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Finite difference approximation of optimal control for systems described by nonlinear differential equation with delay

by

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A finite difference approach to the problem of minimizing an integral cost functional subject to a differential equation with delay constraint is presented in the paper. This problem is viewed as a variational minimization problem subject to nonholonomic constraints and is treated using Lagrange multipliers. Error estimates for the control, state and cost functional are established under appropriate smoothness and boundedness conditions.

1. Introduction

A numerical approximation approach to the problem of minimizing an integral cost functional subject to constraint in the form of differential equation with delay is discussed. The method of finite difference is considered. Main objective of the paper is to derive a priori estimates (in the sense of norm L^2) for the difference between the optimal and approximate solutions —these results are given by Theorems (6.1) and (6.2)). To find these error bounds the method of Lagrange multipliers is used.

The following notation is used in the paper:

$H^1[0, T; R^n]$ and $H^2[0, T; R^n]$ denote Sobolev spaces defined by:

$$H^1[0, T; R^n] \overset{\text{df}}{=} \left\{ x \in L^2[0, T; R^n]; \frac{dx}{dt} \in L^2[0, T; R^n] \right\}, ^{1)} \tag{1.0}$$

$$H^2[0, T; R^n] \overset{\text{df}}{=} \left\{ x \in L^2[0, T; R^n]; \frac{dx}{dt} \in L^2[0, T; R^n], \frac{d^2 x}{dt^2} \in L^2[0, T; R^n] \right\}, \tag{1.1}$$

$$\langle , \rangle \text{ — scalar product in } L^2[0, T; R^n], \tag{1.2}$$

$$\|\cdot\| \text{ — norm in } L^2[0, T; R^n], \tag{1.3}$$

$\langle , \rangle_k$ — scalar product in $L^2[kh[(k+1)h; R^n]$ for $k=0, 1, ..., m-1$ where $h>0$ and $m \overset{\text{df}}{=} T/h$ is assumed to be an integer (1.4)

[1]) All derivatives are understood in the sense of distributions.

$\|\cdot\|_k$ — norm in $L^2\,[(kh), (k+1)\,h;\, R^n]$, (1.5)

$|\cdot|$ — the norm in $L_\infty\,[0, T;\, R^n]$, (1.6)

$|\cdot|_k$ — the norm in $L_\infty\,[kh, (k+1)\,h;\, R^n]$. (1.7)

2. Problem statement

The continuous and discrete (approximate) problems are formulated in this section. Moreover some basic results concerning the solutions to above problems are presented.

2.1. Continuous optimal control problem

Let $x \in H^1\,[0, T;\, R^n]$; $u \in H^1\,[0, T;\, R^m]$.

Given $\varphi \in H^1\,[-h, 0;\, R^n]$,

$A\colon R^n \times R^n \times R^m \to R^n$,

$\Phi\colon R^n \times R^m \to R^1$,

$\Omega \subseteq H^1\,[0, T;\, R^m]$ which is assumed to be a convex closed set with a nonempty interior.

Consider the following (nonlinear) deterministic optimal control problem:

minimize $J(x, u) \overset{\mathrm{df}}{=} \int\limits_0^T \Phi\left(x(t), u(t), u(t)\right) dt$ subject to the constraints:

$$\frac{dx(t)}{dt} + A\left(x(t), x(t-h), u(t)\right) = 0, \qquad t \in [0, T], \tag{2.1.1}$$

$$x(Q) = \varphi(\Theta), \qquad \Theta \in [-h, 0], \tag{2.1.2}$$

$$u \in \Omega, \tag{2.1.3}$$

We shall refer to the above problem as Problem Θ_0.

The existence of the solution (x^o, u^o) to Problem Θ_0 is assumed throughout the paper.

Assume that the following hypothesis are satisfied

H1. A is the continuously differentiable (Frechet) vector function with respect to their arguments.

H2. Φ is the continuously differentiable (Frechet) vector function with respect to their arguments.

H3. J is a radially unbounded functional with respect to u (i.e. $J(x, u)$ tends to infinity uniformly with respect to x with $\|u\| \to \infty$).

H4. A is strongly monotonne operator i.e.:

$$\exists\, \alpha > 0\ \forall x_1, x_2\ y \in L^2\,[0, T;\, R^n]\ \forall u \in L^2\,[0, T;\, R^m],$$

$$\langle A(x_1, y, u) - A(x_2, y, u), x_1 - x_2 \rangle_k \geqslant \alpha\, \|x_1 - x_2\|_k^2 .\ ^{2)}$$

[2]) $A(x, y, u)(t) \overset{\mathrm{df}}{=} A\left(x(t), y(t), u(t)\right)$.

It is a simple matter to demonstrate that the solution of problem Θ_0 must belong to some bounded set. This result is presented in the following Lemma:

Lemma 2.1. Suppose that hypothesis H1, H3, H4 are satisfied. Then $(x^o, u^o) \subset \subset G = G_x \times G_u \subset H^1 [0, T; R^n] \times H^1 [0, T; R^m]$ where G_u and G_x are the "balls" with centers at zero and bounded radiuses ρ_{0x} and ρ_{0u} given by (2.15) and (2.19) respectively.

Proof. Hypothesis H3 implies that:

$$\forall M \; \exists m > 0 \; \|u\| > m \Rightarrow J(x, u) > \mathrm{M}. \tag{2.1.4}$$

Choose $(\bar{x}, \bar{u}) \in H^1 [0, T; R^n] \times \Omega$ such that $(\bar{x}, \bar{u})$ satisfy the state equation (2.1.1.) with the initial condition (2.1.2). Denote $M^* \overset{\mathrm{df}}{=} J(\bar{x}, \bar{u})$. As a result of (2.1.4) we get:

$$\exists m^* > 0 \; \|u\| > m^* \Rightarrow J(x, u) > M^*.$$

Hence, by optimality of u^o we conclude that:
$\|u^o\| \leqslant m^*$, so $u^o \in G_u$ where G_u is a ball with a radius

$$\rho_{0u} \overset{\mathrm{df}}{=} m^*. \tag{2.1.5}$$

In order to complete the proof of the Lemma we have to show that x^o can be a priori estimated. Denote by δ a bounded set belonging to

$$H^1 [0, T; R^n] \times H^1 [0, T; R^n] \times H^1 [0, T; R^m].$$

Hypothesis H2 implies that:

$$\forall \delta \; \exists L_\delta > 0 \; \forall x, y, u \in \delta \; \|A(x, y, u)\| \leqslant L_\delta. \text{ [3]} \tag{2.1.6}$$

Multiplying by $x(t)$ and integrating the state equation (2.1.1.) from 0 to h we get:

$$x^2(h) - x^2(0) + \int_0^h \big(A(x(t), x(t-h), u(t)) - $$
$$- A(0, x(t-h), u(t)), x(t)\big)\, dt + \int_0^h \big(A(0, x(t-h), u(t)), x(t)\big)\, dt = 0.$$

Employing the inequality $ab \leqslant 2\varepsilon a^2 + \frac{1}{4\varepsilon} b^2$ and hypothesis H4 we have:

$$x^2(h) - x^2(0) + \left(\alpha - \frac{1}{4\varepsilon}\right) \|x\|_0^2 \leqslant 2\varepsilon \|A(0, \varepsilon, u)\|_0^2 \leqslant 2\varepsilon (L_{\delta 0})^2 \tag{2.1.7}$$

$\delta_0 \overset{\mathrm{df}}{=} \{(0, \varepsilon, u); u \in G_u\}$ is a bounded set.

Denote $y(t) \overset{\mathrm{df}}{=} x(t-h)$.

Now integrating the state equation from kh to $(k+1)h$ for $k = 1, \ldots, m-1$ we obtain:

$$x^2((k+1)h) - x^2(kh) + \left(\alpha - \frac{1}{4\varepsilon}\right) \|x\|_k^2 \leqslant 2\varepsilon \|A(0, y, u)\|_k^2.$$

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[3]) Although hypothesis H2 may seem unnecessary at this point, (since to obtain (2.1.6) it is sufficient assume A to be bounded operator) this requirement will be essential in the sequel.

Hence

$$x^2((k+1)h)+\left(\alpha-\frac{1}{4\varepsilon}\right)\|x\|_k^2\leqslant 2\varepsilon\,[L_{\delta_k}^2+L_{\delta_{k-1}}^2+\ldots+L_{\delta_0}^2]+x^2(0)$$

where for $\varepsilon>\frac{1}{4\alpha}$

$$\delta_k\stackrel{\mathrm{df}}{=}\left\{(0,y,u);\ \|y\|_k\leqslant\frac{2\varepsilon}{\alpha-\frac{1}{4\varepsilon}}[L_{\delta_0}^2+L_{\delta_1}^2+\ldots+L_{\delta_{k-1}}^2]+\right.$$
$$\left.+\frac{x^2(0)}{\alpha-\frac{1}{4\varepsilon}}\quad\text{and}\quad u\in G_u\right\}. \tag{2.1.8}$$

Then we conclude:

$$\|x^o\|^2=\sum_{k=0}^{m-1}\|x^o\|_k^2\leqslant\frac{2\varepsilon}{\alpha-\frac{1}{4\varepsilon}}\sum_{k=0}^{m-1}(L_{\delta_0}^2+\ldots+L_{\delta k}^2)+\frac{mx^2(0)}{\alpha-\frac{1}{4\varepsilon}}.$$

So x^o belongs to the ball G_x with a radius

$$\rho_{0x}\stackrel{\mathrm{df}}{=}\frac{2\varepsilon}{\alpha-\frac{1}{4\varepsilon}}\sum_{k=0}^{m-1}L_{\delta_0}^2+\ldots+L_{\delta_k}^2+\frac{mx^2(0)}{\alpha-\frac{1}{4\varepsilon}} \tag{2.1.9}$$

where δ_k is defined by the requirence formula (2.1.7) and (2.1.8). Q.E.D.

Additionally assuming some regularity conditions imposed on A the following result concerning the regularity of optimal solution x^o can be obtained.

Lemma 2.2. Assume that:

(i) Hypothesis H1, H3, H4 are satisfied.

(ii) A Satisfies Lipschitz condition on the set $\tilde{G}$ with a constant L^o, where

$$\tilde{G}\stackrel{\mathrm{df}}{=}\{(x,y,u)\in PC\,[0,T;R^n]\times PC\,[0,T;R^n]\times$$
$$\times PC\,[0,T;R^m];\ \|x\|\leqslant\rho_x\ \|y\|\leqslant\rho_x+\|\varphi\|_{-1},\ \|u\|\leqslant\rho_u\}.$$ [4])

where $\rho_x\stackrel{\mathrm{df}}{=}\max(\rho_{0x},\rho_{1x})$ with ρ_{0x}, ρ_{0u} given by (2.1.5), (2.1.9) and ρ_{1x}, ρ_{1u} defined in Appendix C.

$$\rho_u\stackrel{\mathrm{df}}{=}\max(\rho_{0u},\rho_{1u}).$$

(iii) The norms $(A_x(x,y,u)|$, $|Ay(x,y,u)|$, $|A_u(x,y,u)|$ are bounded by M^o for all $(x,y,u)\in\tilde{G}$.

Then:

(i) $\|x^o\|\leqslant\rho_x$.

(ii) $\left\|\frac{dx^o}{dt}\right\|\leqslant g_1$, where g_1 (defined in (A.1)) depends on:

$$\rho_x,\ L_0,\ \|\varphi\|_{-1},\ \rho_u,\ \|A(0,0,0)\|.$$

[4]) $PC\,[0,T;R\,]$ denotes a space of piecewise continuous functions.

(iii) $\left\| \frac{d^2 x}{dt^2} \right\| \leqslant g_2$ where g_2 (defined in (A2)) depends on:

$$M^o, L_0, \rho_x, \rho_u, \|\varphi\|_{-1}, \left\| \frac{d\varphi}{dt} \right\|_{-1}, \left\| \frac{du^o}{dt} \right\|, \|A(0,0,0)\|.$$

The proof of the Lemma is given in Appendix A.

Problem Θ_0 is a problem of minimization of a functional over a Hilbert space subject to constraints. The classical approach to solving problems of this form is with the use of Lagrange multipliers (see [1], [2], [3]). For this purpose consider the Lagrange multiplier $\lambda \in L^2[0, T; R^n]$ and define the Lagrangian $L: H_1[-h, T; R^n] \times H^1[0, T; R^m] \times L^2[0, T; R^n] \to R^1$ as:

$$L(x, u, \lambda) \stackrel{\mathrm{df}}{=} J(x, u) + \left\langle \frac{d}{dt} x + A(x, y, u), \lambda \right\rangle. \tag{2.1.10}$$

In view of this definition, the theory of Lagrange multipliers provides the following result [5]: (see [4]) $\exists \lambda^o \in H_1[0, T); R^n], \lambda(T) \subset 0$ such that:

$$\langle \delta_x L(x^o, u^o, \lambda^o), \delta_x \rangle + \langle \delta_y L(x^o, u^o, \lambda^o), \delta_y \rangle = 0 \tag{2.1.11}$$

where:

$$\delta_x \in H^1[-h, T; R^n]; \ \delta_x(\Theta) = 0 \quad \text{for} \quad \Theta \in [-h, 0];$$

and

$$\delta_y(t) \stackrel{\mathrm{df}}{=} \delta_x(t-h),$$

$$\delta_\lambda L(x^o, u^o, \lambda^o) = 0, \tag{2.1.12}$$

$$\langle \delta_u L(x^o, u^o, \lambda^o), u - u^o \rangle \geqslant 0 \quad \text{for any } u \in \Omega. \tag{2.1.13}$$

In addition to the preceeding hypothesis assume the following one:

H5. The second variation of the Lagrangian is strongly positive in some bounded convex neigborhood of (x^o, u^o, λ^o) that is: there exist neighborhoods: $N(x^o) \subset \subset \{x \in H^1[-h; T, R^n];\ x(\Theta) = \varphi(\Theta), \Theta \in [-h, 0]\}$; $N(\lambda^o) \in L^2[0, T; R^n]$; $N(u^o) \subset \subset \Omega$ such that:

$$\left\langle \begin{bmatrix} L_{x'x}(\tilde{x}, \tilde{u}, \tilde{\lambda}), & L_{ux'}(\tilde{x}, \tilde{u}, \tilde{\lambda}) \\ L_{x'u}(\tilde{x}, \tilde{u}, \tilde{\lambda}), & L_{uu}(\tilde{x}, \tilde{u}, \tilde{\lambda}) \end{bmatrix} \begin{bmatrix} x' \\ u \end{bmatrix}, \begin{bmatrix} x' \\ u \end{bmatrix} \right\rangle \geqslant \gamma \|u\|^2$$

where $\gamma > 0$ and $x' \stackrel{\mathrm{df}}{=} [x, y]$ for any $\tilde{x} \in N(x^o), \tilde{u} \in N(u^o), \tilde{\lambda} \in N(\lambda^o)$.

Observe that hypothesis H5 constitute a local sufficiency condition for the uniqueness of a solution for Problem Θ_0. Now it is a simple matter to demonstrate the Saddle Point Behavior of the Lagrangian.

Lemma 2.3. Suppose that H5 is satisfied. Then the Lagrangian (2.1.10) possesses a degenerated saddle point at (x^o, u^o, λ^o) on the set:

$$\{x \in H^1[-h, T; R^n]; x(\Theta) = \varphi(\Theta), \Theta \in [-h, 0]\} \times \Omega \times L^2[0, T; R^n],$$

that is: $L(x^o, u^o, \lambda) = L(x^o, u^o, \lambda^o) \leqslant L(x, u, \lambda^o)$ for any $x \in N(x^o), u \in N(u^o), \lambda \in N(\lambda^o)$.

[5]) At this point recall that hypothesis H2 is satisfied.

Proof. The left-hand side equality follows directly from noting that (x^o, u^o) satisfy (2.1.1). The right-hand side inequality is result of the strong positivity condition. Indeed, expanding L into a Taylor series about the point (x^o, u^o), we have

$$L(x, u, \lambda^o) = L(x^o, u^o, \lambda^o) + \langle \delta_x L(x^o, u^o, \lambda^o), x - x^o \rangle +$$
$$+ \langle \delta_y L(x^o, u^o, \lambda^o), y - y^o \rangle + \langle \delta_u L(x^o, u^o, \lambda^o), u - u^o \rangle +$$
$$+ \left\langle \begin{bmatrix} L_{x'x'}(\tilde{x}, \tilde{u}, \lambda^o), & L_{ux'}(\tilde{x}, \tilde{u}, \lambda^o) \\ L_{x'u}(\tilde{x}, \tilde{u}, \lambda^o), & L_{uu}(\tilde{x}, \tilde{u}, \lambda^o) \end{bmatrix} \begin{bmatrix} x' - x'^o \\ u - u^o \end{bmatrix}, \begin{bmatrix} x' - x'^o \\ u - u^o \end{bmatrix} \right\rangle .$$

We note that all first order variational terms are greater than zero by virtue of (2.1.11), (2.1.13), and hypothesis H5 implies that

$$L(x, u, \lambda^o) \geqslant L(x^o, u^o, \lambda^o) + \gamma \|u - u^o\|^2 \geqslant L(x^o, u^o, \lambda^o)$$

which establishe the Lemma.

2.2. Discrete problem

In all cases, except very simple ones, it is impossible to determine the optimal solution to problem Θ_0 analytically. Therefore some approximation of this problem must be applied. We are going to use finite difference approach. To accomplish this first we must introduce a space approximating $L^2[-h, T; R^n]$. Let be given a time interval $\tau > 0$ such that $p \overset{\mathrm{df}}{=} \frac{T}{\tau}$; $l \overset{\mathrm{df}}{=} \frac{h}{\tau}$ are integers.

The approximating space $E_\tau[-h, T+\tau; R^n]$ in defined as follows (see [5], [6])

$$E[-h, T+\tau; R^n] \overset{\mathrm{df}}{=} \left\{ x_\tau(t) = \sum_{r=-l}^{p} x_\tau(r_\tau)\, Wr(t);\ x_\tau(r_\tau) \in R^n \right\},$$

where $Wr(t)$ denotes the characteristic function of interval $[r\tau; (r+1)\tau)$.

We shall consider a family of spaces $E_{\tau i}[-h, T+\tau; R^n]$ depending on parameter τ_i such that $\tau_i \to 0$ and $\tau_i = k_{ij}\tau_j$ for any $i > j$ where $k_{ij} > 1$ an integer.

Given an operator $P_\tau: H^1[-h; T; R^n] \to E_\tau[-h; T+\tau; R^n]$ such that:

$$\forall x \in H^1[-h, T; R^n]\ \|P_\tau x - x\|_{-1} + \|P_\tau x - x\| \leqslant \tau \left[\left\| \frac{dx}{dt} \right\|_{-1} + \left\| \frac{dx}{dt} \right\| \right]. \tag{2.2.0}$$

A convex closed set $\mathscr{P}_\tau \Omega \subset E_\tau[0, T; R^m]$ is said to be an approximation of Ω if the following conditions are satisfied:

$$\forall u \in \Omega\ \exists u_\tau \in \mathscr{P}_\tau \Omega\ \|u - u_\tau\| \leqslant \left\| \frac{du}{dt} \right\| \tau, \tag{2.2.1}$$

$$\forall u_\tau \in \mathscr{P}_\tau \Omega\ \exists u \in \Omega\ \|u - u_\tau\| \leqslant \left\| \frac{du}{dt} \right\| \tau. \tag{2.2.2}$$

Let $x_\tau \in E_\tau[0, T+\tau; R^n]$; $u_\tau \in E_\tau[0, T; R^m]$.

We define $\nabla x_\tau(t) \stackrel{\text{df}}{=} \dfrac{x(t+\tau)-x(t)}{\tau}$.

As an approximation of the initial problem Θ_0 the following problem Θ_τ is introduced: minimize $\int_0^T \Phi\left(x_\tau(t), u_\tau(t)\right) dt$ subject to the constraints:

$$\nabla x_\tau(t)+A\left(x_\tau(t), x_\tau(t-h), u_\tau(t)\right)=0, \quad t \in [0, T], \tag{2.2.3}$$

$$x_\tau(\Theta)=\varphi_\tau(\Theta) \stackrel{\text{df}}{=} P_\tau \varphi(\Theta), \quad \Theta \in [-h, 0), \tag{2.2.4}$$

$$x_\tau(0)=\varphi(0), \tag{2.2.5}$$

$$u_\tau \in \mathscr{P}_\tau \Omega. \tag{2.2.6}$$

As a result of the problem assumptions (hypothesis H1-H3) there exists (x_τ^0, u_τ^0) — an optimal solution of the problem Θ_τ (see [12]).

A result analogous to Lemma 2.1 is now established for the discrete optimization problem Θ_τ (the proof is given in Appendix B).

Lemma 2.4. Suppose that hypothesis H1, H3, H4 are satisfied. Then there exists $\tau_0>0$ such that for any $\tau<\tau_0$:

$$(x_\tau^0, u_\tau^0) \in G_1 \subset \{(x, u) \in PC[0, T; R^n] \times PC[0, T; R^m] \; \|x\| \leqslant \rho_{1x}; \; \|u\| \leqslant \rho_{1u}\}.$$

The finite dimensional analogue of L is the functional $L_\tau : E_\tau[-h, T+\tau; R^n] \times \times E_\tau[0, T; R^m] \times E_\tau[0, T+\tau; R^n] \to R^1$ given by

$$L_\tau(x_\tau, u_\tau, \lambda_\tau) \stackrel{\text{df}}{=} J(x_\tau, u_\tau)+\langle \nabla x_\tau + A(x_\tau, y_\tau, u_\tau), \lambda_\tau \rangle \tag{2.2.7}$$

where $y_\tau(t) \stackrel{\text{df}}{=} x_\tau(t-h)$.

The classical theory of Lagrange multipliers [4] applied to $L_\tau(x_\tau, u_\tau, \lambda_\tau)$ implies that due to hypothesis H2 there exists $\lambda_\tau^0 \in E_\tau[0, t+\tau; R^n]$ such that

$$\langle \delta x_\tau L_\tau(x_\tau^0, u_\tau^0, \lambda_\tau^0), \delta x_\tau \rangle + \langle \delta y_\tau L_\tau(x_\tau^0, u_\tau^0, \lambda_\tau^0), \delta y_\tau \rangle = 0, \tag{2.2.8}$$

where $\delta x_\tau \in E_\tau[-h, T+\tau; R^n]$; $\delta x_\tau(\Theta)=0$ for $\Theta \in [-h, 0]$; and $\delta y_\tau(t) \stackrel{\text{df}}{=} \delta x_\tau(t-h)$.

$$\delta_{\lambda_\tau} L_\tau(x_\tau^0, u_\tau^0, \lambda_\tau^0)=0, \tag{2.2.9}$$

$$\langle \delta u_\tau L_\tau(x_\tau^0, u_\tau^0, \lambda_\tau^0), u_\tau - u_\tau^0 \rangle \geqslant 0, \quad \forall u_\tau \in \mathscr{P}_\tau \Omega. \tag{2.2.10}$$

Assume additionally the following hypothesis:

H5′
$$\left\langle \begin{bmatrix} L_{\tau x'x'}(\tilde{x}_\tau, \tilde{u}_\tau, \tilde{\lambda}_\tau), & L_{\tau ux'}(\tilde{x}_\tau, \tilde{u}_\tau, \tilde{\lambda}_\tau) \\ L_{\tau x'u}(\tilde{x}_\tau, \tilde{u}_\tau, \tilde{\lambda}_\tau), & L_{\tau uu}(\tilde{x}_\tau, \tilde{u}_\tau, \tilde{\lambda}_\tau) \end{bmatrix} \begin{bmatrix} x'_\tau \\ u_\tau \end{bmatrix}, \begin{bmatrix} x'_\tau \\ u_\tau \end{bmatrix} \right\rangle \geqslant \gamma \|u_\tau\|^2$$

for all $\tilde{x}_\tau \in N(x_\tau^0)$, $\tilde{u}_\tau \in N(u_\tau^0)$, $\tilde{\lambda}_\tau \in N(\lambda_\tau^0)$ where $N(x_\tau^0)=\{x \in E_\tau[-h, T+\tau; R^n]$; $x_\tau(\Theta)=\varphi_\tau(\Theta)$ $\Theta \in [-h, 0)$; $x_\tau(0)=\varphi(0)\}$; $N(u_\tau^0) \in \mathscr{P}_\tau \Omega$, $N(\lambda_\tau^0) \in E_\tau[0, T+\tau; R^n]$ (hypothesis H5′ implies the uniqueness of a solution for problem Θ_τ).

This hypothesis leads to the finite-dimensional analogue of Lemma 2.3.

Lemma 2.5. Suppose H5′ is satisfied. Then the Lagrangian L_τ has a degenerated saddle-point at $(x_\tau^o, u_\tau^o, \lambda_\tau^o)$ on the set;

$$\{x_\tau \in E_\tau[-h, T+\tau; R^n] \times_\tau(\Theta) = P_\tau \varepsilon(\Theta); \Theta \in [-h, 0); x(0) = \varepsilon(0)\} \times$$
$$\times \mathscr{P}_\tau \Omega \times E_\tau [0, T+\tau; R^n]$$

that is:

$$L_\tau(x_\tau^0, u_\tau^0, \lambda_\tau) = L_\tau(x_\tau^0, u_\tau^0, \lambda_\tau^0) \leqslant L_\tau(x_\tau, u_\tau, \lambda_\tau^0)$$

for all $x_\tau \in N(x_\tau^0)$, $u_\tau \in N(u_\tau^0)$, $\lambda_\tau \in N(\lambda_\tau^0)$.

The proof is almost identical to that of Lemma 2.3 and is therefore omitted.

Observe that an approximating control u^o does not belong to the admissible set of controlers. Therefore by employing of u_τ^o we construct another control u_τ^* close to u^o such that $u_\tau^* \in \Omega$ (it is possible due to (2.2.2)).

Our main purpose is to find the bounds for $\|u^o - u_\tau^*\|$, $\|x^o - x_\tau^*\|$ (where x_τ^* is a solution of (2.1.1) corresponding to u_τ^*) and $J(x_\tau^*, u_\tau^*) - J(x^o, u^o)$. In order to find these error bounds we are going to estimate successively:

1° $\|x^o - x_\tau^o\|$ in terms of $\|u^o - u_\tau^o\|$;

2° $\|\lambda^o - \lambda_\tau^o\|$ in terms of $\|u^o - u_\tau^o\|$ and $\|x^o - x_\tau^o\|$;

3° $\|u^o - u_\tau^o\|$ in terms of $\|x^o - x_\tau^o\|$ and $\|\lambda^o - \lambda_\tau^o\|$ (these estimation is obtained by Saddle Point Theorem with help of hypothesis H5 and H5′)

4° $\|u^o - u_\tau^o\|$ by a constant convergent to zero with τ — what will be easily deduced from 1°, 2°, 3°)

5° $\|u^o - u_\tau^*\|$ on the basis of 4° and condition (2.2.2);

6° $\|x^o - x_\tau^*\|$ using 5° and state equation (2.1.1);

7° $J(x_\tau^*, u_\tau^*) - J(x^o, u^o)$ with help of 5° and 6°.

3. Estimation of the difference between optimal and approximate solutions of the state equations

In this section we are going to estimate $\|x^o - x_\tau^o\|$ in terms of $\|u^o - u_\tau^o\|$. In preparation for the main result of the section we first present a Lemma which will be essential in the sequel.

Lemma 3.1. Suppose that:

(i) $P_\tau : H^1[0, T; R^n] \to E_\tau[0; T+\tau : R^n]$ is such that:

$$P_\tau x(t) \stackrel{\text{df}}{=} \sum_{r=0}^{P} x(t_r) W_r(t)$$

where $t_r \in [r\tau; (r+1)\tau]$ and $t_{r+1} - t_r = \tau$; [6])

(ii) $x \in H^2[0, T; R^n]$.

Then

$$\left\| \nabla P_\tau x - \frac{dx}{dt} \right\| \leqslant \sqrt{2}\, \tau \left\| \frac{d^2 x}{dt^2} \right\|.$$

[6]) It is easy to check that P_τ defined by (i) satisfies condition (2.2.0)).

Proof. On the basis of mean value theorem we obtain

$$\left\|\nabla P_\tau x-\frac{d}{dt}x\right\|^2=\sum_{r=0}^{p-1}\int_{r_\tau}^{(r+1)_\tau}\left|\frac{1}{\tau}(P_\tau x(t+\tau)-P_\tau x(t))-\frac{d}{dt}x(t)\right|^2dt=$$

$$=\sum_{r=0}^{p-1}\int_{r_\tau}^{(r+1)_\tau}\left|\frac{1}{\tau}(x(t_{r+2})-x(t_{r+1}))-\frac{d}{dt}x(t)\right|^2dt=$$

$$=\sum_{r=0}^{p-1}\int_{r_\tau}^{(r+1)_\tau}\left|\frac{dx(t_r)}{dt}-\frac{d}{dt}x(t)\right|^2dt$$

where $t_{r+2}\in[(r+1)_\tau;(r+2)_\tau]$: $t_{r+1}\in[r_\tau;(r+1)_\tau]$, $t_r\in[r_\tau;(r+2)_\tau]$.

After applying Schwartz inequality we get

$$\left\|\nabla P_\tau x-\frac{d}{dt}x\right\|^2=\sum_{r=0}^{p-1}\int_{r_\tau}^{(r+1)_\tau}\left|\int_{\tau}^{t_r}\frac{d^2x(s)}{ds^2}ds\right|^2dt\leqslant$$

$$\leqslant\tau\sum_{r=0}^{p-1}\int_{r_\tau}^{(r+1)_\tau}\int_{r_\tau}^{(r+2)_\tau}\left|\frac{d^2x(s)}{ds^2}\right|^2ds\,dt\leqslant 2\tau^2\left\|\frac{d^2x}{dt^2}\right\|^2. \qquad \text{Q.E.D.}$$

We are now ready to find the bound for the error of approximation of the state.

Theorem 3.1. Suppose that all assumptions of Lemmas 2.2 and 3.1 are satisfied. Moreover assume that τ is chosen in such a way that $\tau<\frac{\alpha}{2L_0^2}$. Then: $\|x^o-x_\tau^0\|^2\leqslant\leqslant C_0\|u^o-u_\tau^o\|^2+C_1\tau^2$ where C_0 given by (3.17) depends on: L_0,α,m; C_1 given by (3.18) depends on: $L_0,\rho_x,\rho_u,\left\|\frac{d\varepsilon}{dt}\right\|_{-1},\left\|\frac{du^o}{dt}\right\|,M^o,m$.

Proof. Theorem 3.1 is proved using step by step method.

Denote: $\tilde{x}_\tau^o\overset{\text{df}}{=}P_\tau x^o$; $\tilde{y}_\tau^o(t)\overset{\text{df}}{=}\tilde{x}_\tau^o(t-h)$, $t\in[0,T]$;

$$\delta_\tau(t)\overset{\text{df}}{=}(\nabla(x_\tau^o(t)-\tilde{x}_\tau^o(t)),\ x_\tau^o(t)-\tilde{x}_\tau^o(t))+$$

$$+\left(A(x_\tau^o(t),y_\tau^o(t),u_\tau^o(t))-A(\tilde{x}_\tau^o(t),y_\tau^o(t),u_\tau^o(t)),\ x_\tau^o(t)-\tilde{x}_\tau^o(t)\right).$$

Observe that:

$$(\nabla x_\tau(t),x_\tau(t))=\frac{\nabla|x_\tau(t)|^2-\tau|\nabla x_\tau(t)|^2}{2} \qquad (3.0)$$

(it is easily deduced from the definition of ∇).

After applying (3.0) and hypothesis H4 we have:

$$\int_0^h\delta_\tau(t)\,dt\geqslant\frac{1}{2}|x_\tau^o(h)-\tilde{x}_\tau^o(h)|^2-\frac{\tau}{2}\|\nabla(x_\tau^o-\tilde{x}_\tau^o)\|_0^2+\alpha\|x_\tau^o-\tilde{x}_\tau^o\|_0^2. \qquad (3.1)$$

Making use of the fact that x^o and x^o_τ are solutions of state equations (2.1.1), (2.2.3) respectively and that A satisfies Lipschitz condition we get:

$$\|\nabla (x^o_\tau - \tilde{x}^o_\tau)\, H\|_0^2 = -\langle A(x^o_\tau, y^o_\tau, u^o_\tau), \nabla (x^o_\tau - \tilde{x}^o_\tau)\rangle_0 +$$

$$+\left\langle \frac{d}{dt} x^o - \nabla \tilde{x}^o_\tau, \nabla (x^o_\tau - \tilde{x}^o_\tau)\right\rangle_0 + \langle A(x^o, y^o, u^o), \nabla (x^o_\tau - \tilde{x}^o_\tau)\rangle_0 \leqslant$$

$$\leqslant \|A(x^o, y^o, u^o) - A(x^o_\tau, y^o_\tau, u^o_\tau)\|_0 \, \|\nabla (x^o_\tau - \tilde{x}^o_\tau)\|_0 +$$

$$+\left\| \frac{d}{dt} x^o - \nabla \tilde{x}^o_\tau \right\|_0 \|\nabla (x^o_\tau - \tilde{x}^o_\tau)\|_0 \leqslant \Big\{ L_0 [\|x^o - x^o_\tau\|_0 +$$

$$+ \|\varphi - \varphi_\tau\|_{-1} + \|u^o - u^o_\tau\|_0] + \tau \sqrt{2} \left\| \frac{d^2 x^o}{dt^2} \right\|_0 \Big\} \|\nabla (x^o_\tau - \tilde{x}^o_\tau\|_0 .$$

The last term in parentheses is obtained due to Lemma 3.1. From this estimation using (2.2.0) we get

$$\|\nabla (x^o_\tau - \tilde{x}^o_\tau)\| \leqslant L_0 [\|x^o - x^o_\tau\|_0 + \|u^o - u^o_\tau\|_0] +$$

$$+\tau \left[L_0 \left(\left\| \frac{d\varphi}{dt} \right\|_{-1} + \left\| \frac{dx^o}{dt} \right\|_0 \right) + \sqrt{2} \left\| \frac{d^2 x^0}{dt^2} \right\|_0 \right]. \tag{3.2}$$

Substituting (3.2) into (3.1) we obtain

$$\int_0^h \delta_\tau(t)\, dt \geqslant \frac{1}{2} |x^o_\tau(h) - \tilde{x}^o_\tau(h)|^2 + (\alpha - 2\tau L_0^2)\, \|x^o_\tau - \tilde{x}^o_\tau\|_0^2 + 2\tau L_0^2\, \|u^o - u^o_\tau\|_0^2 -$$

$$-\tau^3 \left[L_0 \left(\left\| \frac{d\varphi}{dt} \right\|_{-1} + \left\| \frac{dx^o}{dt} \right\|_0 \right) + \sqrt{2} \left\| \frac{d^2 x^o}{dt^2} \right\|_0 \right]^2 . \tag{3.3}$$

Using inequality

$$ab \leqslant 2\varepsilon\, a^2 + \frac{1}{4\varepsilon} b^2 \qquad \text{for any } \varepsilon > 0. \tag{3.4}$$

Lemma 3.1 and Lipschitz condition we obtain:

$$\int_0^h \delta_\tau(t)\, dt = \left\langle \frac{d}{dt} x^o - \nabla \tilde{x}^o_\tau, x^o_\tau - \tilde{x}^o_\tau \right\rangle_0 + \langle A(x^o, y^o, u^o) +$$

$$- A(\tilde{x}^o_\tau, \tilde{y}^o_\tau, \tilde{u}^o_\tau), x^o_\tau - \tilde{x}^o_\tau\rangle_0 \leqslant \frac{1}{2\varepsilon} \Big\{ L_0^2 [\|x^o - \tilde{x}^o_\tau\|_0 + \|\varphi - \varphi_\tau\|_{-1} +$$

$$+ \|u^o - u^o_\tau\|_0]^2 + 2\tau^2 \left\| \frac{d^2 x^o}{dt^2} \right\|_0^2 \Big\} + 4\varepsilon\, \|x^o_\tau - \tilde{x}^o_\tau\|_0^2 \leqslant \frac{1}{2\varepsilon} \Big\{ 2L_0^2 [u^o - u^o_\tau]_0^2 +$$

$$+2\tau^2 \left[L_0^2 \left(\left\| \frac{dx^o}{dt} \right\|_0^2 + \left\| \frac{d\varphi}{dt} \right\|_{-1} \right)^2 + \left\| \frac{d^2 x^o}{dt^2} \right\|_0^2 \right] \Big\} + 4\varepsilon\, [x^o_\tau - \tilde{x}^o_\tau\|_0^2 . \tag{3.5}$$

Combining (3.3) and (3.5) we arrive at:

$$(\alpha-2\tau L_0^2-4\varepsilon)\,\|x^o-\tilde{x}_\tau^o\|_0^2+\frac{1}{2}\,|x_\tau^o(h)-\tilde{x}_\tau^o(h)|^2\leqslant$$

$$\leqslant 2L_0^2\left(\tau+\frac{1}{2\varepsilon}\right)\|u^o-u_\tau^o\|_0^2+2\tau^2 L_0^2\left(\tau+\frac{1}{2\varepsilon}\right)\left[\left\|\frac{dx^o}{dt}\right\|_0+\right.$$

$$\left.+\left\|\frac{d\varphi}{dt}\right\|_{-1}\right]^2+4\tau^2\left(\tau+\frac{1}{4\varepsilon}\right)\left\|\frac{d^2x^o}{dt^2}\right\|_0^2. \tag{3.6}$$

In the same way we can prove for $k=1,\ldots,m-1$ that:

$$\int_{kh}^{(k+1)h}\delta_\tau(t)\,dt\leqslant\frac{1}{2}\,|x_\tau^o(k+1)\,h-\tilde{x}_\tau^o(k+1)\,h|^2-\frac{1}{2}\,|x_\tau^o(kh)-\tilde{x}_\tau^o(kh)|^2+$$

$$+(\alpha-2\tau L_0^2)\,\|x_\tau^o-\tilde{x}_\tau^o\|_k^2-2\tau L_0^2\,\|u^o-u_\tau^o\|_k^2-2\tau L_0^2\,[x_\tau^o-\tilde{x}_\tau^o\|_{k-1}^2+$$

$$-2\tau^3\left[L_0\left(\left\|\frac{dx^o}{dt}\right\|_{k-1}+\left\|\frac{d}{dt}\,x^o\right\|_k\right)+\sqrt{2}\left\|\frac{d^2x^o}{dt^2}\right\|_k\right]^2 \tag{3.7}$$

and

$$\int_{kh}^{(k+1)h}\delta_\tau(t)\,dt\leqslant\frac{1}{2\varepsilon}\left\{2L_0^2\,\|u^o-u_\tau^o\|_k^2+4L_0^2\,\|x_\tau^o-\tilde{x}_\tau^o\|_{k-1}^2+\right.$$

$$\left.+4\tau^2L_0^2\left(\left\|\frac{dx^o}{dt}\right\|_k^2+\left\|\frac{dx^o}{dt}\right\|_{k-1}^2\right)+2\tau^2\left\|\frac{d^2x^o}{dt^2}\right\|_k^2\right\}+4\varepsilon\,\|x_\tau^o-\tilde{x}_\tau^o\|_k^2. \tag{3.8}$$

Combining (3.7) and (3.8) we see that:

$$(\alpha-2\tau L_0^2-4\varepsilon)\,\|x^o-\tilde{x}_\tau^o\|_k^2+\frac{1}{2}\,|x_\tau^o(k+1)\,h-\tilde{x}_\tau^o(k+1)\,h|^2\leqslant$$

$$\leqslant 2L_0^2\left(\tau+\frac{1}{2\varepsilon}\right)\|u^o-u_\tau^o\|_k^2+4L_0^2\left(\tau+\frac{1}{2\varepsilon}\right)\|x_\tau^o-\tilde{x}_\tau^o\|_{k-1}^2+$$

$$+4\tau^2\left[L_0^2\left(\tau+\frac{1}{2\varepsilon}\right)\left(\left\|\frac{d}{dt}\,x^o\right\|_{k-1}+\left\|\frac{dx^o}{dt}\right\|_k\right)^2+\right.$$

$$\left.+\left(2\tau+\frac{1}{4\varepsilon}\right)\left\|\frac{d^2x^o}{dt^2}\right\|^2+\frac{1}{2}\left|x_\tau^o(kh)-\tilde{x}_\tau^o(kh)\right|^2\right. \tag{3.9}$$

Let ε be such that: $\alpha-2\tau L_0^2-4\varepsilon>0$ (it is possible since $\alpha-2\tau L_0^2>0$). Denote:

$$\alpha_0\stackrel{\mathrm{df}}{=}\alpha-2\tau L_0^2-4\varepsilon, \tag{3.10}$$

$$\alpha_1 \overset{\text{df}}{=} 2L_0^2\left(\tau+\frac{1}{2\varepsilon}\right),$$

$$a_0 \overset{\text{df}}{=} \alpha_1 \|u^o-u_\tau^o\|_0^2+\tau^2\alpha_1\left(\left\|\frac{dx^o}{dt}\right\|_0+\left\|\frac{d\varphi}{dt}\right\|_{-1}\right)^2+4\tau^2\left(\tau+\frac{1}{4\varepsilon}\right)\left\|\frac{d^2x^o}{dt^2}\right\|_0^2,$$

$$a_k \overset{\text{df}}{=} \alpha_1\|u^o-u_\tau^o\|_k^2+2\alpha_1\tau^2\left(\left\|\frac{dx^o}{dt}\right\|_{k-1}+\left\|\frac{dx^o}{dt}\right\|_k\right)^2+$$

$$+4\tau^2\left(2\tau+\frac{1}{4\varepsilon}\right)\left\|\frac{d^2x^o}{dt^2}\right\|_k^2 \tag{3.11}$$

for $=k=1, \ldots, m-1$.

Using notations (3.10), (3.11) we rewrite (3.6) and (3.9) in the form:

$$\alpha_0\|x^o-\tilde{x}_\tau^o\|_0^2+\frac{1}{2}|x_\tau^o(h)-\tilde{x}_\tau^o(h)|^2\leqslant a_0, \tag{3.12}$$

$$\alpha_0\|x^o-\tilde{x}_\tau^o\|_k^2+\frac{1}{2}|x_\tau^o(k+1)h-\tilde{x}_\tau^o(k+1)h|^2\leqslant a_k+$$

$$+\frac{1}{2}|x_\tau^o(kh)-\tilde{x}_\tau^o(kh)|^2+2\alpha_1\|x_\tau^o-\tilde{x}_\tau^o\|_{k-1}^2. \tag{3.13}$$

Hence (3.12) and (3.13) imply that:

$$\alpha_0\|x_\tau^o-\tilde{x}_\tau^o\|_k^2+\frac{1}{2}|x_\tau^o(k+1)h-\tilde{x}_\tau^o(k+1)h|^2\leqslant a_0\left(\frac{2\alpha_1}{\alpha_0}+1\right)^k+$$

$$+a_1\left(\frac{2\alpha_1}{\alpha_0}+1\right)^{k-1}+\ldots+a_{k-1}\left(\frac{2\alpha_1}{\alpha_0}+1\right)+a_k \quad \text{for } k=0,1,\ldots,m-1.$$

Hence

$$\|x_\tau^o-\tilde{x}_\tau^o\|^2=\sum_{k=0}^{m-1}\|x_\tau^o-\tilde{x}_\tau^o\|_k^2\leqslant\frac{1}{\alpha_0}\left(\frac{2\alpha_1}{\alpha_0}+1\right)^{m-1}\cdot[(m-1)a_0+$$

$$+(m-2)a_1+\ldots+a_{m-1}]\leqslant\frac{m-1}{\alpha_0}\left(\frac{2\alpha_1}{\alpha_0}+1\right)^{m-1}\sum_{k=0}^{m-1}a_k. \tag{3.14}$$

Recalling the definitions of a_i we obtain from (3.14)

$$\|x_\tau^o-\tilde{x}_\tau^o\|^2\leqslant\frac{m-1}{\alpha_0}\left(\frac{2\alpha_1}{\alpha_0}+1\right)^{m-1}\cdot\left[2\alpha_1\|u^o-u_\tau^o\|^2+2\tau^2\left(\alpha_1\left\|\frac{d\varphi}{dt}\right\|_{-1}^2+\right.\right.$$

$$\left.\left.+4\alpha_1\left\|\frac{dx^o}{dt}\right\|^2+2\left(2\tau+\frac{1}{4\varepsilon}\right)\left\|\frac{d^2x^o}{dt^2}\right\|^2\right)\right]. \tag{3.15}$$

It is easy to see that

$$\|x^o-x_\tau^o\|^2\leqslant 2\|x_\tau^o-\tilde{x}_\tau^o\|^2+2\|\tilde{x}_\tau^o-x^o\|^2\leqslant 2\|x_\tau^o-\tilde{x}_\tau^o\|^2+2\tau^2\left\|\frac{dx^o}{dt}\right\|^2 \tag{3.16}$$

(the last estimation follows from Lemma 3.1).

We arrive at the desired result by combining (3.15) and (3.16). So

$$\|x^o - x^o_\tau\|^2 \leqslant C_0 \|u^o - u^o_\tau\|^2 + C_1 \tau^2$$

where

$$C_0 \stackrel{\text{df}}{=} \frac{4\alpha_1}{\alpha_0}(m-1)\left(\frac{2\alpha_1}{\alpha_0}+1\right)^{m-1}, \tag{3.17}$$

$$C_1 \stackrel{\text{df}}{=} C_0\left(\left\|\frac{d}{dt}\right\|^2_{-1} + 4\left\|\frac{dx^o}{dt}\right\|^2 + \left(\frac{2C_0\left(2\tau+\frac{1}{4\varepsilon}\right)}{\alpha_1}+2\right)\left\|\frac{d^2x^o}{dt^2}\right\|^2\right) \tag{3.18}$$

and α_0, α_1 are given by (3.10), (3.11).

Now applying Lemma 2.2 we complete the proof of the Theorem.

4. Estimation of the difference between the optimal and approximate Lagrange multipliers

An estimation of the norm $\|\lambda^o - \lambda^o_\tau\|$ in term of $\|x^o_\tau - x^o\|$ and $\|u^o_\tau - u^o\|$ is presented in this section. As a result of (2.1.11) adjoint equation of the following form is obtained:

$$\frac{d}{dt}\lambda^o(t) - A_x(x^o(t), x^o(t-h), u^o(t))\,\lambda^o(t) - A_y(x^o(t+h), x^o(t), u^o(t+h)) \times$$

$$\times \lambda^o(t+h) = \Phi_x(x^o(t), u^o(t)) \quad \text{for } t \in [0, T-h] \tag{4.1}$$

with the terminal condition:

$$\frac{d}{dt}\lambda^o(t) - A_x(x^o(t), x^o(t-h), u^o(t))\,\lambda^o(t) = \Phi_x(x^o(t), u^o(t));$$

$$\lambda^o(T) = 0 \quad \text{for} \quad t \in [T-h, T].$$

Some properties of Lagrange multiplier λ^o are given in the following Lemma:

Lemma 4.1. Assume that all hypothesis of Lemma 2.2 and H1 are satisfied. Moreover suppose that:

(i) For all $(x, y, u) \in \tilde{G}$ the norms $|A_{xx}(x,y,u)|$, $|A_{xy}(x, y, u)|$, $A_{xu}(x, y, u)|$, $|\Phi_{xx}(x, u)|$, $|\Phi_{xu}(x, u)|$, $|\Phi_x(x, u)|$, $|\Phi_u(x, u|$ are bounded by $M_1 > 0$ [7]). Then

(i) $|\lambda^o| \leqslant g_3$

where g_3 depends on: M_0, M_1 and is given by (C.1))

(ii) $\left\|\frac{d\lambda^o}{dt}\right\| \leqslant g_4$

where g_4 depends on M_0, M_1 and is given by (C.2)

(iii) $\left\|\frac{d^2\lambda^o}{dt^2}\right\| \leqslant g_5$

where g_5 depends on: ρ_x, ρ_u, $\|\varphi\|_{-1}$, $\left\|\frac{d\varphi}{dt}\right\|_{-1}$, $\left\|\frac{d\lambda^o}{dt}\right\|$, $\left\|\frac{du^o}{dt}\right\|$, M_1, M^o and is given by (C.3).

---

[7]) $\Phi(x, u)(t) \stackrel{\text{df}}{=} \Phi(x(t), u(t))$.

The proof of the Lemma is given in Appendix C. Equality (2.2.8) leads us to the following finite-dimensional analogue of adjoint equation (4.1).

$$\nabla\lambda_\tau^o(t)-A_x\big(x_\tau^o(t),x_\tau^o(t-h),u_\tau^o(t)\big)\,\lambda_\tau^o(t)-$$
$$+A_y\big(x_\tau^o(t+h),x_\tau^o(t),u_\tau^o(t+h)\big)\,\lambda_\tau^o(t+h)=\Phi_x\big(x_\tau^o(t),u_\tau^o(t)\big)$$
$$\text{for } t\in[0,T-h],$$
$$\nabla\lambda_\tau^o(t)-A_x\big(x_\tau^o(t),x_\tau^o(t-h),u_\tau^o(t)\big)\,\lambda_\tau^o(t)=\Phi_x\big(x_\tau^o(t),u_\tau^o(t)\big)$$
$$\lambda_\tau^o(T)=0 \quad \text{for} \quad t\in[T-h,T]. \tag{4.2}$$

We are now ready to present the estimations of errors commited in aproximating the Lagrangies multipliers. This result is given by the following.

Theorem 4.1. Let hypothesis of Lemmas 3.1, 4.1 are satisfied. Moreover suppose that A_x, A_y, Φ_x satisfy Lipschitz condition on $\tilde{G}$ with constants L_1, L_2, L_4 respectively. Then

$$\|\lambda^o-\lambda_\tau^o\|^2\leqslant C_2\,\|x^o-x_\tau^o\|^2+C_3\,\|u^o-u_\tau^o\|^2+C_4\,\tau^2$$

where C_2, C_3 depend on $L_1, L_2, L_4, m, \rho_x, \rho_u, M_1, M_0$ and are given by (4.16), (4.17) and C_4 depends on $L_1, L_2, L_4, m, \rho_x, \rho_u, M_0, M_1, \|\varphi\|_{-1}, \left\|\frac{d\varphi}{dt}\right\|_{-1}, \left\|\frac{du^o}{dt}\right\|$ and is given by (4.18).

Proof. Note that without loss of generality it can be assumed that the matrix $A_x\big(x_\tau^o(t),x_\tau^o(t-h),u_\tau^o(t)\big)$ is positive definite in the sense of L^2-norm i.e.:

$$\int\limits_{kh}^{(k+1\,h}\big(A_x\big(x_\tau^o(t),x_\tau^o(t-h),u_\tau^o(t)\big),\lambda_\tau(t)\big)\,dt\geqslant\beta\,\|\lambda_\tau\|_k^2\ ^{8)} \tag{4.3}$$
$$\text{for } k=0,1,\dots,m-1;\quad \beta>0.$$

In order to proof Theorem step by step method is used:
Denote:

$$\tilde{\lambda}_\tau^o(t)\overset{\text{df}}{=}P_\tau\,\lambda^o(t)\qquad \text{for } t\in[0,T);\quad \text{and } \tilde{\lambda}_\tau(T)=0$$
$$\delta_\tau(t)\overset{\text{df}}{=}\big(\nabla\,(\lambda_\tau^o(t)-\tilde{\lambda}_\tau^o(t)),\lambda_\tau^o(t)-\tilde{\lambda}_\tau^o(t)\big)+$$
$$-\big(A_x\big(x_\tau^o(t),x_\tau^o(t-h),u_\tau^o(t)\big)\,(\lambda_\tau^o(t)-\tilde{\lambda}_\tau^o(t)),\lambda_\tau^o(t)-\tilde{\lambda}_\tau^o(t)\big).$$

Observe that by (3.1)

$$\int\limits_T^{T-h}(\nabla\lambda_\tau(t),\lambda_\tau(t))\,dt=\frac{\int\limits_T^{T-h}\nabla\,|\lambda_\tau((t)|^2\,dt+\tau\int\limits_{T-h}^{T}|\nabla\lambda_\tau(t)|^2\,dt}{2}\geqslant\frac{1}{2}\int\limits_T^{T-h}\nabla\,|\lambda_\tau(t)|^2.$$

[8]) If $A_x\big(x_\tau^o(t),x_\tau^o(t-h),u_\tau^o(t)\big)$ does not satisfy (4.3) we can introduce a transformation of variables putting $\lambda_\tau=\lambda_\tau^*\,P_\tau\,(\exp\alpha^*\,t)$. For variable λ_τ^* equation (4.2) will have the same form as before, but with operator A_x depending on α^* and such that (4.3) is satisfied for α^* large enough

After integrating $\delta_\tau(t)$ from T to $T-h$ and applying (4.3), (4.4) we have:

$$\int_T^{T-h} \delta_\tau(t)\, dr \geqslant \frac{1}{2} |\lambda_\tau^o(T-h) - \tilde{\lambda}_\tau^o(T-h)|^2 - \frac{1}{2} |\lambda_\tau^o(T) - $$
$$-(\tilde{\lambda}_\tau^o T)|^2 + \beta \|\lambda_\tau^o - \tilde{\lambda}_\tau^o\|_{m-1}^2, \qquad (4.5)$$

On the other hand using (4.2), (4.2) and inequality (3.4), Lipschitz condition to be satisfied by A_x and Lemma (3.1) we get:

$$\int_T^{T-h} \delta_\tau(t)\, dt = \langle \Phi_x(x_\tau^o, u_\tau^o) - \Phi_x(x^o, u^o), \tilde{\lambda}_\tau^o - \lambda_\tau^o \rangle_{m-1} +$$
$$+ \left\langle (A_x(x_\tau^o, y_\tau^o, u_\tau^o) - A_x(x^o, y^o, u^o))\, \tilde{\lambda}_\tau^o, \tilde{\lambda}_\tau^o - \lambda_\tau^o \right\rangle_{m-1} +$$
$$+ \langle A_x(x^o, y^o, u^o)(\tilde{\lambda}_\tau^o - \lambda^o), \tilde{\lambda}_\tau^o - \lambda_\tau^o \rangle_{m-1} + \left\langle \frac{d\lambda^o}{dt} - \nabla \tilde{\lambda}_\tau^o, \tilde{\lambda}_\tau^o - \lambda_\tau^o \right\rangle_{m-1} \leqslant$$
$$\leqslant \frac{1}{2\varepsilon} \Big\{ 2L_4^2 [\|x^o - x_\tau^o\|_{m-1}^2 + \|u^o - u_\tau^o\|_{m-1}^2] + 3\, |\lambda^o|^2\, L_1^2 [\|x^o - x_\tau^o\|_{m-1}^2 +$$
$$\|x^o - x_\tau^o\|_{m-2}^2 + \|u^o - u_\tau^o\|_{m-1}^2] + M_0^2 \tau^2 \left\| \frac{d\lambda^o}{dt} \right\|_{m-1}^2 +$$
$$+ 2\tau^2 \left\| \frac{d^2 \lambda^o}{dt^2} \right\|_{m-1}^2 \Big\} + 8\varepsilon \|\lambda_\tau^o - \tilde{\lambda}_\tau^o\|_{m-1}^2. \qquad (4.6)$$

Combining (4.5) and (4.6) we arrive at

$$(\beta - 8\varepsilon) \|\lambda_\tau^o - \tilde{\lambda}_\tau^o\|_{m-1}^2 + \frac{1}{2} |\lambda_\tau^o(T-h) - \tilde{\lambda}_\tau^o(T-h)|^2 \leqslant$$
$$\leqslant \frac{1}{2\varepsilon} \Big\{ (2L_4^2 + 3L_1^2 |\lambda^o|^2) \|x^o - x_\tau^o\|_{m-1}^2 + 3L_1^2 |\lambda^o|^2 \|x^o - x_\tau^o\|_{m-2}^2 +$$
$$+ (2L_4^2 + 3L_1^2 |\lambda^o|^2) \|u^o - u_\tau^o\|_{m-1}^2 + \tau^2 \Big[M_0^2 \left\| \frac{d\lambda^o}{dt} \right\|_{m-1}^2 +$$
$$+ 2 \left\| \frac{d^2 \lambda^o}{dt^2} \right\|_{m-1}^2 \Big] \Big\}. \qquad (4.7)$$

In the same way we estimate

$$\int_{T-kh}^{T-(k+1)h} \delta_\tau(t)\, dt \quad \text{for } k = 1, 2, \ldots, m-1:$$

$$\int_{T-kh}^{T-(k+1)h} \delta_\tau(t)\, dt \geqslant \frac{1}{2} |\lambda_\tau^o(T-(k+1)h) - \lambda_\tau^o(T-(k+1)h)|^2 +$$
$$- \frac{1}{2} |\lambda_\tau^o(T-kh) - \tilde{\lambda}_\tau^o(T-kh)|^2 + \beta \|\lambda_\tau^o - \tilde{\lambda}_\tau^o\|_{m-1}^2 \qquad (4.8)$$

and

$$\int_{T-kh}^{T-(k+1)h} \delta_\tau(t)\,dt = \langle \Phi_x(x_\tau^o, u_\tau^o) - \Phi_x(x^o, u^o), \tilde{\lambda}_\tau^o - \lambda_\tau^o \rangle_{m-k-1} +$$
$$+ \left\langle \left(A_x(x_\tau^o, y_\tau^o, u_\tau^o) - A_x(x^o, y^o, u^o)\right) \tilde{\lambda}_\tau^o, \tilde{\lambda}_\tau^o - \lambda_\tau^o \right\rangle_{m-k-1} +$$
$$+ \langle A_x(x^o, y^o, u^o)(\lambda^o - \tilde{\lambda}_\tau^o), \tilde{\lambda}_\tau^o - \lambda_\tau^o \rangle_{m-k-1} + \left\langle \left(A_y(x_\tau^o, y_\tau^o, u_\tau^o) + \right.\right.$$
$$\left.\left. - A_y(x^o, y^o, u^o)\right) \lambda^o, \tilde{\lambda}_\tau^o - \lambda_\tau^o \right\rangle_{m-k} + \langle A_y(x_\tau^o, y_\tau^o, u_\tau^o)(\lambda_\tau^o - \lambda^o), \tilde{\lambda}_\tau^o - \lambda_\tau^o \rangle_{m-k} +$$
$$+ \left\langle \frac{d\lambda^o}{dt} - \nabla \tilde{\lambda}_\tau^o, \tilde{\lambda}_\tau^o - \lambda_\tau^o \right\rangle_{m-k}. \tag{4.9}$$

After applying inequality (3.4), Lipschitz condition to be satisfied by A_x, A_y, Φ_x and Lemma (3.1) we obtain from (4.9)

$$\int_{T-kh}^{T-(k+1)h} \delta_\tau(t)\,dt \leqslant \frac{1}{2\varepsilon} \Big\{ 2L_4^2 \left[\|x^o - x_\tau^o\|_{m-k-1}^2 + \|u^o - u_\tau^o\|_{m-k-1}^2 \right] +$$
$$+ 3L_1^2 |\lambda^o|^2 \left[\|x^o - x_\tau^o\|_{m-k-1}^2 + \|x^o - x_\tau^o\|_{m-k-2}^2 + \|u^o - u_\tau^o\|_{m-k-1}^2 \right] +$$
$$+ 3L_1^2 |\lambda^o|^2 \left[\|x^o - x_\tau^o\|_{m-k}^2 + \|x^o - x_\tau^o\|_{m-k-1}^2 + \|u^o - u_\tau^o\|_{m-k}^2 \right] +$$
$$+ M_0^2 \|\tilde{\lambda}_\tau^o - \lambda_\tau^o\|_{m-k}^2 + 2\tau^2 \left[M_0^2 \left\| \frac{d\lambda^o}{dt} \right\|^2 + \left\| \frac{d^2 \lambda^o}{dt^2} \right\|_{m-k-1}^2 \right] \Big\} +$$
$$+ 12\varepsilon \|\tilde{\lambda}_\tau^o - \lambda_\tau^o\|_{m-k-1}^2. \tag{4.10}$$

Combining (4.8) and (4.10) we have for $k = 1, \ldots, m-1$

$$\frac{1}{2} |\lambda_\tau^o(T-(k+1)h) - \tilde{\lambda}_\tau^o(T-(k+1)h)|^2 + (\beta - 12\varepsilon) \|\lambda_\tau^o - \tilde{\lambda}_\tau^o\|_{m-k-1}^2 \leqslant$$
$$\leqslant \frac{1}{2} |\lambda_\tau^o(T-kh) - \tilde{\lambda}_\tau^o(T-kh)|^2 + \frac{1}{2\varepsilon} \Big\{ [2L_4^2 +$$
$$+ 3|\lambda^o|^2 (L_1^2 + L_2^2)] \|x^o - x_\tau^o\|_{m-k-1}^2 + 3L_1^2 |\lambda^o|^2 \|x^o - x_\tau^o\|_{m-k-2}^2 +$$
$$3L_2^2 |\lambda^o|^2 \|x^o - x_\tau^o\|_{m-k}^2 + [2L_4^2 + 3L_1^2 |\lambda^o|^2] \|u^o - u_\tau^o\|_{m-k-1}^2 +$$
$$+ 3L_2^2 |\lambda^o| \|u^o - u_\tau^o\|_{m-k}^2 + M_0^2 \|\tilde{\lambda}_\tau^o - \lambda_\tau^o\|_{m-k}^2 +$$
$$+ 2\tau^2 \left[M_0^2 \left\| \frac{d\lambda^o}{dt} \right\|^2 + \left\| \frac{d^2 \lambda^o}{dt^2} \right\|_{m-k-1}^2 \right] \Big\}. \tag{4.11}$$

Denote:

$$\beta_0 \stackrel{\text{df}}{=} \beta - 12\varepsilon; \qquad \beta_1 \stackrel{\text{df}}{=} \frac{1}{2\varepsilon} M_0^2;$$

$$b_{m-1} \stackrel{\text{df}}{=} \frac{1}{2\varepsilon} \Big\{ [2L_4^2 + 3L_1^2 (\lambda^o|^2] \|x^o - x_\tau^o\|_{m-1}^2 + 3L_1^2 |\lambda^o|^2 \|x^o - x_\tau^o\|_{m-2}^2 +$$
$$+ [2L_4^2 + 3L_1^2 |\lambda^o|^2] \|u^o - u_\tau^o\|_{m-1}^2 + \tau^2 \left[M_0^2 \left\| \frac{d}{dt} \lambda^o \right\|_{m-1}^2 + 2 \left\| \frac{d^2 \lambda^o}{dt^2} \right\|_{m-1}^2 \right] \Big\}$$

for $k = 1, \ldots, m-1$;

$$b_{m-k-1} \stackrel{\text{df}}{=} \frac{1}{2\varepsilon} \Big\{ [2L_4^2 + 3L_1^2 |\lambda^o|^2] \, \|x^o - x_\tau^o\|_{m-k-1}^2 +$$

$$+ 3L_1^2 |\lambda^o|^2 \|x^o - x_\tau^o\|_{m-k-2}^2 + 3L_2^2 |\lambda^o|^2 \|x^o - x_\tau^o\|_{m-k}^2 +$$

$$+ [2L_4^2 + 3L_1^2 |\lambda^o|^2] \, \|u^o - u_\tau^o\|_{m-k-1}^2 + 3L_2^2 |\lambda^o|^2 \|u^o - u_\tau^o\|_{m-k}^2 +$$

$$+ 2\tau^2 \left[M_0^2 \left\| \frac{d}{dt} \lambda^o \right\|^2 + \left\| \frac{d^2 \lambda^o}{dt^2} \right\|_{m-k-1}^2 \right] \Big\}. \tag{4.12}$$

Using these notations we rewrite (4.7) and (4.11) in the form:

$$\frac{1}{2} |\lambda_\tau^o(T-h) - \tilde{\lambda}_\tau^o(T-h)|^2 + \beta_0 \|\lambda_\tau^o - \tilde{\lambda}_\tau^o\|_{m-1}^2 \leqslant b_{m-1} \tag{4.13}$$

$$\frac{1}{2} |\lambda_\tau^o(T-(k+1)\,h) - \tilde{\lambda}_\tau^o(T-(k+1)\,h)|^2 +$$

$$+ \beta_0 \|\lambda_\tau^o - \tilde{\lambda}_\tau^o\|_{m-k-1}^2 \leqslant b_{m-k-1} + \beta_1 \|\lambda_\tau^o - \tilde{\lambda}_\tau^o\|_{m-k}^2 +$$

$$+ \frac{1}{2} |\lambda_\tau^o(T-kh) - \tilde{\lambda}_\tau^o(T-kh)|^2 \quad \text{for } k=1, \ldots, m-1. \tag{4.14}$$

Let ε be choosen such that $\varepsilon < \frac{\beta}{12}$ (it is possible since $\beta > 0$).

Combining (4.13) and (4.14) we have:

$$\beta_0 \|\lambda_\tau^o - \tilde{\lambda}_\tau^o\|_{m-k-1}^2 \leqslant \left(\frac{\beta_1}{\beta_0} + 1 \right)^k b_{m-1} + \ldots + \left(1 + \frac{\beta_1}{\beta_0} \right) b_{m-k-2} + b_{m-k-1}$$

for $k=0, 1, \ldots, m-1$.

Hence

$$\|\lambda_\tau^o - \tilde{\lambda}_\tau^o\|^2 = \sum_{k=0}^{m-1} \|\lambda_\tau^o - \tilde{\lambda}_\tau^o\|^2 \leqslant \frac{m}{\beta_0} \left(\frac{\beta_1}{\beta_0} + 1 \right)^{m-1} \sum_{k-0}^{m-1} b_{m-k-1} \leqslant$$

$$\leqslant \frac{m}{2\beta_0\,\varepsilon} \left(\frac{\beta_1}{\beta_0} + 1 \right)^{m-1} \Big\{ [2L_4^2 + |\lambda^o|^2 (6L_1^2 + 3L_2^2)] \, \|x^o - x_\tau^o\|^2 +$$

$$+ [2L_4^2 + (3L_1^2 + 3L_2^2) \, |\lambda^o|^2] \, \|u^o - u_\tau^o\|^2 + \tau^2 \left[3L_1^1 |\lambda^o|^2 \left\| \frac{d\varphi}{dt} \right\|_1^2 + \right.$$

$$\left. + 2M_0^2 \left\| \frac{d\lambda^o}{dt} \right\|^2 + 2 \left\| \frac{d^2 \lambda^o}{dt^2} \right\|^2 \right] \Big\}. \tag{4.15}$$

If we denote:

$$C_2 \stackrel{\text{df}}{=} \frac{m}{2\beta_0\,\varepsilon} \left(\frac{\beta_1}{\beta_0} + 1 \right)^{m-1} [2L_4^2 + (6L_1^2 + 3L_2^2) \, |\lambda^o|^2], \tag{4.16}$$

$$C_3 \stackrel{\text{df}}{=} \frac{m}{2\beta_0\,\varepsilon} \left(\frac{\beta_1}{\beta_0} + 1 \right)^{m-1} [2L_4^2 + (3L_1^2 + 3L_2^2) \, |\lambda^o|^2], \tag{4.17}$$

$$C_4 \overset{\text{df}}{=} \frac{m}{2\beta_0\,\varepsilon}\left(\frac{\beta_1}{\beta_0}+1\right)^{m-1}\left[3L_1^2\,|\lambda^o|^2\left\|\frac{d\varphi}{dt}\right\|_{-1}^2+\right.$$

$$\left.+2M_0^2\left\|\frac{d\lambda^o}{dt}\right\|^2+2\left\|\frac{d^2\,\lambda^o}{dt^2}\right\|^2\right], \tag{4.18}$$

where β_1, β_0 one given by (4.12), then after applying Lemma (4.1) we complete the proof of Theorem 4.1.

5. Estimation of the difference between the optimal and approximate controls

In this section the difference between u^o and u^o_τ is estimated by a constant convergent to zero with τ. In order to find this error bound the Lagrange multipliers method is used. Two cases are considered independently. One is unconstrained problem where Ω is equal to the whole space $H^1\,[0,\,T;\,R^m]$. The other constrained optimization problem where Ω is a real subset of $H^1\,[0\ T,\,R^m]$.

5. 1. Unconstrained optimization problem

In this case conditions (2.1.13), (2.2.10) take on the form:

$$\delta u\,L\,(x^o,\,u^o,\,\lambda^o)=0\,, \tag{5.1.1}$$

$$\delta u_\tau L_\tau (x^o_\tau,\,u^o_\tau,\,\lambda^o_\tau)=0\,,$$

(it is easily deduced from standard arguments in the calculus of variations). This fact will be essential to the proof of the main result of this paper given by the following.

Theorem 5.1. Assume:

(i) $(x^o,\,u^o)$ and $(x^o_\tau,\,u^o_\tau)$ are the solutions of problems Q_0 and Q_τ respectively.

(ii) Hypothesis H1—H5 are satisfied.

(iii) The operator $A,\,A_x,\,A_y,\,A_u,\,\Phi_x,\,\Phi_u$ satisfy on $\tilde{G}$ Lipschitz condition with the constants $L_0, L_1, L_2, L_3, L_4, L_5$ respectively.

(iv) The norms $|A_x\,(x,\,y,\,u)|$, $|A_y\,(x,\,y,\,u)|$, $|A\,(x,\,y,\,u)|$, $\Phi_x\,(x,\,u)|$, $|\Phi_u\,(x,\,u)|$, $|A_{xx}\,(x,\,y,\,u)|$, $|A_{xy}\,(x,\,y,\,u)|$, $|A_{xu}\,(x,\,y,\,u)|$, $|\Phi_{xx}\,(x,\,u)|$, $|\Phi_{xu}\,(x,\,u)|$ are bounded by a constant $M\overset{\text{df}}{=}\max\,(M_0,\,M_1)$ for all $(x,\,y,\,u)\in\tilde{G}$.

(v) Operator $\mathscr{P}_\tau$ satisfies assumption of Lemma (3.1).

(vi) $\tau<\dfrac{\alpha}{2L_0^2}$.

(vii) $\Omega=H^1\,[0,\,T;\,R^m]$.

Then $\|u^o-u^o_\tau\|\leqslant C\tau$ where C depends on $\left(\alpha,\,\gamma,\,L_0,\,L_1,\,L_2,\,L_3,\,L_4,\,L_5,\,M,\,\|\varepsilon\|_{-1},\,\left\|\frac{d\varepsilon}{dt}\right\|_{-1},\,\left\|\frac{du^o}{dt}\right\|,\,m,\right)$ and is given by (5.1.13), (5.1.12), (5.1.10), (5.1.8).

Proof. Denote $\tilde{u}^o_\tau \overset{\text{df}}{=} \mathscr{P}_\tau u^o$. On the basis of Saddle Point Theorem (see Lemma 2.5) for $L_\tau(x_\tau, u_\tau, \lambda_\tau)$ we get:

$$J(x^o_\tau, u^o_\tau) = L_\tau(x^o_\tau, u^o_\tau, \lambda^o_\tau) \leqslant L_\tau(\tilde{x}^o_\tau, \tilde{u}^o_\tau, \lambda^o_\tau). \tag{5.1.2}$$

On the other hand recalling the result of Lemma (2.5), expanding $L_\tau(x^o_\tau, u^o_\tau, \tilde{\lambda}^o_\tau)$ by Taylor's formula about $(\tilde{x}^o_\tau, \tilde{u}^o_\tau, \tilde{\lambda}^o_\tau)$ and applying hypothesis H5′ we obtain the following lower bound for $J(x^o_\tau, u^o_\tau)$

$$\begin{aligned} J(x^o_\tau, u^o_\tau) &= L_\tau(x^o_\tau, u^o_\tau, \lambda^o_\tau) = L_\tau(x^o_\tau, u^o_\tau, \tilde{\lambda}^o_\tau) \geqslant L_\tau(\tilde{x}^o_\tau, \tilde{u}^o_\tau, \tilde{\lambda}^o_\tau) + \\ &+ \langle \delta_x L_\tau(\tilde{x}^o_\tau, \tilde{u}^o_\tau, \tilde{\lambda}^o_\tau), x^o_\tau - \tilde{x}^o_\tau \rangle + \langle \delta_y L_\tau(\tilde{x}^o_\tau, \tilde{u}^o_\tau, \tilde{\lambda}^o_\tau), y^o_\tau - \tilde{y}^o_\tau \rangle + \\ &+ \langle \delta_u L_\tau(\tilde{x}^o_\tau, \tilde{u}^o_\tau, \tilde{\lambda}^o_\tau), u^o_\tau - \tilde{u}^o_\tau \rangle + \gamma \, \|u^o_\tau - \tilde{u}^o_\tau\|^2. \end{aligned} \tag{5.1.3}$$

Adding (5.2) and (5.3) we obtain:

$$\begin{aligned} \gamma \, \|u^o_\tau - \tilde{u}^o_\tau\|^2 &\leqslant L_\tau(\tilde{x}^o_\tau, \tilde{u}^o_\tau, \lambda^o_\tau) - L_\tau(\tilde{x}^o_\tau, \tilde{u}^o_\tau, \tilde{\lambda}^o_\tau) + \\ &+ \langle \delta_x L_\tau(\tilde{x}^o_\tau, \tilde{u}^o_\tau, \tilde{\lambda}^o_\tau), x^o_\tau - \tilde{x}^o_\tau \rangle + \langle \delta_y L_\tau(\tilde{x}^o_\tau, \tilde{u}^o_\tau, \tilde{\lambda}^o_\tau), y^o_\tau - \tilde{y}^o_\tau \rangle + \\ &+ |\delta_u L_\tau(\tilde{x}^o_\tau, \tilde{u}^o_\tau, \tilde{\lambda}^o_\tau)| \, \|u^o_\tau - \tilde{u}^o_\tau\|. \end{aligned} \tag{5.1.4}$$

In order to proof Theorem the following expressions must be estimated:

$$L_\tau(\tilde{x}^o_\tau, \tilde{u}^o_\tau, \lambda^o_\tau) - L_\tau(\tilde{x}^o_\tau, \tilde{u}^o_\tau, \tilde{\lambda}^o_\tau); \tag{5.1.5}$$

$$\langle \delta_x L_\tau(\tilde{x}^o_\tau, \tilde{u}^o_\tau, \tilde{\lambda}^o_\tau), \alpha_1 \rangle + \langle \delta_y L_\tau(\tilde{x}^o_\tau, \tilde{u}^o_\tau, \tilde{\lambda}^o_\tau), \alpha_2 \rangle, \tag{5.1.6}$$

where $\alpha_2(t) \overset{\text{df}}{=} \alpha_1(t-h)$ and $\alpha_1(\Theta) = 0$ for $\Theta \in [-h, 0]$;

$$|\delta_u L_\tau(\tilde{x}^o_\tau, \tilde{u}^o_\tau, \tilde{\lambda}^o_\tau)|. \tag{5.1.7}$$

By definition of L_τ, Lemma 3.1, Schwartz inequality and Lipschitz condition we obtain:

$$\begin{aligned} L_\tau(\tilde{x}^o_\tau, \tilde{u}^o_\tau, \lambda^o_\tau) - L_\tau(\tilde{x}^o_\tau, \tilde{u}^o_\tau, \tilde{\lambda}^o_\tau) &= \langle \nabla \tilde{x}^o_\tau + A(\tilde{x}^o_\tau, \tilde{y}^o_\tau, \tilde{u}^o_\tau), \lambda^o_\tau - \tilde{\lambda}^o_\tau \rangle = \\ &= \left\langle \nabla \tilde{x}^o_\tau - \frac{dx^o}{dt}, \lambda^o_\tau - \tilde{\lambda}^o_\tau \right\rangle + \langle A(\tilde{x}^o_\tau, \tilde{y}^o_\tau, \tilde{u}^o_\tau) - \\ &- A(x^o, y^o, u^o), \lambda^o_\tau - \tilde{\lambda}^o_\tau \rangle \leqslant \tau \left[\sqrt{2} \left\| \frac{d^2 x^o}{dt^2} \right\| + L_0 \left(2 \left\| \frac{dx^o}{dt} \right\| + \right.\right. \\ &\left.\left. + \left\| \frac{d\varphi}{dt} \right\|_{-1} + \left\| \frac{du^o}{dt} \right\| \right) \right] \|\lambda^o_\tau - \tilde{\lambda}^o_\tau\|. \end{aligned} \tag{5.1.5}$$

Recalling that $\langle \delta_x L(x^o, u^o, \lambda^o), \alpha_1 \rangle + \langle \delta_y L(x^o, u^o, \lambda^o), \alpha_2 \rangle = 0$ [see condition (2.1.11)] and after applying again Lemma 3.1, Schwartz inequality and (4.1) we see that

$$\begin{aligned} &\langle \delta_x L_\tau(\tilde{x}^o_\tau, \tilde{u}^o_\tau, \tilde{\lambda}^o_\tau), \alpha_1 \rangle + \langle \delta_y L_\tau(\tilde{x}^o_\tau, \tilde{u}^o_\tau, \tilde{\lambda}^o_\tau), \alpha_2 \rangle = \langle \Phi_x(\tilde{x}^o_\tau, \tilde{u}^o_\tau) - \\ &- \Phi_x(x^o, u^o); \alpha_1 \rangle + \langle A_x(x^o, y^o, u^o)(\tilde{\lambda}^o_\tau - \lambda^o), \alpha_1 \rangle + \langle (A_x(\tilde{x}^o_\tau, \tilde{y}^o_\tau, \tilde{u}^o_\tau) - \end{aligned}$$

$$-A_x(x^o, y^o, u^o)\,\tilde{\lambda}^o_\tau, \alpha_1\rangle + \left\langle \frac{d}{dt}\lambda^o - \nabla\tilde{\lambda}^o_\tau, \alpha_1 \right\rangle +$$

$$+\langle A_y(x^o, y^o, u^o)(\tilde{\lambda}^o_\tau - \lambda^o), \alpha_2\rangle + \big\langle (A_y(\tilde{x}^o_\tau, \tilde{y}^o_\tau, \tilde{u}^o_\tau) -$$

$$- A_y(x^o, y^o, u^o))\,\tilde{\lambda}^o_\tau, \alpha_2\big\rangle \leqslant \tau \left[L\left(\left\| \frac{dx^o}{dt} \right\| + \left\| \frac{du^o}{dt} \right\| \right) + M^o \left\| \frac{d\lambda^o}{dt} \right\| + \right.$$

$$\left. + L_1 |\lambda^o| \left[2\left\| \frac{dx^o}{dt} \right\| + \left\| \frac{d\varphi}{dt} \right\|_{-1} + \left\| \frac{du^o}{dt} \right\| \right] + \sqrt{2} \left\| \frac{d^2\lambda^o}{dt^2} \right\| \right] \|\alpha_1\| +$$

$$+\tau \left[M^o \left\| \frac{d\lambda^o}{dt} \right\| + L_2 |\lambda^o| \left[2\left\| \frac{dx^o}{dt} \right\| + \left\| \frac{d\varphi}{dt} \right\|_{-1} + \left\| \frac{du^o}{dt} \right\| \right] \right] \|\alpha_2\|. \quad (5.1.6)$$

Since condition (5.1.1) is satisfied then:

$$|\delta_{u_\tau} L_\tau(\tilde{x}^o_\tau, \tilde{u}^o_\tau, \tilde{\lambda}^o_\tau)| = \left| \int_0^T \left[\Phi_u(\tilde{x}^o_\tau(t), \tilde{u}^o_\tau(t)) - \Phi_u(x^o(t), u^o(t)) \right] \right| -$$

$$+\langle A_u(\tilde{x}^o_\tau, \tilde{y}^o_\tau, \tilde{u}^o_\tau) - A_u(x^o, y^o, u^o), \lambda^o_\tau\rangle + \langle A_u(x^o, y^o, u^o), \tilde{\lambda}^o_\tau - \lambda^o\rangle| \leqslant$$

$$\leqslant \tau \left[L_5 \left(\left\| \frac{dx^o}{dt} \right\| + \left\| \frac{du^o}{dt} \right\| \right) + L_3 \left(2\left\| \frac{dx^o}{dt} \right\| + \left\| \frac{d\varphi}{dt} \right\|_{-1} + \right.\right.$$

$$\left.\left. + \left\| \frac{du^o}{dt} \right\| \right) |\lambda^o| + M^o \left\| \frac{d\lambda^o}{dt} \right\| \right]. \quad (5.1.7)$$

Denote:

$$C_5 \stackrel{\text{df}}{=} \sqrt{2} \left\| \frac{d^2 x^o}{dt^2} \right\| + L_0 \left(2\left\| \frac{dx^o}{dt} \right\| + \left\| \frac{d\varphi}{dt} \right\|_{-1} + \left\| \frac{du^o}{dt} \right\| \right);$$

$$C_6 \stackrel{\text{df}}{=} L_4 \left(\left\| \frac{dx^o}{dt} \right\| + \left\| \frac{du^o}{dt} \right\| \right) + 2M^o \left\| \frac{d\lambda^o}{dt} \right\| + (L_1 |\lambda|^o +$$

$$+ L_2 |\lambda^o|) \left[2\left\| \frac{dx^o}{dt} \right\| + \left\| \frac{d\varphi}{dt} \right\|_{-1} + \left\| \frac{du^o}{dt} \right\| + \sqrt{2} \left\| \frac{d^2\lambda^o}{dt^2} \right\|; \right.$$

$$\bar{C}_7 \stackrel{\text{df}}{=} L_5 \left(\left\| \frac{dx^o}{dt} \right\| + \left\| \frac{du^o}{dt} \right\| \right) + L_3 \left(2\left\| \frac{dx^o}{dt} \right\| + \left\| \frac{d\varphi}{dt} \right\|_{-1} + \right.$$

$$\left. + \left\| \frac{du^o}{dt} \right\| \right) |\lambda^o| + M^o \left\| \frac{d\lambda^o}{dt} \right\|;$$

$$\bar{C}_8 \stackrel{\text{df}}{=} \left[M^o \left\| \frac{d\lambda^o}{dt} \right\| + L_2 |\lambda^o| \left(2\left\| \frac{dx^o}{dt} \right\| + \left\| \frac{d\varphi}{dt} \right\|_{-1} + \left\| \frac{du^o}{dt} \right\| \right) \right] \left\| \frac{d\varepsilon}{dt} \right\|_{-1}. \quad (5.1.8)$$

Substituting (5.1.5), (5.1.6), (5.1.7) into (5.1.4) and using notation (5.1.8) we get:

$$\gamma\,\|u^o-\tilde{u}^o_\tau\|^2\leqslant\tau\,[C_5\,\|\lambda^o_\tau-\tilde{\lambda}^o_\tau\|+C_6\,\|x^o_\tau-\tilde{x}^o_\tau\|+\bar{C}_7\,\|u^o_\tau-\tilde{u}^o_\tau\|]+$$
$$+\bar{C}_8\,\tau^2\leqslant\tau\,[C_5\,|\lambda^o_\tau-\lambda^o_\tau|+C_6\,\|x^o_\tau-x^o_\tau\|+\bar{C}_7\,\|u^o_\tau-u^o\|]+$$
$$+\tau^2\left[\bar{C}_8+C_5\left\|\frac{d\lambda^o}{dt}\right\|+C_6\left\|\frac{dx^o}{dt}\right\|+\bar{C}_7\left\|\frac{du^o}{dt}\right\|\right].$$

Since

$$\gamma.\|u^o_\tau-u^o\|^2\leqslant\gamma\,\|u^o_\tau-u^o_\tau\|^2+\gamma^2_\tau\left\|\frac{du^o}{dt}\right\|^2+\tau\,2\gamma\,\|u^o_\tau-u^o\|\left\|\frac{du^o}{dt}\right\|$$

then:

$$\gamma\,\|u^o_\tau-u^o\|^2\leqslant\tau\Bigg[\bar{C}_5\,\|\lambda^o_\tau-\lambda^o\|+C_6\,\|x^o_\tau-x^o\|+$$
$$+\bar{C}_7\,2\gamma\left\|\frac{du^o}{dt}\right\|\|u^o_\tau-u^o\|\Bigg]+\tau^2\Bigg[\bar{C}_8+\bar{C}_5\left\|\frac{d\lambda^o}{dt}\right\|+C_6\left\|\frac{dx^o}{dt}\right\|+$$
$$+\bar{C}_7\left\|\frac{du^o}{dt}\right\|+\gamma\left\|\frac{du^o}{dt}\right\|^2\Bigg].\qquad(5.1.9)$$

Set

$$C_7\stackrel{\text{df}}{=}\bar{C}_7\,2\gamma\left\|\frac{du^o}{dt}\right\|,$$
$$C_8\stackrel{\text{df}}{=}\bar{C}_8+C_5\left\|\frac{d\lambda^o}{dt}\right\|+C_6\left\|\frac{dx^o}{dt}\right\|+\bar{C}_7\left\|\frac{du^o}{dt}\right\|+\gamma\left\|\frac{du^o}{dt}\right\|^2.\qquad(5.1.10)$$

Using notations (5.1.10) we rewrite (5.1.9) in the form

$$\gamma\,\|u^o-u^o_\tau\|^2\leqslant\tau\,[C_5\,\|\lambda^o_\tau-\lambda^o\|+C_6\,\|x^o-x^o_\tau\|+C_7\,\|u^o-u^o_\tau\|]+C_8\,\tau^2.\qquad(5.1.11)$$

Applying Theorems 3.1 and 4.1 to the terms $\|\lambda^o-\lambda^o_\tau\|$ and $\|x^o-x^o_\tau\|$ respectively we conclude that:

$$\gamma\,\|u^o-u^o_\tau\|^2\leqslant\tau\,[C_5\,(\sqrt{C_2}\,\|x^o-x^o_\tau\|+\sqrt{C_3}\,\|u^o-u^o_\tau\|+\sqrt{C_4}\,\tau)+$$
$$+C_6\,(\sqrt{C_0}\,\|u^o-u^o_\tau\|+\sqrt{C_1}\,\tau)+C_7\,\|u^o-u^o_\tau\|+C_8\,\tau^2\leqslant$$
$$\leqslant\tau\,\|u^o-u^o_\tau\|\,(C_5\sqrt{C_2\,C_0}+C_5\sqrt{C_3}+C_6\sqrt{C_0}+C_7)+$$
$$+\tau^2\,(C_5\sqrt{C_1\,C_2}+C_5\sqrt{C_4}+C_6\sqrt{C_1}+C_8).$$

Denote:

$$B\stackrel{\text{df}}{=}C_5\sqrt{C_0\,C_2}+C_5\sqrt{C_3}+C_6\sqrt{C_0}+C_7,$$
$$B_1\stackrel{\text{df}}{=}C_5\sqrt{C_1\,C_2}+C_5\sqrt{C_4}+C_6\sqrt{C_1}+C_8.\qquad(5.1.12)$$

So we get:

$$\gamma\,\|u^o-u^o_\tau\|^2-\tau B\,\|u^o-u^o_\tau\|\leqslant\tau^2\,B_1.$$

Completing the square on the left-hand side of the inequality, and taking the square root of it we have:

$$\|u^o - u^o_\tau\| \leqslant C\tau \tag{5.1.13}$$

where $C \overset{\text{df}}{=} \frac{1}{\gamma}\left[\left(B_1 + \frac{B^2}{4\gamma}\right)^{\frac{1}{2}} + \frac{B}{2\gamma}\right]$ with B, B_1 defined by (5.1.12), (5.1.10), (5.1.8).

Using this fundamental error bound, it is a simple matter to show that similar bounds hold for the state and cost functional. These results are summarized in the following:

Corollary 5.1. If the assumptions of Theorems 5.1 are satisfied then:

$$\|x^o - x^o_\tau\| \leqslant \tau\left[\sqrt{C_0}\, C + \sqrt{C_1}\right]$$

and

$$J(x^o_\tau, u^o_\tau) - J(x^o, u^o) \leqslant \tau M\left(C + \sqrt{C_0}\, C + \sqrt{C_1}\right)$$

where C_0, C_1, C_2 are defined in Lemmas 3.1. and 5.1.

5.2. Constrained optimization problem

In this case the following estimation takes place

Theorem 5.2. Assume:

(i) (x^o, u^o) and (x^o_τ, u^o_τ) are the solutions of problem Θ_0 and Θ_τ respectively;

(ii) Hypothesis H1—H5 are satisfied;

(iii) Operators A, A_x, A_y, A_u, Φ_x satisfy Lipschitz condition with the constants. L_0, L_1, L_2, L_3, L_4 respectively on $\tilde{G}$;

(iv) The norms $|A_x(x, y, u)|$, $|A_y(x, y, u)|$, $|YA_u(x, y, u)|$, $|\Phi_x(x, u)|$, $|\Phi_u(x, u)|$ are bounded by $M>0$ for all $(x, y, u) \in \tilde{G}$;

(v) Operator P_τ stisfies assumption of Lemma 3.1.

(vi) $\Omega \neq H^1[0, T; R^m]$;

then

$$\|u^o - u^o_\tau\| \leqslant C_\tau t$$

where C defined by (5.2.16), depends on:

$$\left(\alpha, \gamma, L_0, L_1, L_2, L_3, L_4, M, \|\varepsilon\|_{-1}, \left\|\frac{d\varphi}{dt}\right\|_{-1}, \left\|\frac{du^o}{dt}\right\|, m,\right).$$

Proof. By Lemma 2.5 we have:

$$L_\tau(x^o_\tau, u^o_\tau, \lambda^o_\tau) \leqslant L_\tau(\tilde{x}^o_\tau, \tilde{u}^o_\tau, \lambda^o_\tau). \quad (5.2.1($$

Using x^o_τ and u^o_τ we construct the functions: $\hat{x} \in H^1[-h, T; R^n]$ and $\hat{u} \in \Omega$ satisfying the following conditions

$$\hat{x}(\Theta) = \varphi(\Theta),\ \Theta \in [-h; 0], \tag{5.2.2}$$

$$\|\hat{x} - x^o_\tau\| \leqslant \tau d_1, \tag{5.2.3}$$

$$\left\| \frac{d\hat{x}}{dt} - \nabla x_\tau^o \right\| \leqslant \tau d_2, \text{ }^{9)} \tag{5.2.4}$$

$$\|\hat{u} - u_\tau^o\| \leqslant \tau d_3, \text{ }^{10)} \tag{5.2.5}$$

where d_1, d_2, d_3 are given constants.

Denote $\hat{y}(t) \overset{\mathrm{df}}{=} \hat{x}(t-h), \quad t \in [0, T]$.

Conditions (2.1.11), (2.1.13) imply that:

$$\langle \delta_x L(x^o, u^o, \lambda^o), \hat{x} - x^o \rangle + \langle \delta_y L(x^o, u^o, \lambda^o)\, \hat{y} - y^o \rangle = 0, \tag{5.2.6}$$

$$\langle \delta_u L(x^o, u^o, \lambda^o), \hat{u} - u^o \rangle \geqslant 0. \tag{5.2.7}$$

Applying Taylor's formula to $L(\hat{x}, \hat{u}, \lambda^o)$ about (x^o, u^o, λ^o) and taking into account conditions (5.2.6), (5.2.7) as well as hypothesis H5 we obtain:

$$\begin{aligned} L(\hat{x}, \hat{u}, \lambda^o) \geqslant L(x^o, u^o, \lambda^o) + \langle \delta_u L(x^o, u^o, \lambda^o), \hat{u} - u^o \rangle + \\ + \langle \delta_x L(x^o, u^o, \lambda^o), \hat{x} - x^o \rangle + \langle \delta_y L(x^o, u^o, \lambda^o), \hat{y} - y^o \rangle + \\ + \gamma \|\hat{u} - u^o\|^2 \geqslant L(x^o, u^o, \lambda^o) + \gamma \|\hat{u} - u^o\|^2. \end{aligned} \tag{5.2.8}$$

By (5.2.8) off nad Lemma 2.5 we have:

$$\begin{aligned} L_\tau(x_\tau^o, u_\tau^o, \lambda_\tau^o) = L_\tau(x_\tau^o, u_\tau^o, \tilde{\lambda}_\tau^o) = L_\tau(x_\tau^o, u_\tau^o, \tilde{\lambda}_\tau^o) + \\ - L(\hat{x}, \hat{u}. \lambda^o) + L(\hat{x}, \hat{u}, \lambda^o) \geqslant L_\tau(x_\tau^o, u_\tau^o, \tilde{\lambda}_\tau^o) - L(\hat{x}, \hat{u}, \lambda^o) + \\ + L(x^o, u^o, \lambda^o) + \gamma \|\hat{u} - u^o\|^2. \end{aligned} \tag{5.2.9}$$

Combining (5.2.1.) and (5.2.9) we arrive at:

$$\gamma \|\hat{u} - u^o\|^2 \leqslant L_\tau(\tilde{x}_\tau^o, \tilde{u}_\tau^o, \lambda_\tau^o) - L(x^o, u^o, \lambda^o) + L(\hat{x}, \hat{u}, \lambda^o) - L_\tau(x_\tau^o, u_\tau^o, \tilde{\lambda}_\tau^o). \tag{5.2.10}$$

After same inequality manipulations similar to those performed in the proof of Theorem 5.1. we obtain:

$$\begin{aligned} L_\tau(\tilde{x}_\tau^o, \tilde{u}_\tau^o, \lambda_\tau^o) - L(x^o, u^o, \lambda^o) = \int_0^T [\Phi(\tilde{x}_\tau^o(t), \tilde{u}_\tau^o(t)) + \\ - \Phi(x^o(t), u^o(t))]\, dt + \langle \nabla \tilde{x}_\tau^o - \frac{d}{dt} x^o, \lambda_\tau^o \rangle + \langle A(\tilde{x}_\tau^o, \tilde{y}_\tau^o, \tilde{u}_\tau^o) + \\ - A(x^o, y^o, u^o), \lambda_\tau^o \rangle \leqslant \tau \left[M \left(\left\| \frac{dx^o}{dt} \right\| + \left\| \frac{du^o}{dt} \right\| \right) + \sqrt{2} \left\| \frac{d^2 x^o}{dt^2} \right\| \|\lambda_\tau^o\| + \right. \\ \left. + L_0 \left(2 \left\| \frac{dx^o}{dt} \right\| + \left\| \frac{d\varphi}{dt} \right\|_{-1} + \left\| \frac{du^o}{dt} \right\| \right) \|\lambda_\tau^o\| \leqslant \tau \left[M \left(\left\| \frac{dx^o}{dt} \right\| + \left\| \frac{du^o}{dt} \right\| \right) + \right. \\ \left. + \frac{2mM}{\beta} \right] \sqrt{2} \left\| \frac{d^2 x^o}{dt^2} \right\| + L_0 \left(2 \left\| \frac{dx^o}{dt} \right\| + \left\| \frac{d\varphi}{dt} \right\|_{-1} + \left\| \frac{du^o}{dt} \right\| \right). \text{ }^{11)} \end{aligned} \tag{5.2.11}$$

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9) For x we can take for example a piece-wise linear function such that $\hat{x}(r\tau) = x_\tau^o(r\tau)$ for $r = 0, 1, \ldots, p$. This time $d_1 = \left\| \frac{dx}{dt} \right\|$ and $d_2 = 0$.

10) Such $\hat{u}$ exists due to condition (2.2.2).

11) Observe that $\|\lambda_\tau^o\| \leqslant \frac{2mM}{B}$ (the proof of this fact is almost identical to that of Lemma 2.4).

Employing conditions (5.2.3) and (5.2.4) we have

$$L(\hat{x}, \hat{u}, \lambda^o) - L_\tau(x_\tau^o, u_\tau^o, \tilde{\lambda}_\tau^o) = \int_0^T [\Phi(\hat{x}(t), \hat{u}(t)) - \Phi(x^o(t), u^o(t))]\, dt +$$

$$+ \left\langle \frac{d\hat{x}}{dt} - \nabla x_\tau^o, \lambda^o \right\rangle + \left\langle (A(\hat{x}, \hat{y}, \hat{u}) - A(x_\tau^o, y_\tau^o, u_\tau^o)), \lambda^o \right\rangle \leqslant$$

$$\leqslant \tau\, [M(d_1+d_3) + d_2\, |\lambda^o| + L_0(d_1+d_3+\|\varphi\|_{-1})\, |\lambda^o| . \quad (5.2.12)$$

Denote:

$$C_9 \overset{\mathrm{df}}{=} M\left(\left\| \frac{dx^o}{dt} \right\| + \left\| \frac{du^o}{dt} \right\| + \frac{2mM}{\beta} \left[\sqrt{2} \left\| \frac{d^2 x^o}{dt^2} \right\| + L_0 \left(2 \left\| \frac{dx^o}{dt} \right\| + \right.\right.\right.$$

$$\left.\left.\left. + \left\| \frac{d\varphi}{dt} \right\|_{-1} + \left\| \frac{du^o}{dt} \right\| \right) + M(d_1+d_3) + d_2\, |\lambda^o| + L_0(d_1+d_3+\|\varphi\|_{-1})\, |\lambda^o| \right] . \quad (5.2.13)$$

Substituting (5.2.11), (5.2.12) into (5.2.10) we get:

$$\gamma\, \|\hat{u} - u^o\|^2 \leqslant \tau C_9 . \quad (5.2.14)$$

Since

$$\|u_\tau^o - u^o\| \leqslant \|u_\tau^o - \hat{u}\| + \|u^o - \hat{u}\|$$

then

$$\|u_\tau^o - u^o\| \leqslant \left(\frac{C_9}{\gamma} \right)^{\frac{1}{2}} \tau^{\frac{1}{2}} + \tau d_3 \leqslant \tau^{\frac{1}{2}}\, \bar{C} \quad (5.2.15)$$

where

$$\bar{C} \overset{\mathrm{df}}{=} \left(\frac{C_9}{\gamma} \right)^{\frac{1}{2}} + \tau^{\frac{1}{2}} d_3 . \quad (5.2.16)$$

Using results of Lemmas 2.2., 4.1 on the terms with λ^o, $\frac{dx^o}{dt}$, $\frac{d^2 x^o}{dt}$ in (5.2.13) we complete the proof of the Theorem. Theorem 5.2 implies the following analogue to Collorary 5.1.

Corollary 5.2. If the assumptions of Theorem 5.2 are satisfied then:

$$\|x^o - x_\tau^o\| \leqslant (\sqrt{C_0}\, \bar{C} + \sqrt{C_1}\, \tau^{\frac{1}{2}})\, \tau^{\frac{1}{2}}$$

and

$$J(x_\tau^o, u_\tau^o) - J(x^o, u^o) \leqslant M\, (\bar{C} + \sqrt{C_0}\, C_1 + \sqrt{C_1}\, \tau^{\frac{1}{2}})\, \tau^{\frac{1}{2}}$$

where C_0, C_1, $\bar{C}$ are defined by (3.17), (3.18), (5.2.16) respectively:

As it was stated earlier the main object of the paper is to find the error bounds for $\|u^o - u_\tau^*\|$ where $u_\tau^* \in \Omega$ is some admissible control in $H^1[0, T; R^m]$ constructed using u_τ^o. This result is given in the next section.

6. Error estimation for the difference between an optimal and approximate in H^1 $[0, T; R^m]$ control

Let u_τ^* be admissible control corresponding to u_τ^o constructed according to condition (2.2.2).

Observe that

$$\|u_\tau^* - u^o\| \leqslant \|u_\tau^* - u_\tau^o\| + \|u_\tau^o - u^o\|.$$

For the case of unconstrained optimization as a result of Theorem 5.1 and condition (2.2.2) we have:

$$\|u_\tau^* - u^o\| \leqslant \tau \left[C + \left\| \frac{du^o}{dt} \right\| \right], \tag{6.1}$$

where C is given by (5.1.13).

Denote by x_τ^* solution of equation (2.1.1.) corresponding to u_τ^* and satisfying initial condition (2.1.2).

We wish to estimate $J(x^o, u^o) - J(x_\tau^*, u_\tau^*)$. To accomplish this, the difference between x^o and x_τ^* must be estimated. This result is given in the following Lemma:

Lemma 6.1. Assume that:

(i) Hypothesis H2, H3, H4 are satisfied,

(ii) A satisfied Lipschitz condition on the set $\tilde{G}$ with a constant L_0

Then

$$\|x^o - x_\tau^*\| \leqslant \tilde{C}\, \|u^o - u_\tau^*\|$$

where

$$\tilde{C} \stackrel{\text{df}}{=} \frac{4m}{\varepsilon} L_0^2 \left(1 + \frac{2L}{\alpha - 2\varepsilon}\right)^{m-1} \quad \text{and } \varepsilon < \frac{\alpha}{2}.$$

Proof. To prove this Lemma step by step method is used. Recal that x^o and x_τ^* satisfy the following equations:

$$\frac{dx^o}{dt} + A\left(x^o(t), x^o(t-h), u^o(t)\right) = 0; \quad x(\Theta) = \varphi(\Theta), \ \Theta \in [-h, 0], \tag{6.1}$$

$$\frac{dx_\tau^*(t)}{dt} + A\left(x_\tau^*(t), x_\tau^*(t-h), u_\tau^*(t)\right) = 0; \quad x_\tau^*(\Theta) = \varphi(\Theta), \ \Theta \in [-h, 0]. \tag{6.2}$$

After substructing (6.1) and (6.2), multiplying the result by $x^o - x_\tau^*$ and integrating from 0 to h we have

$$\frac{1}{2}|x^o(h) - x_\tau^*(h)|^2 + \int_0^h \left(A\left(x^o(t), \varphi(t-h)\right), u^o(t)\right) +$$
$$- A\left(x_\tau^*(t), \varphi(t-h), u_\tau^*(t), x^o(t) - x_\tau^*(t)\right) dt = 0. \tag{6.3}$$

Adding and substructing term $\int_0^h \left(A\left(x_\tau^*(t),\, x^o(t-h),\, u^o(t)\right),\, x^o(t)-x_\tau^*(t)\right) dt$, applying hypothesis H4, Lipschitz condition and inequality (3.4) we obtain:

$$\frac{1}{2}|x^o(h)-x_\tau^*(h)|^2+(\alpha-2\varepsilon)\,\|x^o-x_\tau^*\|_0^2\leqslant\frac{1}{2\varepsilon}L_0^2\,[u_\tau^*-u^o\|_0^2. \tag{6.4}$$

Using the same argument for $k=1, \ldots, m-1$ we get:

$$\frac{1}{2}\left|x^o((k+1)\,h)-x\,((k+1)+h)\right|^2-\frac{1}{2}|x^o(kh)-x_\tau^*(kh)|^2+$$
$$+(\alpha-2\varepsilon)\,\|x^o-x_\tau^*\|_k^2\leqslant\frac{1}{2\varepsilon}L_0^2\left(\|x^o-x_\tau^*\|_{k-1}^2+\|u^o-u_\tau^*\|_k^2\right). \tag{6.5}$$

From (6.4) and (6.5) it follows that

$$\frac{1}{2}|x^o(k+1)\,h-x_\tau^*(k+1)\,h|^2+(\alpha-2\varepsilon)\,\|x^o-x_\tau^*\|_k^2\leqslant$$
$$\leqslant\frac{1}{2\varepsilon}L_0^2\,\|u_\tau^*-u^o\|_k^2+\ldots+\frac{1}{2\varepsilon}L_0^2\left(1+\frac{2L_0}{\alpha-2\varepsilon}\right)^k\|u_\tau^*-u^o\|_0^2=$$
$$=\frac{1}{2\varepsilon}L_0^2\left[\left(1+\frac{2L_0}{\alpha-2\varepsilon}\right)^k\|u_\tau^*-u^o\|+\ldots+\|u_\tau^*-u^o\|_k^2\right].$$

Hence

$$\|x^o-x_\tau^*\|^2\leqslant\sum_{k=0}^{m-1}\|x^o-x_\tau^*\|_k^2\leqslant\left(1+\frac{2L_0}{\alpha-2\varepsilon}\right)^{m-1}\frac{4mL_0^2}{\varepsilon}\|u_\tau^*-u^o\|^2 \tag{6.6}$$

where ε is chosen such that $\alpha-2\varepsilon>0$. Q.E.D.

Lemma 6.1 implies the following result.

Theorem 6.1. If all assumptions of Theorem 5.1 are satisfied then:

$$\|x^o-x_\tau^*\|\leqslant\tilde{C}\left(C+\left\|\frac{du^o}{dt}\right\|\right)\tau \tag{6.6}$$

($\tilde{C}$, C are given by Lemma 6.1 and Theorem 5.1), and

$$J(x_\tau^*,\, u_\tau^*)-J(x^o,\, u^o)\leqslant M\left(C+\left\|\frac{du^o}{dt}\right\|\right)(1+\tilde{C})\,\tau.$$

Proof. The estimation (6.7) is obtained by direct substitution of (6.1) into inequality result of Lemma 6.1. The second inequality is obtained by expanding $J(x_\tau^*, u_\tau^*)$ in Taylor series about (x^o, u^o) and by employing (6.1) and (6.7). For constrained case the result analogous to Theorem 6.1 is obtained:

Theorem 6.2. If all assumptions of Theorem 5.2 are satisfied then:

$$\|x^o-x_\tau^*\|\leqslant\tilde{C}\left(\bar{C}+\left\|\frac{du^o}{dt}\right\|\tau^{\frac{1}{2}}\right)\tau^{\frac{1}{2}} \tag{6.8}$$

and

$$J(x_\tau^*, u_\tau^*) - J(x^o, u^o) \leqslant M\left(\bar{C} + \left\|\frac{du^o}{dt}\right\| \tau^{\frac{1}{2}}\right)(1+\bar{C})\,\tau^{\frac{1}{2}} \tag{6.9}$$

where $\bar{C}$ is defined in Theorem 5.2.

The proof uses arguments the same as those in Theorem 6.1 (In order to obtain (6.8), (6.9) observe that condition (6.1) must be replaced by the following one:

$$\|u_\tau^* - u^o\| \leqslant \left(\bar{C} + \left\|\frac{du^o}{dt}\right\| \tau^{\frac{1}{2}}\right)\tau^{\frac{1}{2}}.$$

Conclusions and remarks

A finite difference approximation of optimal control problem for systems with delay was investigated in the paper. Such an approximation problem can be effectively solved using computer. In this way we obtain a suboptimal control for real system. As examplified in the work [7, 8, 9] finite difference methods usually produce simple computational algorithm. A number of useful itterative methods are available for treating problem of this type [10, 11]. A priori estimates for differential between such suboptimal and optimal controls were derived.

The cases of unconstrained and constrained optimization problems were considered independently.

I. It was shown that for unconstrained problem the obtained suboptimal control converges to the optimal one with the rate $0(\tau)$ [see Theorems 5.1, 6.1], whereas for constrained problem this rate is equal to $0(\tau^{\frac{1}{2}})$ [see Theorems 5.2, 6.2]

II. In order to obtain convergence of suboptimal control to optimal (without estimation of its rate) the weaker assumptions than those of Theorems 5.1 and 5.2 are required.

It is enough to assume that hypothesis H1—H5 (and H5′) are satisfied and that A_x, A_y, Φ_x are bounded operators in the sense of $L_\infty(\tilde{G})$.

III. Results given in the paper can be extended to the case where A and Φ depend explicity on time t. In order to obtain the error bound in this case let us consider the following "auxiliary" approximation of initial problem minimize $J_\tau(x, u) \stackrel{\text{df}}{=} \int_0^T \Phi_\tau(x(t), u(t))\, dt$ subject to the constraints:

$$\frac{dx(t)}{dt} + A_\tau(x(t), x(t-h), u(t)) = 0, \qquad t \in [0, T]$$

$$x(\Theta) = \varepsilon(Q) \;\; \Theta \in [-h, 0]$$

$$u \in \Omega, \;\; x \in H^1[0, T; R^n]$$

where $\Phi_\tau(x, u)$ and $A_\tau(x, y, u)$ are defined as

$$\Phi_\tau(x(t), u(t)) \stackrel{\text{df}}{=} \Phi(x(t), u(t), t_r)$$

and for t

$$A_\tau(x(t), y(t), u(t)) \stackrel{\mathrm{df}}{=} A(x(t), y(t), u(t), t_{r1})$$

where $t_r, t_{r1}, \in [r\tau; (r+1)\tau]$.

We shall refer to above problem as problem Θ_1. It easy to see that problem Θ_1 can be vieved as an optimization problem of the same form as before (since A and Φ do not depend on time). Then to obtain the desired result the difference between the optimal solutions of initial optimization problem and problems Θ_1 must be estimated additionally. The error bound commited in such approximation (approximation Θ_1) is derived on analogous way to that given in the proof of Lemma 5.1 and therefore is omitted here.

IV. Results given in the paper are available for the case where A doesn't depend on $x(t)$. This time strong monotonne condition (hypothesis H4) is not headed at all.

Appendix A

Proof of Lemma 2.2

(i) follows directly from Lemma 2.1.

We are going to prove (ii).

(ii) Taking advantage of Lipschitz condition we get from (2.1.1)

$$\left\|\frac{dx^o}{dt}\right\| = \|A(x^o, y^o, u^o)\| \leqslant \|A(x^o, y^o, u^o) - A(0, 0, 0)\| + \\ + \|A(0, 0, 0)\| \leqslant L_0 [\|x^o\| + \|y^o\| + \|u^o\|] + \|A(0, 0, 0)\|.$$

Applying this time the result of Lemma 2.1 to terms $\|x^o\|$, $\|y^o\|$ we obtain

$$\left\|\frac{dx^o}{dt}\right\| \leqslant L_0 [2\|x^o\| + \|\varphi\|_{-1} + \|u^o\|] + \|A(0, 0, 0)\| \leqslant \\ \leqslant L_0 [2\rho_x + \rho_u + \|\varphi\|_{-1} + \|A(0, 0, 0)\|.$$

Denoting

$$g_1 \stackrel{\mathrm{df}}{=} L_0 [2\rho_x + \rho_u + \|\varphi\|_{-1}] + \|A(0, 0, 0)\| \tag{A1}$$

we arrive at (ii)

(iii) is proved by differentiating tha state equation (2.1.1) with respect to t. Namely

$$\left\|\frac{d^2 x^o}{dt^2}\right\| \leqslant |A_x(x^o, y^o, u^o)| \left\|\frac{dx^o}{dt}\right\| + |A_y(x^o, y^o, u^o)| \left\|\frac{dy^o}{dt}\right\| + \\ + |A_u(x^o, y^o, u^o)| \left\|\frac{du^o}{dt}\right\| \leqslant M^o \left[\left\|\frac{dx^o}{dt}\right\| + \left\|\frac{dy^o}{dt}\right\| + \left\|\frac{du^o}{dt}\right\|\right] \leqslant \\ \leqslant M^o \left[2\left\|\frac{dx^o}{dt}\right\| + \left\|\frac{d\varphi}{dt}\right\|_{-1} + \left\|\frac{du^o}{dt}\right\|\right].$$

Denote

$$g_2 \overset{\text{df}}{=} M_o\left[2g_1 + \left\|\frac{d\varphi}{dt}\right\|_{-1} + \left\|\frac{du^o}{dt}\right\|\right] \tag{A2}$$

then by virtue of (ii) we get:

$\left\|\frac{d^2 x^o}{dt^2}\right\| \leqslant g_2$ what completes the proof of the Lemma.

Appendix B

Proof of Lemma 2.4

First we wish to show that there exists some constant ρ_{1u} such that $\|u^o\| \leqslant \rho_{1u}$ for all $\tau < \tau_0$. In order to prove this, let us denote by $(x^o_{\tau 0}, u^o_{\tau 0})$ and $(x^o_{\tau 1}, u^o_{\tau 1})$ the optimal solutions of problem $\Theta_{\tau 0}$ and $\Theta_{\tau 1}$ respectively.

Suppose that τ_0 is a fixed discretization and τ_1 is less than τ_0. Let $x_{\tau 1}$ be a solution of (2.2.3) with the initial condition (2.2.4), (2.2.5) (for the step of discretization equal to τ_1) corresponding to control $u^o_{\tau 0}$ (since $\tau_1 < \tau_0$ then $u^o_{\tau 0}$ may be considered as an element of $E_{\tau 1}\,[0, T; R^m]$).

Then by optimality we obtain

$$J(x^o_{\tau 1}, u^o_{\tau 1}) \leqslant J(x_{\tau 1}, u^o_{\tau 0}). \tag{B.0}$$

We are now going to show that $|J(x_{\tau 1}, u^o_{\tau 0})|$ is bounded independently on τ_1. Indeed, note that $x_{\tau 1}$ satisfies the following equation

$$\begin{aligned} &\nabla x_{\tau 1}(t) + A\left(x_{\tau 1}(t), x_{\tau 1}(t-h), u^o_{\tau 0}(t)\right) = 0 \\ &x_{\tau 1}(\Theta) = \varphi_{\tau 1}(\Theta), \quad \Theta \in [-h, 0), \\ &x_{\tau 1}(0) = \varphi(0). \end{aligned} \tag{B.1}$$

Denote

$$\bar{x}_{\tau 1}(t) \overset{\text{df}}{=} \frac{x_{\tau 1}(t+\tau_1) + x_{\tau 1}(t)}{2}.$$

After multiplying (B.1) by $\bar{x}_{\tau 1}(t)$, and integreting from 0 to $+h$ we obtain

$$\begin{aligned} &\int_0^h (\nabla x_{\tau 1}(t), \bar{x}_{\tau 1}(t))\, dt + \int_0^h \big(A(\bar{x}_{\tau 1}(t), \varphi_{\tau 1}(t-h), u^o_{\tau 0}(t)) - \\ &\quad - \big(A(0, \varphi_{\tau 1}(t-h), u^o_{\tau 0}(t)), \bar{x}_{\tau 1}(t)\big)\, dt + \int_0^h \big(A(x_{\tau 1}(t), \varphi_{\tau 1}(t-h), u^o_{\tau 0}(t)) - \\ &\quad - A(\bar{x}_{\tau 1}(t), \varphi_{\tau 1}(t-h), u^o_{\tau 0}(t)), \bar{x}_{\tau 1}(t)\big)\, dt + \int_0^h \big(A(0, \varphi_{\tau 1}(t-h), u^o_{\tau 0}(t))\big) = 0. \end{aligned}$$

By employing hypothesis H4, inequality (3.4) and continuity of A we see that

$$\frac{1}{2}|x_{\tau 1}(h)|^2-\frac{1}{2}|x_{\tau 1}(0)|^2+\left(\alpha-\frac{1}{2\varepsilon}\right)\|\bar{x}_{\tau 1}\|\leqslant 2\varepsilon\,[h\varepsilon_0^2+L_{\gamma_0}^2]$$

where

$$\gamma\overset{\text{df}}{=}\{(0,\varphi,u_{\tau 0}^o)\in E_\tau[0,h;R^n]\times E_\tau[(-h;0);R^n]\times E_\tau[0,h;R^m]\} \tag{B2}$$

and L_{γ_0} denotes the maximal value of A over the set γ_0. Since $\|x_{\tau 1}-\bar{x}_{\tau 1}\|_{\tau_1\to 0}\to 0$ then by continuity of A for any fixed $\varepsilon_0>0$ there exist τ_1 such that for any $\tau_1<\bar{\tau}_1$

$$\|A(x_{\tau 1},\varphi_{\tau 1},u_{\tau 0}^o)-A(\bar{x}_{\tau 1},\varphi_{\tau 1},u_{\tau 0}^o)\|\leqslant\varepsilon_0.$$

By the identical arguments we prove that for $p=1,\ldots,m-1$

$$\frac{1}{2}|x_{\tau 1}(k+1)h|^2-\frac{1}{2}|x_{\tau 1}(kh)|^2+\left(\alpha-\frac{1}{2\varepsilon}\right)\|x_{\tau 1}\|_k^2\leqslant 2\varepsilon\,[\varepsilon_0^2h+L_{\gamma_k}^2]$$

where

$$\gamma_k\overset{\text{df}}{=}\Big\{(0,x,u_{\tau_0}^o)\in E_\tau[kh,(k+1)h;R^n]\times E_\tau[(k-1)h;kh;R^n]\times$$
$$\times E_\tau[kh;(k+1)h;R^m],\ \|x\|_{k-1}\leqslant\frac{2}{\alpha-\frac{1}{2\varepsilon}}[k\varepsilon_o^2h+L_{\gamma_0}^2+L_{\gamma_{k-1}}^2+x^2(0)]\Big\}. \tag{B3}$$

Combining the above results we arrive at

$$\|x_{\tau 1}\|^2\leqslant\frac{2\varepsilon}{\alpha-\frac{1}{2\varepsilon}}\left[\varepsilon_0^2T+m\,(L_{\gamma_0}^2+\ldots+L_{\gamma_k}^2+x^2(0))\right],\qquad \varepsilon<\frac{1}{2\alpha}, \tag{B4}$$

where $\gamma_0\ldots\gamma_k$ are defined by requrence formula (B.2) and (B.3). Since J is a bounded operator then (B,0) and (B,4) imply that $|J(x_{\tau 1}^o,u_{\tau 1}^o)|$ is bounded independently on τ_1. Furthermore by hypothesis H3 we conclude that there exists $\rho_{1u}<\infty$ such that:

$$\|u_{\tau 1}^o\|\leqslant\rho_{1u}\qquad\text{for all }\tau_1<\tau_0 \tag{B.5}$$

(the proof of this fact uses arguments similar to those given in the proof of Lemma 2.1)

In the some way as was obtained (B.4) it is now a simple matter to show that there exists $\rho_{1x}>0$ such that

$$\|x_{\tau 1}^o\|\leqslant\rho_{1x}\qquad\text{for all }\tau_1<\tau_0.\qquad\text{Q.E.D.} \tag{B.6}$$

Appendix C

Proof of Lemma 4.1

Using Gronwall's inequality [13] we obtain from (4.1).

$$|\lambda^o|_{m-1}\leqslant h\,\|\Phi_x(x^o,u^o)\|_{m-1}\left(\exp\left[\|A_x(x^o,y^o,u^o)\|_{m-1}h\right]\right)^2.$$

Furthermore for $k=1, \ldots, m-1$ we have:

$$|\lambda^o|_{m-k-1} \leqslant h\, \|\Phi_x(x^o, u^o)\|_{m-k-1} \left((\exp[\|A_x(x^o, y^o, u^o)\|_{m-k-1}\, h])^2 + \right.$$
$$\left. + \left(h\, \|A_y(x^o, y^o, u^o)\|_{m-k} \left((\exp[\|A_x(x^o, y^o, u^o)\|_{m-k-1}\, h]\right)^2 + 1\right) |\lambda^o|_{m-k}.$$

Hence for $k=0, 1, \ldots, m-1$

$$|\lambda^o|_{m-k-1} \leqslant hM(\exp(M_0 h))^2 \sum_{l=0}^{k} [hM(\exp(M_0 h))^2 + 1]^l.$$

Then

$$|\lambda^o| \leqslant hM(\exp(M_0 h))^2 \sum_{l=0}^{m-1} [hM(\exp(M_0 h))^2 + 1]^l.$$

Denoting

$$g_3 \stackrel{\mathrm{df}}{=} hM(\exp(M_0 h))^2 \sum_{l=0}^{m-1} [hM(\exp(M_0 h))^2 + 1]^l \qquad \text{(C.1)}$$

we obtain (i).

We are going to prove (ii). To accomplish this we estimate:

$$\left\| \frac{d\lambda^o}{dt} \right\|_{m-1} \leqslant \|\Phi_x(x^o, u^o)\|_{m-1} + \|A_x(x^o, y^o, u^o)\|_{m-1} |\lambda^o| \leqslant M + M^o g_3$$

and

$$\left(\int_0^{T-h} \left| \frac{d}{dt} \lambda^o \right|^2 dt \right)^{\frac{1}{2}} \leqslant M + M^o g_3 +$$
$$+ \left[\int_0^{T-h} [(A_y(x^o(t+h), x^o(t), u^o(t+h))\, \lambda^o(t+h))]^2\, dt \right]^{\frac{1}{2}} \leqslant M + 2M^o g_3.$$

So

$$\left\| \frac{d}{dt} \lambda^o \right\| \leqslant g_4$$

where

$$g_4 \stackrel{\mathrm{df}}{=} 2M + 3M^o g_3. \qquad \text{(C.2)}$$

In order to prove (iii) we differentiate (4.1) with respect to t and obtain

$$\left\| \frac{d^2 \lambda^o}{dt^2} \right\| \leqslant |\lambda^o| \left[|A_{xx}(x^o, y^o, u^o)| \left\| \frac{dx^o}{dt} \right\| + |A_{xy}(x^o, y^o, u^o)| \left\| \frac{dy^o}{dt} \right\| + \right.$$
$$\left. + |A_{xu}(x^o, y^o, u^o)| \left\| \frac{du^o}{dt} \right\| \right] + \Phi_{xx}(x^o, y^o, u^o)| \left\| \frac{dx^o}{dt} \right\| + \Phi_{xu}(x^o, u^o)| \left\| \frac{du^o}{dt} \right\| +$$
$$+ \left\| \frac{d\lambda^o}{dt} \right\| |A_x(x^o, y^o, u^o)| + |\lambda^o| \left[|A_{yx}(x^o, y^o, u^o)| \left\| \frac{dx^o}{dt} \right\| + \right.$$
$$\left. + |A_{yy}(x^o, y^o, u^o)| \left\| \frac{dx^o}{dt} \right\| + |A_{yu}(x^o, y^o, u^o)| \left\| \frac{du^o}{dt} \right\| \right] + \left\| \frac{d\lambda^o}{dt} \right\| |A_y(x^o, y^o, u^o)|.$$

Hence

$$\left\|\frac{d^2\lambda^o}{dt}\right\| \leqslant Mg_3\left[2\left\|\frac{dx^o}{dt}\right\|+\left\|\frac{d\varphi}{dt}\right\|_{-1}+\left\|\frac{du^o}{dt}\right\|\right]+$$

$$+M\left[\left\|\frac{dx^o}{dt}\right\|+\left\|\frac{du^o}{dt}\right\|\right]+2g_4\, M+Mg_3\left[2\left\|\frac{dx^o}{dt}\right\|+\left\|\frac{du^o}{dt}\right\|\right].$$

After applying Lemma 2.2. we arrive at:

$$\left\|\frac{d^2\lambda^o}{dt^2}\right\| \leqslant Mg_3\left[4g_2+\left\|\frac{d\varphi}{dt}\right\|_{-1}+2\left\|\frac{du^o}{dt}\right\|\right]+M\left[\left\|\frac{du^o}{dt}\right\|+g_2+2g_4\right].$$

Denoting

$$g_5 \overset{\mathrm{df}}{=} Mg_3\left[4g_2+\left\|\frac{d\varphi}{dt}\right\|_{-1}+2\left\|\frac{du^o}{dt}\right\|\right]+M\left[\left\|\frac{du^o}{dt}\right\|+g_2+2g_4\right] \tag{C.3}$$

and after recalling the result of Lemma 2.2 we arrive at (iii). (Q.E.D)

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Aproksymacja różnicowa sterowania optymalnego dla układów opisanych nieliniowymi równaniami różniczkowymi z opóźnieniem

Artykuł jest poświęcony aproksymacji różnicowej problemu sterowania optymalnego dla układów opisywanych nieliniowymi równaniami różniczkowymi z opóźnieniem. Problem optymalizacji rozważono stosując teorię mnożników Lagrange'a. Podano oszacowania różnicy normy (w sensie przestrzeni L^2) sterowania i stanu optymalnego dla problemu dokładnego i aproksymowanego.

О разностной аппроксимации оптималного управления для систем описываемых нелинейными дифференциальными уравнениями с запаздыванием

Статья посвящена разностной аппроксимации задачи оптимельного управления для системы описываемой нелинейными дифференциальными уравнениями с запаздыванием. Задача оптимизации рассматривается при использовании теории множителей Лагранжа. Дается оценка разности нормы (в смысле пространства L^2) управления и оптимального состояния для точного и аппроксимированного решения задачи.