km} & k>p\end{cases} \tag{3.8c}$$

Recall that we are looking for the function $u_v\in H^r(0,T)$ hence, for a function of class $C^{r-1}(0,T)$. This implies that

$$\delta u_v^{(m)}(i\tau)=0; \qquad i=1,2,\ldots,M-1; \qquad m=0,1,\ldots,r-1 \tag{3.9a}$$

Substituting (3.9a) to (3.6) and taking into account (3.8) we find that the system (3.6) reduces to the homogeneous system of $(r-1)$ equations with $(r-1)$ unknowns $\delta u^{(r)}(i\tau)$, $\delta u^{(r+1)}(i\tau)$, ..., $\delta u^{(2r-2)}(i\tau)$. The matrix of this system is a triangle matrix with elements on diagonal of the form $(-1)^{k-1}a_{rr}$. Hence taking into account (2.2b) we get from (3.6)

$$\delta u_v^{(m)}(i\tau)=0; \qquad i=1,2,\ldots,M-1; \qquad m=r,r+1,\ldots,2r-2. \tag{3.9b}$$

Summing up the above results we conclude that if the function u_v is of class C^{2r} on each interval $(i\tau,(i+1)\tau)$ then it must satisfy the following conditions

(i) on each interval $(i\tau, (i+1)\tau)$ the function u_v is a solution of homogeneous ordinary differential equation (3.4) of order $2r$

(ii) the initial conditions in the form of the system of equations (3.5) are satisfied

(iii) there are satisfied the conditions $u_v(i\tau)=v(i\tau)$ as well as smoothness condition in the form of requirement that the derivatives $u_v^{(m)}$, $m=1, 2, \ldots, 2r-2$ are continuous at the points $i\tau$, $i=, \ldots, M-1$, in other words $u_v \in C^{2r-2}(0, T)$

(iv) the final conditions in the form of the system of equations (3.7) are satisfied.

It is more convenient to rewrite (3.4) in the form of a normal system [2] of $2r$ equations of first order

$$\dot{\tilde{u}}_v(t)=Q\tilde{u}_v(t); \qquad t \in \big(i\tau, (i+1)\tau\big); \qquad i=0, 1, \ldots, M-1 \tag{3.10}$$

where Q is Frobenius matrix.

Note, that it follows from (iii) that all components of the vector $\tilde{u}_v^T$ except the last one have to be continuous at the points $i\tau$. The last component $u_v^{(2r-1)}$ can have jumps at these points.

Let us denote

$$c_i=\delta u_v^{(2r-1)}(i\tau)=u_v^{(2r-1)}(i\tau_+)-u_v^{(2r-1)}(i\tau_-); \qquad i=1, 2, \ldots, M-1 \tag{3.11}$$

Hence taking into account (3.11) and the form of matrix Q we conclude that in the whole interval $(0, T)$ u_v must satisfy the following inhomogeneous differential equation

$$\dot{\tilde{u}}_v(t)=Q\tilde{u}_v(t)+e_{2r-2}\sum_{i=1}^{M-1} c_i\,\delta(i\tau) \qquad t \in (0, T) \tag{3.12}$$

where

$e_j^T=[0, 0, \ldots, 0, 1, 0, \ldots, 0]$ is $2r$-dimensional vector, the $(j+1)$-st component of which is equal to 1 and the other to 0,

$\delta(t)$ — denotes Dirac's measure.

Any solution of (3.12) has the form

$$\tilde{u}_v(t)=\Phi(t)\left[\tilde{u}_v(0)+\int_0^t \Phi^{-1}(s)\,e_{2r-2}\sum_{i=1}^{M-1} c_i\,\delta(i\tau)\right]=$$

$$=\Phi(t)\left[\tilde{u}_v(0)+\sum_{i=1}^{M-1}\Phi^{-1}(i\tau)\,e_{2r-2}\,\mathbf{1}(t-i\tau)\,c_i\right] \tag{3.13}$$

where $\Phi(t)$ is fundamental matrix of solutions [2] of system (3.10) satisfying condition $\Phi(0)=I$, and

$$\mathbf{1}(t)=\begin{cases}1 & \text{for} \quad t>0\\ 0 & \text{for} \quad t\leqslant 0.\end{cases}$$

We are looking for solutions satisfying boundary conditions (3.5), (3.7).

These conditions impose some restrictions on vector $\tilde{u}_v(0)$. Indeed from (3.5) we have

$$D\tilde{u}_v(0)=0 \tag{3.14a}$$

and (3.7) together with (3.13) imply

$$G\Phi(T)\,\tilde{u}_v(0)+G\Phi(T)\sum_{i=1}^{M-1}\Phi^{-1}(i\tau)\,e_{2r-2}\,c_i=0. \tag{3.14b}$$

The system (3.14) we rewrite in the matrix form

$$C\tilde{u}_v(0)=\sum_{j=1}^{M-1}\xi_i\,c_i \tag{3.15}$$

where

$$C=\begin{bmatrix} D \\ G\Phi(T)\end{bmatrix} \tag{3.15a}$$

is $(2r\times 2r)$-dimensional matrix
and

$$\xi_i=\begin{bmatrix}\theta \\ \tilde{\xi}_i\end{bmatrix} \tag{3.15b}$$

is $2r$-dimensional vector, such that θ is r-dimensional zero vector and

$$\tilde{\xi}_i=-G\Phi(T)\,\Phi^{-1}(i\tau)\,e_{2r-2}. \tag{3.15c}$$

We shall assume that

$$\operatorname{rank} C=2r. \tag{3.16}$$

Then from (3.15) we obtain

Note that the condition $u_v(0)=v(0)=0$ is fulfilled for the initial conditions (3.5) are satisfied.

We can rewrite the system (3.20) in the matrix form

$$\boldsymbol{v}=Y\boldsymbol{c} \tag{3.21}$$

where $\boldsymbol{v}=[v(\tau), v(2\tau), ..., v((M-1)\tau)]$ is a $(M-1)$ — dimensional vector, and Y is a $(M+1)\times(M+1)$ — dimensional matrix, the verses of which are given by

$$y_k^T=\boldsymbol{\psi}^T(k\tau). \tag{3.21a}$$

If for a given vector $\boldsymbol{v}$ the equation (3.21) has a solution $\boldsymbol{c}$, then we can find the function u_v substituting $\boldsymbol{c}$ to (3.19).

We shall show that (3.21) has the solution for any v. Indeed, suppose that it is not so. It would mean that the matrix Y is singular, and therefore for some vectors (namely those for which Cappella's condition [11] is satisfied) equation (3.21) would have more than one solution $\boldsymbol{c}$. It would imply that to these vectors $\boldsymbol{v}$ (and hence to functions $v\in E_\tau(0, T)$) would correspond, according to (3.19), more then one function u_v. This is impossible since the operator $\bar{\mathscr{R}}_\tau^{-1}$ in (2.12) is unique.

Hence the matrix Y is nonsingular. Therefore for any arbitrary $\boldsymbol{v}$ we get from (3.19) and (3.21)

$$u_v(t)=\psi^T(t)\,Y^{-1}\boldsymbol{v}. \tag{3.22}$$

This formula defines the operator $\bar{\mathscr{R}}_{1\tau}^{-1}$ which we were looking for. The above results can be summarized in the following

THEOREM 2. If

(i) the set of admissible control is given by (2.1), where condition (2.2) is fulfilled

(ii) the operator $\mathscr{R}_\tau$ is given by (2.4a)

(iii) condition (3.16) is satisfied

then the set $V_{\tau \mathrm{ad}}$ approximating U_{ad} is given by (2.14), where the linear operator $\bar{\mathscr{R}}_\tau^{-1}$ is defined by (3.22).

Note that to verify condition (3.16) we have to know the fundamental matrix of solutions $\Phi(t)$. However for some cases it is possible to show that (3.22) takes place without verification of (3.16).

Let us assume that the elements d_{km}, g_{km} of matrices D and G satisfy conditions

$$\sum_{\nu=0}^{r-1}[d_{k,\nu}\,d_{m,2r-\nu-1}-d_{k,2r-\nu-1}\,d_{m,\nu}]=\sum_{\nu=0}^{r-1}[g_{k,\nu}\,g_{m,2r-\nu-1}-$$
$$-g_{k,2r-\nu-1}\,g_{m,\nu}];\quad k,m=0,1,...,r-1 \tag{3.23}$$

and let us consider ordinary differential operator

$$\Lambda(u)=\sum_{j=1}^{r}b_{2j}\,u^{(2j)}+(b_0+1)\,u \tag{3.24}$$

defined on the domain

$$\mathscr{D}(\Lambda)=\{u\in H^{2r}(0,T): Du(0)=0;\; Gu(T)=0\}. \tag{3.24a}$$

Since it is assumed that conditions (3.23) are satisfied, then the operator A is ([3] p. 1331, [4] p. 235) a regular self-adjoint formal differential operator. Therefore the spectrum of A is a sequence of points of the real axis with no finite limit point and the resolvent R_λ exists for every λ which is not an eigenvalue of A.

Therefore if $\lambda=1$ is not an eigenvalue of A, then the operator

$$(A-I)(u)=\sum_{j=0}^{r} b_{2j}u^{(2j)} \tag{3.25}$$

has the inverse R_1. This implies that the inhomogeneous equation of the form (3.12) along with the boundary conditions (3.5), (3.7) has the unique solution for any arbitrary inhomogeneous term. In the case where this term does not belong to the domain of R_1 (like in (3.12)) it is the generalized solution.

We shall see that $\lambda=1$ can not be an eigenvalue of A. Indeed if $\lambda=1$ would have been such an eigenvalue, then there would had existed a non zero solution u of the homogeneous equation

$$(A-I)u=\sum_{j=0}^{r} b_{2j}u^{(2j)}=0 \tag{3.26}$$

along with boundary conditions (3.5), (3.7).

But then it is easy to check, that we would get

$$J(u)=\int_0^T \boldsymbol{u}^T(t)\,A\boldsymbol{u}(t)\,dt=0.$$

Hence it follows from (2.2) that $u(t)\equiv 0$. Therefore $\lambda=1$ is not an eigenvalue of A and equation (3.12) along with boundary conditions (3.5), (3.7) has the unique solution for arbitrary values of parameters c_i, $i=1, 2, \ldots, M-1$. This imply that condition (3.16) must be satisfied.

The above consideration yields the following

COROLLARY 1. Theorem 2 is true if instead of (iii) we substitute (iii′) conditions (3.23) are satisfied.

4. Construction of the set $V_{\tau ad}$ in the case $R_\tau=R_{2\tau}$

Like in Chapter 3 we start from the definition of the set $\mathscr{R}^{-1}(\theta)$ which in this case has the form

$$\mathscr{R}_{2\tau}^{-1}(\theta)=\Big\{u\in H^r(0,T): \frac{1}{\tau}\int_{i\tau}^{(i+1)\tau} u(t)\,dt=0;\ i=0, 1, \ldots$$
$$\ldots, M-1;\ u^{(m)}(0)=0;\ m=0, 1, \ldots, p\Big\}. \tag{4.1}$$

Assuming, like in Chapter 3, that u_v is of class C^{2r} on each interval $(i\tau, (i+1)\tau)$ and integrating (3.2) by parts, we obtain the condition of optimality in the form

(3.2a), but this time the set $\mathscr{R}^{-1}(\theta)$ is given by (4.1). Taking advantage of the form of this set we get conditions on u_v analogous to (3.3)

$$\sum_{j=0}^{r} (-1)^j \boldsymbol{a}_j^T \boldsymbol{u}_v^{(j)}(t) = c_i; \qquad t \in \big(i\tau, (i+1)\tau\big); \qquad i=0, 1, \ldots, M-1 \tag{4.2}$$

$$\sum_{j=k+1}^{r} (-1)^{j-k-1} \boldsymbol{a}_j^T \boldsymbol{u}_v^{(j-k-1)}(0) = 0 \qquad k=p+1, \ldots, r-1 \tag{4.2a}$$

$$\sum_{j=k+1}^{r} (-1)^{j-k-1} \boldsymbol{a}_j^T \delta \boldsymbol{u}_v^{(j-k-1)}(i\tau) = 0; \qquad \begin{array}{l} i=1, 2, \ldots, M-1; \\ k=0, 1, \ldots, r-1 \end{array} \tag{4.2b}$$

$$\sum_{j=k+1}^{r} (-1)^{j-k-1} \boldsymbol{a}_j^T \boldsymbol{u}_v^{(j-k-1)}(T) = 0 \qquad k=0, 1, \ldots, r-1 \tag{4.2c}$$

where c_i are arbitrary constants.

In exactly the same way as in the cases of (3.4), (3.5) and (3.6) we obtain from (4.2)

$$\sum_{j=0}^{r} b_{2j} u_v^{(2j)} = c_i; \qquad t \in \big(i\tau, (i+1)\tau\big); \qquad i=0, 1, \ldots, M-1 \tag{4.3}$$

where b_{2j} are given by (3.4a),

$$D\tilde{\boldsymbol{u}}_v(0) = 0 \tag{4.4}$$

$$G\delta\tilde{\boldsymbol{u}}_v(i\tau) = 0 \qquad i=1, 2, \ldots, M-1 \tag{4.5}$$

$$G\tilde{\boldsymbol{u}}_v(T) = 0\,. \tag{4.6}$$

The components of $r \times 2r$ — dimensional matrices D and G are given by (3.8a) and (3.8c) respectively.

Using the same argument, as that used to obtain (3.9), and taking into account that (4.2b) must be satisfied also for $k=0$, which was not the case in (3.3b), (in (4.5) the matrix G is substituting for F in (3.6)) we obtain

$$\delta u_v^{(m)}(i\tau) = 0; \qquad i=1, 2, \ldots, M-1; \qquad m=0, 1, \ldots, 2r-1\,. \tag{4.7}$$

Conditions (4.7) together with (4.3) show that in the whole interval $(0, T)$ the function u_v is a solution of the same inhomogeneous differential equation. Passing from one subinterval $[i\tau, (i+1)\tau)$ to the other changes only the value of constant c_i on the right-hand side of the equation. Therefore we can write

$$\sum_{j=0}^{r} b_{2j} u_v^{(2j)}(t) = \sum_{j=0}^{M-1} c_i W_i(t), \qquad t \in (0, T). \tag{4.8}$$

Summing up the above results we conclude that if the function u_v is of class C^{2r} on each interval $\big(i\tau_1 (i+1)\tau\big)$ then it must satisfy the following conditions

(i) u_v is a solution of (4.8)

(ii) the initial conditions (4.4) are satisfied

(iii) on each interval $(i\tau, (i+1)\tau)$; $i=0, 1, \ldots, M-1$ the following equality takes place

$$v(i\tau)=\frac{1}{\tau}\int_{i\tau}^{(i+1)\tau} u_v(t)\,dt \qquad i=0, 1, \ldots, M-1 \tag{4.9}$$

(iv) the terminal conditions (4.6) are satisfied.

Like in Chapter 3 we rewrite (4.8) in the form of inhomogeneous normal system of $2r$ equations of first order

$$\dot{\tilde{u}}_v(t)=Q\tilde{u}_v(t)+e_{2r-1}\sum_{i=0}^{M-1} c_i W_i(t). \tag{4.10}$$

Repeating the argument of Chapter 3 we conclude that if (3.16) is satisfied then the solution of (4.10) satisfying boundary conditions (4.4), (4.6) has the form

$$\tilde{u}_v(t)=\Phi(t)\left\{\sum_{i=0}^{M-1}\left[\tilde{u}_{v_i}(0)+\int_0^t \Phi^{-1}(s)\,e_{2r-1}W_i(s)\,ds\right]c_i\right\} \tag{4.11}$$

where

$$\tilde{u}_{v_i}(0)=C^{-1}\eta_i \tag{4.11a}$$

the matrix C is given by (3.15a),

$$\eta_i=\begin{bmatrix}\theta\\ \tilde{\eta}_i\end{bmatrix} \tag{4.11b}$$

is $2r$- dimensional vector, such that θ is r-dimensional zero vector and

$$\tilde{\eta}_i=-G\Phi(T)\int_0^T \Phi^{-1}(s)\,e_{2r-1}\,W_i(s)\,ds=$$

$$=-G\Phi(T)\int_{i\tau}^{(i+1)\tau} \Phi^{-1}(s)\,e_{2r-1}\,ds. \tag{4.11c}$$

The function u_v, which we are looking for, is given by:

$$u_v(t)=e_0^T\Phi(t)\left\{\sum_{i=0}^{M-1}\left[\tilde{u}_{v_i}(0)+\int_0^t \Phi^{-1}(s)\,e_{2r-1}W_i(s)\,ds\right]c_i\right\}=\chi^T(t)\,c \tag{4.12}$$

where

$$c^T=[c_0, c_1, \ldots, c_{M-1}] \quad \text{and} \quad \chi^T(t)=[\chi_0(t), \chi_1(t), \ldots, \chi_{M-1}(t)]$$

are M-dimensional vectors and

$$\chi_i(t)=e_0^T\Phi(t)\left[\tilde{u}_{v_i}(0)+\int_0^t \Phi^{-1}(s)\,e_{2r-1}W_i(s)\,ds\right]. \tag{4.12a}$$

Using (4.12) we rewrite (4.9) as

$$v(i\tau)=\frac{1}{\tau}\int_{i\tau}^{(i+1)\tau}\chi^T(t)\,dt\,c; \qquad i=0, 1, \ldots, M-1 \tag{4.13}$$

or in the matrix form

$$\boldsymbol{v}=Zc \tag{4.14}$$

where $\boldsymbol{v}^T=[v(0), v(\tau), \ldots, v((M-1)\tau)]$ is M-dimensional vector and Z is $M\times\times M$-dimensional matrix, the verses of which are given by

$$Z_i^T=\frac{1}{\tau}\int_{i\tau}^{(i+1)\tau} \chi^T(t)\,dt. \tag{4.14a}$$

Using the same reasoning as in Chapter 3 we conclude that the matrix Z is non-singular and from (4.12), (4.14) we find

$$u_v(t)=\chi^T(t)\,Z^{-1}\boldsymbol{v}. \tag{4.15}$$

This formula defines the needed operator $\bar{\mathscr{R}}_\tau^{-1}$.
The above results are summarized in the following

THEOREM 3. If

(i) the set of admissible control is given by (2.1), where condition (2.2) is fulfilled

(ii) the operator $\mathscr{R}_\tau$ is given by (2.4b)

(iii) conditions (3.16) is satisfied

then the set $V_{\tau ad}$ approximating U_{ad} is given by (2.14), where the linear operator $\bar{\mathscr{R}}_\tau^{-1}$ is defined by (4.15).
In exactly the same way as in the case of Corollary 1 we get

COROLLARY 2. Theorem 3 is true if instead of (iii) we substitute (iii′) conditions (3.23) are satisfied.

Note, that in the case where none initial conditions are imposed on control functions, i.e. if $H_0^r(0, T)=H^r(0, T)$ we have $D=G$ and conditions (3.23) are satisfied.

5. Examples

5.1. The case of operator $\mathscr{R}_{1\tau}$

Example 1. Let

$$U_{ad}=\Big\{u\in H^r(0, T):\ r\geqslant 1;\ u^{(m)}(0)=0;\ 0\leqslant m\leqslant r-1;\ J(u)= \\ =\int_0^T (u^{(r)}(t))^2\,dt\leqslant 1\Big\}. \tag{5.1}$$

It is easy to check that condition (2.2) is satisfied.
Equation (3.4) takes on the form

$$u_v^{(2r)}(t)=0;\quad t\in(i\tau, (i+1)\tau);\quad i=0, 1, \ldots, M-1 \tag{5.2}$$

and the elements of matrices of boundary conditions D and G are given respectively by

$$d_{k,m}=\delta_{k,m}; \qquad k=0, 1, ..., r-1; \qquad m=0, 1, ..., 2r-1, \tag{5.2a}$$

$$g_{k,m}=\delta_{k+r,m}; \qquad k=0, 1, ..., r-1; \qquad m=0, 1, ..., 2r-1. \tag{5.2b}$$

Hence conditions (3.23) are satisfied.

It follows from (5.2) that on each interval $(i\tau, (i+1)\tau)$ the functions u_v must be polynomials of $(2r-1)$ — order. Moreover condition (3.9b) implies that u_v are of class $C^{2r-2}(0, T)$. The functions of this type are called spline functions of order $(2r-1)$ [9].

They are usually defined along with boundary conditions of type (5.2a) at both ends of the interval $(0, T)$ and it is convenient to express them in terms of Hermite polynomials of order $(2r-1)$.

Below two simplest cases $r=1$ and $r=2$ are considered in details

$r=1$

It follows from (5.2) that functions u_v are piece-wise linear. They must satisfy conditions

$$u_v(i\tau)=v(i\tau) \qquad i=0, 1, 2, ..., M-1 \tag{5.3a}$$

and the terminal condition following from (5.2b), which has the form

$$u_v^{(1)}(T)=0.$$

From this last condition and from linearity of function u_v we get

$$u_v(T)=u_v((M-1)\tau). \tag{5.3b}$$

Conditions (5.3) fully define function u_v.
In particular we have

$$J(u_v)=\int_0^T (u_v^{(1)}(t))^2\, dt=\sum_{i=0}^{M-2}\frac{1}{\tau}[v((i+1)\tau)-v(i\tau)]^2$$

and

$$V_{\tau ad}=\Big\{v\in E_\tau(0, T): v(0)=0;\ J(u_v)=$$

$$=\sum_{i=0}^{M-2}\frac{1}{\tau}[v((i+1)\tau)-v(i\tau)]^2\leqslant 1\Big\}. \tag{5.4}$$

$r=2$

In this case the functions u_v belong to the class of cubic splines, therefore they can be expressed [9] in the form

$$u_v(t)=\sum_{i=0}^{M}(s_i h_i(t)+s_i^1 h_i^1(t)) \tag{5.5}$$

where

$$s_i = u_v(i\tau) \qquad i=0, 1, ..., M$$
$$s_i^1 = u_v^{(1)}(i\tau) \qquad i=0, 1, ..., M \tag{5.6a}$$

$h_i(t)$ and $h_i^1(t)$ are given by

$$h_i(t) = H\left(\frac{t}{\tau} - i\right) \qquad i=0, 1, ..., M$$
$$h_i^1(t) = H^1\left(\frac{t}{\tau} - i\right) \qquad i=0, 1, ..., M \tag{5.6b}$$

$H(t)$ and $H^1(t)$ denote here cubic Hermite polynomials. Using (5.5) and performing some tedious computations we get

$$J(u_v) = \int_0^T (u_v^{(2)}(t))^2\, dt = \frac{4}{\tau} \sum_{i=0}^{M-1} \left[\frac{3}{\tau^2} (s_{i+1} - s_i)^2 + (s_i^1)^2 + \right.$$
$$\left. + (s_{i+1}^1)^2 - \frac{3}{\tau} (s_{i+1} - s_i)(s_i^1 + s_{i+1}^1) + s_i^1 s_{i+1}^1 \right]. \tag{5.7}$$

Note that (5.6a) implies

$$s_i = v(i\tau), \qquad i=0, 1, ..., M-1, \tag{5.8}$$
$$s_0^1 = 0.$$

The other $(M+1)$ unknown coefficients s_M, s_i^1, $i=1, 2, ..., M$ can be computed from the condition of minimizing (5.7).

To do this note that due to (5.2b) the terminal condition (3.7) takes on the form

$$u^{(m)}(T) = 0 \qquad m=2, 3. \tag{5.9}$$

Differentiating (5.7) with respect to unknown coefficients, putting results equal to zero and taking into account (5.5), (5.6), (5.8) and (5.9) we find that these unknown coefficients are given by linear equation

$$z = B^{-1} k \tag{5.10}$$

where elements of vectors $z=[z_1, z_2, ..., z_{M+1}]$, $k=[k_1, k_2, ..., k_{M+1}]$ and of matrix $B=[b_{ij}]$, $i, j=1, 2, ..., M+1$ are given by

$$z_i = \begin{cases} s_i^1 & i=1, 2, ..., M \\ s_M & i=M+1 \end{cases}$$

$$k_i = \begin{cases} \frac{3}{\tau}[v((i+1)\tau) - v((i-1)\tau)] & i=1, ..., M-2 \\ -\frac{3}{\tau} v((i-1)\tau)) & i=M-1, M \\ \frac{2}{\tau} v((M-1)\tau) & i=M+1 \end{cases}$$

$$b_{ij}=\begin{cases} 4 & 1\leqslant j=i\leqslant M-1 \\ 2 & j=i=M \\ \dfrac{\tau}{2} & j=i=M+1 \\ 1 & 1\leqslant j=i-1\leqslant M-1 \quad \text{and} \quad 2\leqslant j\leqslant i+1\leqslant M \\ -1 & j=M+1;\ i=M-1,\ M \\ \dfrac{3}{\tau} & j=M-1,\ M;\ i=M+1 \\ 0 & \text{otherwise} \end{cases}$$

Substituting (5.10) to (5.7) we find $J(u_v)$ as a quadratic form of $v(i\tau)$, and then we find $V_{\tau ad}$.

Example 2. Let

$$U_{ad}=\left\{u\in H^1(0,T):\ u(0)=0;\ J(u)=\int_0^T [(u(t))^2+(u^{(1)}(t))^2]\,dt\leqslant 1\right\}. \tag{5.11}$$

In this case equation (3.4) takes on the form

$$u_v^{(2)}(t)-u_v(t)=0; \qquad t\in(i\tau,(i+1)\tau); \qquad i=0,1,\dots,M-1 \tag{5.12}$$

and the boundary conditions (3.5), (3.7) are reduced to

$$u_v(0)=0; \qquad u_v^{(1)}(T)=0. \tag{5.12a}$$

Any solution of (5.12) is given by

$$u_v(t)=\alpha\,\mathrm{ch}\,t+\beta\,\mathrm{sh}\,t.$$

For each interval $[i\tau,(i+1)\tau]$, $i=0,\dots,M-2$ boundary conditions $u_v(i\tau)=v(i\tau)$ have to be satisfied. On the other hand it follows from (5.12) that for interval $[(M-1)\tau, M_\tau]$ we must have

$$u_v((M-1)\tau)=v((M-1)\tau); \qquad u_v^{(1)}(M\tau)=0.$$

Hence we find that the function u_v is given by

$$u_v(t)=\alpha_i\,\mathrm{ch}\,t+\beta_i\,\mathrm{sh}\,t; \qquad t\in[i\tau,(i+1)\tau] \tag{5.13}$$

where

$$\alpha_i=\frac{v(i\tau)\,\mathrm{sh}\,(i+1)\tau-v((i+1)\tau)\,\mathrm{sh}\,i\tau}{\mathrm{sh}\,\tau}$$

$$\beta_i=\frac{v((i+1)\tau)\,\mathrm{ch}\,i\tau-v(i\tau)\,\mathrm{ch}\,((i+1)\tau)}{\mathrm{sh}\,\tau} \qquad i=0,1,\dots,M-2 \tag{5.13a}$$

and

$$\alpha_{M-1}=v((M-1)\tau)\frac{\mathrm{ch}\,M\tau}{\mathrm{ch}\,\tau},$$

$$\beta_{M-1}=-v((M-1)\tau)\frac{\mathrm{sh}\,M\tau}{\mathrm{ch}\,\tau}. \tag{5.13b}$$

Using (5.13) we get eventually

$$J(u_v)=\int_0^T [(u_v(t))^2+(u_v^{(1)}(t))^2]\,dt=2\operatorname{cth}\tau\sum_{i=1}^{M-2} v^2(i\tau)+$$

$$+(\operatorname{cth}\tau+\operatorname{th}\tau)\,v^2((M-1)\,\tau)-\frac{2}{\operatorname{sh}\tau}\sum_{i=0}^{M-2} v(i\tau)\,v((i+1)\,\tau) \tag{5.14}$$

and thus we can find the set $V_{\tau\mathrm{ad}}$.

5.2. The case of operator $\mathscr{R}_{2\tau}$

Example 3. Let

$$U_{\mathrm{ad}}=\left\{u\in H^1(0,T): u(0)=0,\ J(u)=\int_0^T (u^{(1)}(t))^2\,dt\leqslant 1\right\}. \tag{5.15}$$

In this case equation (4.8) reduces to

$$u_v^{(2)}(t)=\sum_{i=0}^{M-1} c_i W_i(t) \tag{5.16}$$

along with boundary conditions

$$u_v(0)=0,\quad u_v^{(1)}(T)=0. \tag{5.16a}$$

Moreover conditions (4.9) must be satisfied.

Like in (4.10) we rewrite (5.16) in the form of inhomogeneous normal system of first order equations. We have

$$\begin{bmatrix}\dot u_v(t)\\ \dot u_v^{(1)}(t)\end{bmatrix}=\begin{bmatrix}0, & 1\\ 0, & 0\end{bmatrix}\begin{bmatrix}u_v(t)\\ u_v^{(1)}(t)\end{bmatrix}-\begin{bmatrix}0\\ 1\end{bmatrix}\sum_{i=0}^{M-1} c_i W_i(t). \tag{5.17}$$

The solution of this system is given by

$$\begin{bmatrix}u_v(t)\\ u_v^{(1)}(t)\end{bmatrix}=\begin{bmatrix}1, & t\\ 0, & 1\end{bmatrix}\begin{bmatrix}u_v(0)\\ u_v^{(1)}(0)\end{bmatrix}-\begin{bmatrix}1, & t\\ 0, & 1\end{bmatrix}\int_0^t\begin{bmatrix}-s\\ 1\end{bmatrix}\sum_{i=0}^{M-1} c_i W_i(s)\,ds. \tag{5.18}$$

The initial condition $u_v(0)$ is given by (5.16a) and the initial condition $u_v^{(1)}(0)$ can be found from terminal condition (5.16a). Using this condition we obtain from (5.18)

$$u_v^{(1)}(0)=\tau\sum_{i=0}^{M-1} c_i. \tag{5.19}$$

Therefore the solution (5.18) in the interval $(k\tau,(k+1)\,\tau)$ we can rewrite in the form

$$\begin{bmatrix}u_v(t)\\ u_v^{(1)}(t)\end{bmatrix}=\begin{bmatrix}\tau t\\ \tau\end{bmatrix}\sum_{i=0}^{M-1} c_i-\sum_{i=0}^{k-1} c_i\begin{bmatrix}t\tau-(i+\frac{1}{2})\,\tau^2\\ \tau\end{bmatrix}-\begin{bmatrix}\frac{1}{2}(t-k\tau)^2\\ t-k\tau\end{bmatrix}c_k$$

or

$$u_v(t)=-\tfrac{1}{2}c_k(t-k\tau)^2+t\tau\sum_{i=k}^{M-1} c_i+\tau^2\sum_{i=0}^{k-1}(i+\tfrac{1}{2})\,c_i, \tag{5.20a}$$

$$\dot{u}_v^{(1)}(t) = -c_k(t-k\tau) + \tau \sum_{i=k}^{M-1} c_i . \tag{5.20b}$$

Substituting (5.20a) to (4.9) we get

$$\tau^2 \sum_{i=0}^{k-1} (i+\tfrac{1}{2})\, c_i + \tau^2 (k+\tfrac{1}{3})\, c_k +$$
$$+\tau^2 \sum_{i=k+1}^{M-1} (k+\tfrac{1}{2})\, c_i = v(k\tau), \qquad k=0, 1, \ldots, M-1. \tag{5.21}$$

The system of equations (5.21) constitutes matrix equation (4.14) from which we find c_k in terms of $v(k\tau)$, $k=0, 1, \ldots, M-1$. To find the form of $J(u_v)$ let us note that

$$J(u_v) = \int_0^T (u_v^{(1)}(t))^2\, dt = u_v(T)\, u_v^{(1)}(T) - u_v(0)\, u_v^{(1)}(0) - \int_0^T u_v^{(2)}(t)\, u_v(t)\, dt .$$

Using (4.9), (5.16) and (5.16a) we get

$$J(u_v) = -\int_0^T u_v^{(2)}(t)\, u_v(t)\, dt = \int_0^T \sum_{i=0}^{M-1} c_i W_i(t)\, u_v(t)\, dt =$$
$$= \sum_{i=0}^{M-1} c_i \int_{i\tau}^{(i+1)\tau} u_v(t)\, dt = \tau \sum_{i=0}^{M-1} c_i\, v(i\tau) .$$

Substituting to this formula the values of c_i expressed in terms of $v(i\tau)$ we obtain $J(u_v)$ as a function of $v(i\tau)$, $i=0, 1, 2, \ldots, M-1$ and thus we can find $V_{\tau ad}$.

Appendix

Already after the submition of the manuscript to the printer, the author noted that the condition (3.16) must be always satisfied, so the assumptions (iii) in Theorem 2 and Theorem 3 can be removed.

The proof of this fact is given below.

Let us consider the problem of minimizing the form

$$J(u) = \int_0^T \underline{u}^T(t)\, A\underline{u}(t)\, dt \tag{A.1}$$

on the subspace $H_0^r(0, T)$ given by (2.2a), without any additional constraints. It is clear that due to (2.2a) this problem has the unique solution

$$u_0(t) \equiv 0. \tag{A.2}$$

On the other hand using the same argument as in the proof of Theorem 3 we show that any solution $u_0(t)$ of (4.8) with homogeneous right-hand side and along with the boundary conditions (4.4), (4.6) minimizes $J(u)$. Such solution exists if the vector $\underline{\tilde{u}}_0(0)$ of initial conditions satisfies (3.15) with homogeneous right-hand side, i.e. if

$$C\underline{\tilde{u}}_0(0) = 0. \tag{A.3}$$

But by (A.2) the only possible vector of initial conditions is the zero vector. Hence (A.3) can not have any non-zero solution, which implies that the matrix C satisfies (3.16). Q.E.D.

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O aproksymacji dyskretnej pewnej klasy zbiorów sterowań dopuszczalnych

Formułuje się problem aproksymacji dyskretnej domkniętych i wypukłych zbiorów sterowań dopuszczalnych.
Pokazuje się, że dla szerokiej klasy takich zbiorów aproksymacja dyskretna istnieje i jest określona w sposób jednoznaczny.

Rozważa się dwa typy dyskretyzacji i dla każdego z nich podaje się metodę konstruowania aproksymacji zbiorów sterowań dopuszczalnych. Konstrukcja taka wymaga rozwiązania zadania dwubrzegowego dla równania różniczkowego zwyczajnego. Uzyskane wyniki ilustrowane są przykładami.

О дискретной аппроксимации некоторого класса множеств допустимых управлений

Формулируется задача дискретной аппроксимации замкнутых и выпуклых множеств допустимых управлений.

Показано, что существует дискретная аппроксимация для широкого класса таких множеств, и что она определена однозначно.

Рассматриваются два типа дискретизации и для каждого из них дается метод построения управлений. Такое построение требует решения двухточной краевой задачи для обыкновенного дифференциального уравнения. Полученные результаты иллюстрируются на примерах.