A classification of controllability concepts for infinite-dimensional linear systems

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The paper explores relations between exact, approximate and null controllability and their numerous combinations and variants. In particular it studies the existence of universal time in situations where the controllability time depends on initial and final states.

1. Introduction

The paper attempts to classify some notions of controllability that occur or may occur in the theory of infinite-dimensional linear autonomous systems. In our opinion the profusion of publications concerned with controllability that have been appearing recently ([2] [9] [5] [11] [12] [15] [16] [18] [19] [20] [21] [22] 23]) justifies such attempts. The character of the paper requires the use of extensive terminology (which is mostly operational), but as a result of known facts and the present considerations an amount of terms may be abandoned as the names of properties that turned out to be equivalent to some others.

Definitions of controllability precise sets of initial and final states, a controlling time, exact or approximate reaching, sets of admissible controls (sequences of controls), etc. These elements may be put together in many ways. If we consider a few basic variants, their combinations and permutations of quantifiers (that could be used in the defining phrases) generate a large variety of controllability concepts. Some formal differences are superfluous, others remain essential and our task is to discuss some of them.

A control process is given by

$$x(t) = S(t) x_0 + C_t u, \tag{1}$$

where $x(t)$ is a state at time t, x_0 is an initial state and u is a control. Both x_0 and $x(t)$ are in a Banach space X (state space), where a strongly continuous semigroup $S(t)$ of bounded linear operators is defined. C_t is controllability operator:

$$C_t u = \int_0^t S(t-s) Bu(s) ds, \tag{2}$$

where B is a continuous linear map from a Banach space W to X and $u(\cdot) \in U \overset{\mathrm{df}}{=}$ $\overset{\mathrm{df}}{=} L_p(0, \infty; W)$, $p \geqslant 1$. Of course (1)–(2) describe at least a mild solution of an evolution equation

$$\frac{dx}{dt} = Ax + Bu, \quad x(0) = x_0 \, \varepsilon X \tag{3}$$

where A is the generator of $S(t)$.

We note that if $t_1 \leqslant t_2$, then

$$R(C_{t_1}) \subset R(C_{t_2}) \tag{4}$$

where $R(C)$ denotes the range of C. Another simple formula is

$$S(\tau) R(C_t) \subset R(C_{t+\tau}). \tag{5}$$

In fact, a bounded map commutes with the Bochner integral [10]

$$S(\tau) \int\limits_0^t S(t-s) Bu(s) \, ds = \int\limits_0^t S(t+\tau-s) Bu(s) \, ds. \tag{6}$$

Since u is $L_p(0, \infty; W)_{; \, 1 \leqslant p}$ we take a function $\tilde{u}$ which is equal to u on $(0, t)$ and is null otherwise. Hence, (6) is equal to $\int\limits_0^{t+\tau} S(t+\tau-s) B\tilde{u}(d) \, ds \in R(C_{t+\tau})$.

Actually these are the properties of a process we use in the sequel and thus the theory includes also processes that are represented otherwise than (1), (2) for instance boundary value problems [15].

As far as the notation is concerned we use the following convention. An elementary propositional function is

$$\Phi(x_0, x_1, t, \varepsilon, u) \equiv \|x_1 - S(t) x_0 - C_t u\| \leqslant \varepsilon \tag{7}$$

and in order to form a proposition we use the quantifiers: $\bigvee\limits_t, \bigwedge\limits_{x_0}, \bigwedge\limits_{x_1}, \bigwedge\limits_\varepsilon, \bigvee\limits_u$, where $\bigvee$ denotes the existential quantifier (there exists) and $\bigwedge$ the general one (for all). The basic symbol (C) means:

$$\bigvee\limits_t \bigwedge\limits_{x_0} \bigwedge\limits_{x_1} \bigvee\limits_u \Phi(x_0, x_1, t, 0, u).$$

If we add (A) before (C), the quantifier $\bigwedge\limits_\varepsilon$ enters before $\bigvee\limits_u$.

(L) means that $\bigvee\limits_t$ is put after $\bigwedge\limits_{x_0} \bigwedge\limits_{x_1}$. A subzero after a symbol (e.g. C_0) means that $\bigwedge\limits_{x_0}$ is removed and $x_0 = 0$.

(N) means that x_1 is equal to zero.

(∞) means that $\bigvee\limits_t$ appears after $\bigwedge\limits_{x_0} \bigwedge\limits_{x_1}$ and $\bigwedge\limits_\varepsilon$.

We shall also mention other notions that cannot be defined in this way but we shall not study them thoroughly.

A mathematical tool to show the equivalence of certain types of controllability is mainly the Banach open mapping theorem (or the Baire category theorem).

The differences between other types are presented with the aid of few special examples.

The results of the paper (and other known facts) are collected in a diagram, where implications are denoted by arrows →; the lack of an arrow from one point to another means that this implication does not hold (p. 43).

2. Exact, approximate and null controllability

We say that a process is *(exactly) controllable* at a time T, if for each x_0 and x_1 in X there is a control u such that

$$x(T) = x_1 \qquad\qquad\qquad (C)$$

where $x(T)$ is given by (1).

When we drop "at a time T", we understand that there is a T such the (C) holds. The definition of exact controllability may take form

$$R(C_T) = X. \qquad\qquad\qquad (8)$$

We observe that it is the same as to require that for each x_1 there be a u such that

$$C_T u = x_1. \qquad\qquad\qquad (C_0)$$

This property sometimes called *null — reachability* means that each state may be reached from the origin. If each state can be carried to the origin we talk about *null — controllability* (at a time T):

for each x_0 there is a control u with

$$S(T) x_0 + C_T u = 0. \qquad\qquad\qquad (NC)$$

In other words (NC) amounts to requiring

$$R(C_T) \supset R(S(T)). \qquad\qquad\qquad (9)$$

Example 1. We are given a heat process in $L^2(0, 1)$

$$x(t) = \sum_n c_n \, \Phi_n \, e^{-\lambda_n t} + \sum_n \Phi_n \int_0^t e^{-\lambda_n(t-s)} \, b_n \, u(s) \, ds$$

where u_0 has the expansion $\sum c_n \, \Phi_n$, $\{\Phi_n\}$ are the normalized eigenvectors and $\{\lambda_n\}$ are the eigenvalues of the process. The map B is one- dimensional: $Bu(t) = u(t) \, (\sum_n b_n \, \Phi_n)$. If $T > -\lim \inf \dfrac{1}{\lambda_n} \log |b_n|$ then the process is null — controllable (NC) at a time T in view of [6], [7] and duality theory [9]. It is never exactly controllable (C) for each C_t is compact ([23]).

There follow approximate versions of (C) and (BC) both often met in the control theory.

A process is *approximately controllable* at a time T if for every states x_0 and x_1 and for each $\varepsilon > 0$ there is a u with

$$\|x_1 - x(T)\| \leqslant \varepsilon. \tag{AC}$$

A process is *approximately null — controllable* at T if for each x_0 and ε there is a u such that

$$\|x(T)\| = \|S(T)x_0 + C_T u\| < \varepsilon. \tag{ANC}$$

Using as before the notion of the range we reformulate (AC)

$$\overline{R(C_T)} = X \tag{10}$$

and for (ANC) we have:

$$\overline{R(C_T)} \supset R(S(T)), \tag{11}$$

where the bar denotes the closure.

First we remark that in view of (10) *approximate null — reachability* (AC_0) is equivalent to (AC).

We may also observe that in virtue of (10) and (11) (AC) is implied by (NC) whenever $R(S(T))$ is dense. But in general (AC) and (NC) intersect.

Example 2. The process as in Example 1. We set $b_n = e - \lambda_n^2$ and the critical time formula [7] gives $- \liminf \frac{1}{n} \log |b_n| = \infty$, therefore the process is not null — controllable (NC) at any T. On the other hand, it is approximately controllable (AC) at each T ([22], Corollary 4.4).

Example 3. We consider a "free" process, that is a process without control: $B = 0$. The semigroup $S(t)$ of right shifts is defined on $L^2(0, 1)$.

We observe that $R(C_1) = \{0\}$ and consequently the approximate controllability (AC) is excluded. But also $R(S(1)) = \{0\}$. Therefore (9) is verified and we have null — controllability (NC).

We shall only mention of other notions of approximate controllability without thorough examination. Roughly speaking these types of controllability require that a dense set of initial states can be transferred to prescribed final states:

for each x_0 and x_1 and for every $\varepsilon > 0$ there are a state x and a control u with

$$S(T)x + C_T u = x_1 \tag{AC'}$$

and

$$\|x - x_0\| < \varepsilon$$

for each x_0 and for $\varepsilon > 0$ there are x and u that satisfy

$$\|x - x_0\| < \varepsilon \tag{ANC'}$$

$$S(T)x + C_T u = 0.$$

We see that (AC') and (ANC') are stronger than (AC) and (ANC), respectively. For instance (ANC') means that for each x_0 there is a sequence $\{x_n\}$ convergent

to x_0 and such that $S(T) x_n \in R(C_T)$. Ba continuity of $S(T)$, $S(T) x_0 \in \overline{R(C_T)}$. Hence $R(S(T)) \subset \overline{R(C_T)}$.

If in addition to (AC) we require that there is a number m (depending on x_0 and x_1) such that there is a sequence $\{u_n\}$ with

$$\|u_n\| \leqslant m$$

(and satisfying (AC)), then we have to do with bounded controllability (BC).

In [8] we consider (BC) and its variants. It is shown that (BC) is equivalent to (C).

3. Long-term controllability. First type.

The general method we are going to present was used by S. Rolewicz to prove the existence of universal controllability time for some problems [18].

We are concerned with two classes of bounded linear maps $C_t : U \to X$ and $S_t : X \to X$. Here U and X are Banach spaces.

We assume that $R(C_t)$ is increasing and also

$$\{x : S_t\, x \in R(C_t)\} \text{ is increasing} \tag{12}$$

Of course the corresponding maps $S(t)$, C_t in a control process (1), (2) are of this type in view of (4) and (5).

THEOREM 1 (Rolewicz [18]). If the set

$$\bigcup_{t \geqslant 0} \{x : S_t\, x + C_t\, u = 0 \text{ for some } u\} \tag{13}$$

is equal to the whole space X and if (12) holds, then there is a T such that

$$R(S_T) \subset R(C_T).$$

Proof. Consider a (closed) operator $S_t + C_t$ defined on $X \times U$ into X. Its kernel $\{(x, u) : S_t\, x + C_t\, u = 0\}$ is a closed subspace of $X \times U$. The projection map P on X restricted to $\{(x, u) : S_t\, x + C_t\, u = 0\}$ is continuous and its image is either the whole space X or a meager set (the Banach open mapping theorem, [10]). On the other hand $P\{(x, u) : S_t\, x + C_t\, u = 0\} = \{x : S_t\, x \in R(C_t)\}$. By assumptions of the theorem we have

$$X = \bigcup_{n=1}^{\infty} \{x : S_n\, x \in R(C_n)\}. \tag{14}$$

By the Baire theorem [10] one of these sets is not meager and thus is the whole space.

REMARK. The only aim of (12) is to guarantee that (13) may be replaced by a countable union (14).

We shall apply this theorem in the sequel.

By *long-term controllability (of the first type)* we understand the following property:

for each x_0 and x_1 in X there are a time t and a control u (both dependent on x_0 and x_1) that verify

$$x_1 = S_t\, x_0 + C_t\, u. \tag{15}$$

(LC)

In an analogous way we define long-term versions of (AC), (NC) and (ANC)

for each x_0 and x_1 there is a t such that for all $\varepsilon > 0$ there exists a u with

$$\|x_1 - S(t)\, x_0 - C_t\, u\| < \varepsilon; \tag{LAC}$$

for each x_0 there are a t and a u such that

$$S(t)\, x_0 + C_t\, u = 0; \tag{LNC}$$

for each x_0 there exists a t such that for every $\varepsilon > 0$ one has a u satisfying

$$\|S(t)\, x_0 + C_t\, u\| < \varepsilon. \tag{LANC}$$

Proposition 1. (LC), (LAC), (LNC), (LANC) are equivalent to (C), (AC), (NC), (ANC) respectively (for some time T).

Before proving this proposition we introduce the *long-term null-reachability* (LC$_0$) that is long-term controllability with x_0 replaced by zero. Propositions 1 and 2 together entail the equivalence of (LC) and (LC$_0$).

Proposition 2. (Rolewicz [18], Zabczyk [24]) (LC$_0$) is equivalent to null-reachability (C$_0$).

Proof. In Theorem 1 we set $S_t = I$ (identity) and the hypothesis of the theorem becomes

$$\bigcup_{t>0} R\,(C_t) = X,$$

which is also a definition of (LC$_0$). Hence there is a T such that $R\,(C_T) = X$.

Proof of Proposition 1:

(LC)→(C): In Theorem 1 we replace X by $X \times X$, and also we set $S_t\,(x_0, x_1) = (x_1 - S(t)\, x_0,\, 0)$.

By assumption

$$X \times X = \bigcup_{t>0} \{(x_0, x_1): S_t\,(x_0, x_1) + C_t\, u = 0 \text{ for some } u\}.$$

We shall prove that this union may be replaced by a countable union of such sets. It is sufficient to show that for each x_0 and x_1 there is an integer n such that $x_1 - S\,(n)\, x_0 \in R\,(C_n)$.

By the assumption of long-term controllability (LC) there is a t_1 with $x_1 \in R\,(C_{t_1})$ and in view of (4) $x_1 \in R\,(C_t)$ for $t \geqslant t_1$.

As well, there is a t such that $S(t) x_0 \in R(C_{t_0})$ and because of integral representation of C_t ((5), (6)), $S(t) x_0 \in R(C_t)$ for $t \geqslant t_0$. Thus $x_1 - S(n) x_0$ is in $R(C_n)$ if only n is larger than max (t_0, t_1).

We use Theorem 1 and Remark that follows it to obtain a universal T with $x_1 - S(T) x_0 \in R(C_T)$ for all pairs (x_0, x_1) hence controllability (C).

(LAC)→(AC): Take arbitrary x_0 and x_1. Long-term approximate controllability guarantees that $x_1 \in \overline{R(C_{t_1})}$ for some t_1 and $S(t_0) x_0 \in \overline{R(C_{t_0})}$. Actually $x_1 \in R(C_t)$ for $t \geqslant t_1$ in view of (4). Let n be the smallest integer $\geqslant$ max (t_0, t_1) and set $\kappa = \max_{0 \leqslant t \leqslant n} \|S(t)\|$.

For each $\in$ we are able to find an $h_0 \in R(C_{t_0})$ such that $\|h_0 - S(t_0) x_0\| \leqslant \dfrac{\varepsilon}{2\kappa}$.

We can also find an $h_1 \in R(C_n)$ satisfying $\|h_1 - x_1\| \leqslant \dfrac{\varepsilon}{2}$. Therefore one has estimates

$$\|(x_1 - S(n) x_0) - (h_1 - S(n - t_0) h_0)\| \leqslant \|x_1 - h_1\| + \|S(n - t_0)\| \|S(t_0) x_0 - h_0\| \leqslant$$

$$\leqslant \frac{\varepsilon}{2} + \frac{\|S(n - t_0)\|}{2\kappa} \varepsilon \leqslant \varepsilon.$$

Furthermore $h_1 - S(n - t_0) h_0$ belongs to $R(C_n)$. In fact $S(n - t_0) h_0 \in S(n - t_0) \cdot R(C_{t_0}) \subset R(C_n)$ in view of (5).

Since ε was arbitrary we are sure that $x_1 - S(n) x_0$ is in the closure of $R(C_n)$.

As a consequence $X \times X = \bigcup_{n=1}^{\infty} \{(x_0, x_1) : x_1 - S(n) x_0 \in \overline{R(C_n)}\}$. Each of the components of this union constitutes a (closed) subspace of $X \times X$, and regarding the Baire category argument [10] one of them must be the whole of $X \times X$. We have obtained (AC).

(LNC)→(NC) (Rolewicz [21]). Set $S_t = S(t)$ and use Theorem 1.

(LANC)→(ANC) Similar to the proof of (LAC→AC).

REMARK. From what we observed in the preceding paragraph and having in mind the equivalence of (LAC) and (AC) we may easily conclude that (LAC$_0$) (long-term aproximate reachability) is the same as (LAC).

4. Long-term controllability. Second type

Of course, every definition we encountered may be modified by the substitution of a linear set (manifold) Γ for the space X. For example a system is Γ-controllable at a time T if Γ is included in $R(C_T)$.

In particular a system is null-controllable if $\Gamma = R(S(T))$ and it is approximately controllable if Γ is dense (compare [9]). As an immediate consequence of Theorem 1 we get

Proposition 3. For each *closed* map F from a subset of X to X $R(F)$ — long--term controllability implies $R(F)$ — controllability. (sure enough, $R(F)$ — long-term

controllability means that for each $x \in R(F)$ there exists a t and a u verifying $C_t u = x$).

We are ready now to begin the discusion of long-term controllability of the second type, a notion that seems to be a real novelty with respect to "fixed-time" concepts. It concerns approximate variants of our collection.

The point is that, unlike for the first-type notions, a time t and a control u depend on a distance ε (in the first type it was only controls that depended on ε, time being uniquely a function of states x_0, x_1). Let us define the *long-term (approximate controllability* of the second type, a notion widely explored by many authors (e.g. Fattorini [11], Triggiani [22] and others):

for all x_0 and x_1 and for each $\varepsilon > 0$ there are a t and a u so that   (∞AC)

$$\|x_1 - S(t) x_0 - C_t u\| < \varepsilon.$$

Actually the definition of complete controllability of Fattorini was a simplication of (∞AC) by fixing $x_0 = 0$. This property we denote by (∞AC$_0$) in agreement with the adopted convention. We shall see that (∞AC) and (∞AC$_0$) are different properties.

for each x_1 and ε there are a t and a u with

$$\|x_1 - C_t u\| < \varepsilon.$$ (∞AC$_0$)

Once again making use of the ranges of operators we describe (∞AC$_0$) by

$$\overline{\bigcup_{t > 0} R(C_t)} = X.$$ (16)

We ask a question whether it is possible to reduce (∞AC) or (∞AC$_0$) to approximate controllability at a fixed time (AC) as we did in the preceding paragraph. H. O. Fattorini mentions in [11] that it cannot be done but he does not give any counter-example.

Example 4. As a state space X we take $L^2(-\infty, 0)$ (the Hilbert space of square integrable functions). We may define a semigroup of left shifts by

$$(S(t) f)(x) = f(x + t)$$ (17)

where f is understood as an element of its equivalence class and the equality holds almost everywhere. This semigroup is strongly continuous and since $L^2(-\infty, 0)$ is reflexive the dual semigroup $S^*(t)$ is also strongly continuous [2]. $S^*(t)$ turns out to be a semigroup of right shifts with truncation, that is

$$(S^*(t) g)(x) = g(x - t) \cdot \chi(x),$$

where χ is a characteristic function of $(-\infty, 0]$.

An operator B (of control distribution) is the injection of $L^2(-\kappa, 0)$ into $L^2(-\infty, 0)$.

A control space is $U=L^2(0,\infty; L^2(-\kappa, 0))$. Finally the controllability operator C_t is given by

$$C_t(u) = \int_0^t S(t-s) Bu(s)\, ds \qquad (2)$$

with $S(t)$ and B defined before.

We shall see that (∞AC_0) holds. To prove this it is enough to show that the process is approximately controllable for each time $t > \kappa$ when the state space is replaced by $L^2(-t-\kappa, 0)$ (and the semigroup is slightly modified by truncation).

The dual map $(C_t^* g)(x, s) = \chi_{(-\kappa, 0)} \cdot g(x-s)$; $C_t^* g$ is an element of $L^2(0, t; L^2(-\kappa, 0))$ and we denote the "time" variable by s and the "space" variable by x. This dual is in fact a dual of (2) where $S(t-s)$ is replaced by $S(s)$ (compare [9]).

The duality theory allows to check injectivity of C_t^* instead of resolving the original problem ([13], [22]).

Suppose that $\|C_t^* g\| = 0$ and $\|g\| > 0$. Since $\|g\| = \int_{-t-\kappa}^0 |g(x)|^2\, dx = \sum_{n=1}^N \int_{E_n} |g(x)|^2\, dx$ (where E_n are disjoint intervals of the same length $\leqslant \frac{\kappa}{2}$), it must be $\delta = \int_{E_n} |g(x)|^2\, dx > 0$ for one n at least. Hence $\|C_t^* g\| = \int_0^t \int_{-\kappa}^0 |g(x-s)|^2\, dx\, ds \geqslant \delta \frac{\kappa}{2}$ which is the contradiction. We conclude that C_t^* is injective and C_t is approximately controllable in $L^2(-t-\kappa, 0)$.

Now for each t the range of C_t is in $L^2(-t-\kappa, 0)$ that is, the effects of the controlling action do not get out of the support $(-t-\kappa, 0)$.

If we take any element f of $L^2(-\infty, 0)$ and any $\varepsilon > 0$, then we shall have a number T such that $\int_{-\infty}^{-T-\kappa} |f(x)|^2\, dx < \frac{\varepsilon}{2}$.

From what we just have done there is a u such that $\int_{-T-\kappa}^0 |f(x) - (C_T u)(x)|^2\, dx < \frac{\varepsilon}{2}$ and $\int_{-\infty}^{-T-\kappa} |(C_T u)(x)|^2\, dx = 0$.

The process is not (∞ANC). Take an initial function f_0 with $\|f_0\| = 1$ and a final function $f_1 = 0$. Assume further that $\operatorname{supp} f_0 \subset (-\infty, -\kappa)$. At any time t $\operatorname{supp} S(t) f_0 \subset (-\infty, -\kappa-t)$ and we recall that $R(C_t) \subset L^2(-\kappa-t, 0)$. Thus one gets $\inf \|f_1 - S(t) f_0 - C_t u\|^2 \geqslant \int_{-\infty}^{-\kappa-t} |(S(t) f_0)(x)|^2\, dx = 1$.

Since (∞AC) implies (∞ANC), the process is not ∞AC.

Example 5. Take the semigroup $S(t)$ from the preceding example and define $T(t) = \exp(-t) S(t)$.

All other elements of the process remain invariant. This process is (∞AC) and consequently (∞AC_0) and (∞ANC).

To see this we observe that for any two functions f_0 and f_1 in $L^2(-\infty, 0)$ $f_1 - T(t) f_0$ tends to f_1 as t grows. On the other hand $\widetilde{C}_t u = C_t(u \cdot \exp(t - \cdot))$ where

$\tilde{C}_t$ is the controllability operator from Example 4 and C_t corresponds to $T(t)$. Hence for any f_1 and f_0 we may pick out a t and a u to transfer f_0 as closely to f_1 as we wish.

The process is not (ANC) at any time. We act similarly as in Example 4 to obtain the estimate

$$\inf_u \|S(t)f_0 + C_t u\|^2 \geqslant e^{-t}$$

Example 6. We take a stable semigroup $S(t)$ $\left(\|S(t)\| \leqslant e^{-\gamma t}, \ \gamma > 0\right)$ and we put B equal to null operator (no control). Then $R(C_t) = \{0\}$ for all t and the process is not (∞AC_0). At the same time it is (∞ANC) since any initial state drifts to the origin as t tends to infinity.

We have seen that (∞ANC) is a very weak property and has little to do with control, when a semigroup happens to be stable. This makes it similar to *stabilizability*.

A standard definition of stabilizability is the following ([4], [21], [24]).

for any x_0 there is control u such that

$$\int_0^\infty \|u(t)\|^2 \, dt + \int_0^\infty \|S(t)x_0 + C_t u\|^2 \, dt < \infty \qquad \text{(S)}$$

Evidently (S) is a consequence of null-controllability and it implies (∞ANC). A process of Example 6 is stabilizable, but it may not be (ANC) even if X is one dimensional.

5. On the range of closed maps

We ask a question whether in a Banach space X each dense linear set Γ contains the image $R(F)$ of some closed linear map in X. Our controllability results permits to answer this question partly.

Proposition 4. Let H be a separable Hilbert space. There is a linear dense subset Γ of H such that no bounded linear operator F in H has the dense range $R(F)$ included in Γ.

Proof. Consider the process from Example 4 and set

$$\Gamma = \bigcup_{t > 0} R(C_t).$$

This is a subset of $L^2(-\infty, 0)$ which in view of (∞AC_0) (long-term null-reachability of the second type) is dense.

If it were a map F with dense range such that $R(F) \subset \Gamma$, then by Proposition 3 we would have (AC$_0$) at some time T. But from Example 4 we see that it is not so. $L^2(-\infty, 0)$ may be identified with any Hilbert space H of the same cardinality.

We gather some relations between various controllability notions in the following diagram:
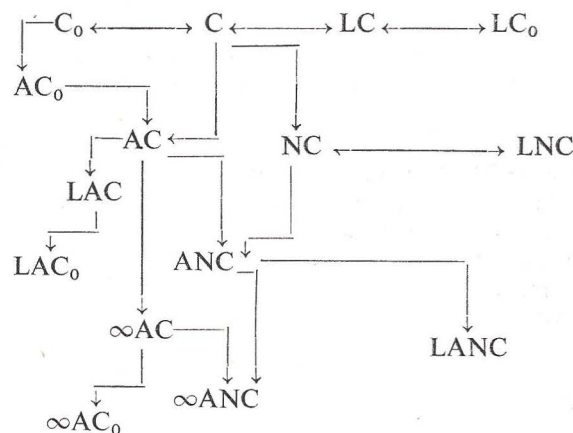
References of some implications that do not seem to be evident: $NC \not\to C$ (Ex. 1); $AC \not\to NC$ (Ex. 2); $NC \not\to AC$ (Ex. 3); $C_0 \leftrightarrow LC_0$ (Pr. 2); $C \leftrightarrow LC$, $AC \leftrightarrow LAC$, $NC \leftrightarrow LNC$, $ANC \leftrightarrow LANC$ (Pr. 1); $\infty AC_0 \not\to \infty ANC$ (Ex. 4); $\infty AC \not\to ANC$ (Ex. 5); $\infty ANC \not\to \infty AC_0$ (Ex. 6).

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Klasyfikacja pojęć sterowalności dla liniowych układów nieskończenie wymiarowych

W pracy zbadano powiązania sterowalności dokładnej, aproksymatywnej i „do zera”, ich liczne kombinacje i warianty. Rozważono zwłaszcza istnienie czasu uniwersalnego w zagadnieniach, w których a priori czas sterowalności zależy od stanów początkowego i końcowego.

Классификация понятий управляемости для линейных бесконечномерных систем

Работа поцвящена соотношениям нежду различным видами управляемости. В частности уделено внимание вопросам универсального времени в случае, когда время управляемости зависит, априорно, от начального и конечного состояний.