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Weak balance: A combination of Heider's theory and cycle and path-balance

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The cycle- and path-balance of a structure are characterized by a new concept: weak balance. By means of the weak balance, the balance of a structure can be reduced to the balance of the elementary structure considered by Heider [2]. A natural measure of balance is constructed based on those elementary structures.

1. Introduction and basic concepts

The purpose of this paper is to show a way of reducing the concept of cycle- and path-balance of a structure defined by Cartwright and Harry [1] to the balance of elementary structures considered by Heider [2]. The characterization uses the concepts of isomorphisms and signed transitive closure. The theorem proved in the paper concerns directed signed graphs (structures) only, but it has an immediate analogy in the case of undirected graphs.

In this paper we mean by a structure a graph with a non-empty, finite set of points. The basic concepts and notations of graph theory used here are mainly those defined by Harary et al. [3], and we recall a few of them. A directed graph $D=(P(D), L(D))$, briefly digraph, is a collection of points $P(D)$ and lines $L(D)$. An ordered pair (x, y) of two points x and y of D belongs to $L(D)$, if there is in D a directed line from x to y. The structures considered here are signed digraphs S. We shall say that S is a signed digraph whose underlying digraph is D, if S is obtained from D by designating each line as being either positive or negative. Following the usage of Harary et al. [3] we use the notation that solid lines are positive and dashed lines are negative. As an illustration, see the signed digraph S of D in Figure 1. In this paper we shall consider weakly connected, finite signed digraphs without loops and multiple lines only.

Two digraphs D_1 and D_2 are isomorphic, if they have the same number p of points and if one can order their points respectively $x_1, x_2, ..., x_p$ and $w_1, w_2, ..., w_p$

so that for any index pair i, j a line (x_i, x_j) belongs to $L(D_1)$ if and only if $(w_i, w_j) \in L(D_2)$. Further, two signed digraphs S_1 and S_2 are isomorphic, if the underlying digraphs D_1 and D_2 are isomorphic and if for any index pair i, j a line (x_i, x_j) in S_1 is positive (negative) if and only if (w_i, w_j) is positive (negative) in S_2.

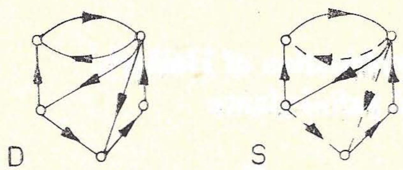


Fig. 1

A collection of distinct points $v_1, v_2, ..., v_n$ of D is a path from v_1 to v_n, if there are in D the lines $(v_1, v_2), (v_2, v_3), ..., (v_{n-1}, v_n)$. A cycle is a path where $n \geqslant 2$ and $(v_n, v_1) \in L(D)$. A semipath joining v_1 and v_n is a collection of distinct points $v_1, v_2, ..., v_n$ together with $n-1$ lines one from each pair (v_1, v_2) or (v_2, v_1); (v_2, v_3) or (v_3, v_2); (v_3, v_4) or (v_4, v_3); ...; (v_{n-1}, v_n) or (v_n, v_{n-1}). A semicycle is obtained from a semipath on adding a line joining the points v_1 and v_n of the semipath. A point v is reachable from a point u if there is a path from u to v in D. The number number of lines in a path is called its length. The closure of a digraph D is the digraph D', where $L(D)D) = P(D')$ and $(v_i, v_j) \in L(D')$ whenever $v_i \neq v_j$ and there is a path from v_i to v_j.

The balance in a structure is defined by Cartwright and Harary [1] as follows: The sign of a semicycle and a semipath in a S is the product of the signs of its lines A signed digraph S is called balanced, if every semicycle of S is positive. Further, according to Harary et al. [3], a signed digraph S is said to be cycle-balanced, if every cycle of S is positive, and path-balanced, if for every pair of points, all paths from one to the other have the same sign.

Some examples given in the monograph of Harary et al. [3] show that the cycle- and path-balance together do not imply balance.

2. A characterization of cycle- and path-balance

We shall first define what we shall mean by a signed closure S^c of a signed digraph S, and thereafter we shall give the definition of the weak balance in a structure. The weak balance is equivalent to the concept of cycle- and path-balance as it will be shown in the theorem below.

Definition 1. The sign of a cycle and a path in a S is the product of the signs of its lines. The signed closure S^c of S is the closure of the underlying digraph of S where the signs of the lines follow the rule: If $(x, y) \in L(S)$, (x, y) has the same sign in S^c as in S. If y is reachable from x in S, $x \neq y$ and $(x, y) \notin L(S)$, the line (x, y) of S^c has the sign of the shortest path from x to y in S; if there are several shortest paths from x to y in S having different signs, the line (x, y) of S^c has the negative sign.

The weak balance of a structure has the following definition:

Definition 2. Let S be a given signed digraph and let S^c be the signed closure of S. S is weakly balanced, if for every two points $x, y \in P(S)$, $(x, y), (y, x) \in L(S^c)$, the signed subdigraph induced by x and y in S^c is isomorphic to one of the signed digraphs (1) and (2) in Figure 2, and if for every three points x, y, z of S, (x, y), (y, z), $(x, z) \in L(S^c)$, the signed subdigraph induced by x, y, z in S^c is isomorphic to one of the signed digraphs (3), (4), (5) and (6) in Figure 2.

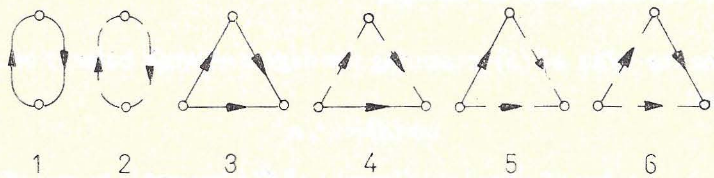


Fig. 2

Clearly Definition 2 offers a way of generalizing the concept of balance in elementary structures considered by Heider (see the characterization of Heider's results in the paper of Cartwright and Harary [1]) to weakly connected structures having more points than the structures in Figure 2. The following theorem gives a connection between the weak balance of Definition 2 and the well known concepts of Harary et al. [3].

Theorem. A signed digraph is weakly balanced if and only if S is path- and cyle-balanced.

Proof. Let S be weakly balanced. We show first that S is path-balanced.

Let y be reachable from x in S, $(x, y) \notin L(S)$, (x, y) positive in S^c, and let $x=z_0, z_1, z_2, \ldots, z_{n-1}, z_n=y$ be a negative path of S. Let k be the greatest value of index i, $0<i\leqslant n$, with the property (z_{k-1}, z_k) is negative in S. Since S is weakly balanced, the lines (x, z_j), $j=k, \ldots, n$, are positive in S^c, and hence (x, z_{k-1}) is negative in S^c. Furthermore, $k-1\neq 0$, since in the other case the elementary signed subdigraph of S^c induced by the points x, z_k, z_{k+1} gives a contradiction to the weak balance of S. As the path $z_0, z_1, \ldots, z_n$ is negative in S, and if $k-2=0$, (z_{k-2}, z_{k-1}) is a positive line. As (x, z_k) was positive in S^c, the elementary subdigraph induced by the points z_{k-2}, z_{k-1}, z_k, where $z_{k-2}=x$, gives a contradiction to the weak balance. Hence $k-2\neq 0$. If (z_{k-2}, z_{k-1}) is positive in S, and since (x, z_{k-1}) is negative in S^c, (x, z_{k-2}) is negative in S^c, and so $k-3\neq 0$, as the path was negative. Similarly, if (z_{k-2}, z_{k-1}) is negative in S, (x, z_{k-2}) is positive in S^c, and so $k-3\neq 0$. By continuing this process we obtain a contradiction in the case $k=m$, as $k-m=0$.

The proof is similar when (x, y) is negative in S^c and $x=z_0, z_1, \ldots, z_{n-1}, z_n=y$ a positive path of S.

Consider a negative cycle $x=z_0, z_1, z_2, \ldots, z_{n-1}, z_n=x$ of S, and let (z_{n-1}, z_n) be negative in S. The positive sign of the path $z_0, z_1, \ldots, z_{n-1}$ in S and the path balance proved above imply that (x, z_{n-1}) is positive in S^c. But the the elementary signed subgraph induced by the points x and z_{n-1} in S^c gives a contradiction to the weak balance. Thus let (z_{n-1}, z_n) be positive in S and k the greatest value of i, $0<i\leqslant n-1$, for which (z_{k-1}, z_k) is negative. Since $z_0, z_1, \ldots, z_k$ is a negative path and $z_k, \ldots, z_n$ a positive path of S and S is path-balanced, the signed subdigraph of the points x and z_k in S^c gives a contradiction. Hence S is cyclobalanced.

The converse part of the proof is obvious and hence we omit it.

3. A way of measuring and an example

We define the index $wb(S)$ measuring the degree of weak balance of a structure S as follows:

$$wb(S)=e^+/e,$$

where e^+ is the number of signed subdigraphs of S^c isomorphic to one of the signed digraphs in Figure 2 and e is the number of subdigraphs of S^c isomorphic to one of the digraphs in Figure 3 without any attention to the signs of the lines in S^c. Clearly $0\leqslant wb(S)\leqslant 1$, and if $wb(S)=0$, we shall say that S is totally unbalanced with respect to weak balance in a structure.

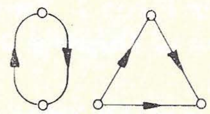


Fig. 3

A maximal subdigraph D' of a digraph D is a block, if it does not contain any point v so that $D'-\{v\}$ is disconnected. Thus S' and S'' are the blocks of the signed digraph S in Figure 4.

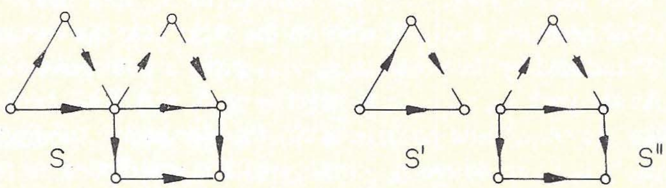
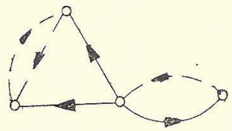


Fig. 4

In the monograph of Harary et al. ([3], p. 346), a balance index $\beta(S)$ has been defined as follows: $\beta(S)=b^+/b$, where b^+ is the number of positive semicycles and b the number of semicycles in the signed digraph S under consideration. According to the definition of balance index $\beta(S)$, a signed digraph S is balanced if and only if all its blocks are balanced separately (see Harary et al. [3]), i.e. $\beta(S)$ does not take into account the interaction between the bloks caused by the paths going from a block to another.

Consider the signed digraphs S of Figure 4 and S_1 of Figure 5. $\beta(S)=\beta(S_1)=3/4$, but $wb(S)=21/26>13/22=wb(S_1)$.



S_1

Fig. 5

Since the index $wb(S)$ uses the structure of S^c by deciding the degree of weak balance, it takes into account the interaction between the blocks of S and the number of lines (relations) forming a path of at least length two. These things explain the inequality above.

The things which $wb(S)$ takes into account by deciding the degree of balance in a structure seem to be remarkable. Unfortunately, the determination of $wb(S)$ is laborious, but for signed digraphs with moderate low number of points, for which the index $\beta(S)$ may be fairly rough, $wb(S)$ may be useful.

REFERENCES

1. CARTWRIGHT D., HARARY F., Structural balance: A generalization of Heider's theory. *Psycholog. Rev.* **65** (1956) 277—293.
2. HEIDER F., Attitudes and cognitive organization. *Psycholog.* **21** (1946) 107—112.
3. HARARY F., NORMAN R. Z., CARWRIGHT D., Structural models: An introduction to the theory of directed graphs. New York 1965.

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Równowaga słaba. Połączenie teorii Heidera oraz równowagi cyklu i drogi

Równowagę cyklu i drogi struktury charakteryzuje się za pomocą nowego pojęcia — równowagi słabej. Za pomocą równowagi słabej równowagę struktury można sprowadzić do równowagi struktur elementarnych rozpatrywanych przez Heidera [2]. Ustalono pewną miarę naturalną równowagi na podstawie tych struktur elementarnych.

Слабое равновесие. Объединение теории Гейдера и равновесия цикла и пути

Равновесие цикла и пути структуры характеризуются посредством нового понятия — слабого равновесия. С помощью слабого равновесия равновесие структуры может быть сведено к равновесию элементарных структур, рассматриваемых Гейдером [2].

Устанавливается некоторая естественная мера равновесия, базирующая на этих элементарных структурах.