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# A note on the quadratic integral evaluations of transient motion

by

RADOSŁAW ŁADZIŃSKI

Technical University of Warsaw
Institute of Automatic Control

The paper deals with evaluation of integral (6) treated as the performance index of the linear, asymptotically stable system governed by Eq. (5) and subject to a step function input. The approach is general, applicable to systems of any order and is based on utilization of matrices (2) and (10) and the relationship between the Liapunov function and the quadratic performance index. Solution of the problem is essentially related with the determination of some auxiliary matrix (32). With its aid the detailed, working formulae (33)–(35), valid for systems of order $n \leqslant 5$ and for performance indices with $k \leqslant 2$ (Eq. 6), are stated. Their generalization for any $n$ and any $k$, due to a universal character of the matrix (32), is in fact straight-forward. Special attention is given to the choice of the weighting factors occurring in formulae (6) and (44) leading finally to the suggested form of the performance index expressed by Eq. (49). Application of this form is illustrated by four, simple examples of parametric optimization of dynamic systems — electric network, servomechanism, industrial controller, and systems with the so-called "optimum" transfer functions.

## I. Introduction

In the paper of Kalman and Bertram [1] it was shown that for the linear autonomous system the stability considerations based on the second method of Liapunov are specially simplified if in the state equation of the system

$$\dot{x}=Ax, \tag{1}$$

its matrix $A$ takes the, so-called, Schwarz canonical form

$$A=\begin{bmatrix} & 1 & & & & \\ -a_n & & 1 & & & \\ & -a_{n-1} & & \ddots & & \\ & & \ddots & & 1 & \\ & & & -a_3 & & 1 \\ & & & & -a_2 & -a_1 \end{bmatrix}^{1)}. \tag{2}$$

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[1]) In this and in all following expressions, the empty element of the matrix should be considered as equal to zero.

This form as well as the well-known relationship between the Liapunov function and the quadratic performance index was utilized later on by Parks [2] for evaluating the simple quadratic measure of the transient motion

$$J_0 = \int_0^\infty [z(t) - z(\infty)]^2 \, dt, \tag{3}$$

n which $z(t)$ is the step response of a li ear, asymptotically stable system described by the following transfer function:

$$F(s) = \frac{1}{s^n + \alpha_1 s^{n-1} + \ldots + \alpha_{n-1} s + \alpha_n}. \tag{4}$$

The purpose of the present paper is to show a slightly different approach to Parks' problem and at the same time to generalize his results by taking into considerations systems described by transfer functions with polynomial numerators

$$F(s) = \frac{\mathscr{L}[z(t)]}{\mathscr{L}[v(t)]} = \frac{\beta_0 s^n + \beta_1 s^{n-1} + \ldots + \beta_{n-1} s + \beta_n}{s^n + \alpha_1 s^{n-1} + \ldots + \alpha_{n-1} s + \alpha_n}. \tag{5}$$

and by assuming for the performance index its more general and, from the engineering point of view, more practical form

$$J_k = \int_0^\infty \{[z(t) - z(\infty)]^2 + [\tau_1 \dot{z}(t)]^2 + [\tau_2^2 \ddot{z}(t)]^2 + \ldots + [\tau_k^k \overset{(k)}{z}(t)]^2\} \, dt \tag{6}$$

with $z(t)$ having the same meaning as in Eq. (3) and with $\tau_1, \tau_2, \ldots, \tau_k$ being constant parameters (weighting factors) with dimensionality of time.

## II. Linear autonomous system in the canonical form

The linear, single output, autonomous system is said to be in the canonical form if it is governed by Eqs. (1), (2) and by the following output equation:

$$y = \mathbf{c}' \mathbf{x} \tag{7}$$

with

$$\mathbf{c}' = [0, \ldots, 0, 1]. \tag{8}$$

Calculating for any trajectory of this system the time derivative of the quadratic form

$$V[\mathbf{x}(t)[ = \mathbf{x}' \mathbf{P} \mathbf{x} \tag{9}$$

with its matrix $\mathbf{P}$ given by

$$\mathbf{P} = \frac{1}{2a_1} \, \mathbf{diag} \begin{bmatrix} a_n \; a_{n-1} \ldots a_3 \; a_2 \\ a_{n-1} \ldots a_3 \; a_2 \\ \vdots \\ a_3 \; a_2 \\ a_2 \\ 1 \end{bmatrix} \tag{10}$$

we get

$$\dot{V}\,[x(t)]=\dot{x}'Px+x'P\dot{x}=x'(A'P+PA)\,x=-x'\,\mathbf{diag}\,(0,\,\ldots,\,0,\,1)\,x=-y^2\,. \tag{11}$$

Thus, on the basis of the well-known Kalman-Bertram modification of the fundamental Liapunov's theorem we have the following important result [2]): The state $x=0$ of the system described by Eqs. (1) and (2) is asymptotically stable if and only if

$$a_i>0 \text{ for } i=1,\,\ldots,\,n. \tag{12}$$

The quadratic form $V(x)$ plays, of course, the role of Liapunov function of the system.

Assuming that condition (12) is satisfied, we can use relation (11) for evaluating the improper time integral of the squared output signal of the system. The result is

$$\int_0^\infty y^2(t)\,dt=-\int_0^\infty \dot{V}\,[x(t)]\,dt=V\,[x(0)]=x'(0)\,Px(0)\,, \tag{13}$$

i.e. the sought integral is equal to Liapunov function taken at the initial state of the system.

Now, let us determine the Laplace transform of the function $y(t)$. According to Eqs. (1) and (7), we can write

$$\mathscr{L}\,[y(t)]=c'(sI-A)^{-1}\,x(0)=\frac{c'\,\mathbf{adj}\,(sI-A)\,x(0)}{\det\,(sI-A)}\,. \tag{14}$$

For the system matrix $A$ given by Eq. (2), we have

$$sI-A=\left[\begin{array}{c|c|c|c|c|c} s & -1 & & & & \\ \hline a_n & s & -1 & & & \\ \hline & a_{n-1} & s & \ddots & & \\ \hline & & \ddots & \ddots & -1 & \\ \hline & & & a_3 & s & -1 \\ \hline & & & & a_2 & s+a_1 \end{array}\right] \tag{15}$$

and in consequence

$$\mathbf{adj}\,(sI-A)=$$

$$=\left[\begin{array}{cccc|c} \ddots & & & & A_1 \\ \cline{5-5} & \ddots & & & A_2 \\ \cline{5-5} & & \ddots & & \vdots \\ \cline{5-5} & & & \ddots & A_{n-2} \\ \cline{5-5} & & & \ddots & A_{n-1} \\ \hline (-)^{n-1}\,a_n\ldots a_2\,A_1 & (-)^{n-2}\,a_{n-1}\ldots a_2\,A_2 & \ldots & a_3\,a_2\,A_{n-2} \;\big|\; -a_2\,A_{n-1} & A_n \end{array}\right], \tag{16}$$

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[2]) Modification related with the semi-definiteness of $\dot{V}$; vide Ref. [1] p. 378, Corollary 1.3 and p. 382—383, Example 7.

then the Laplace transform and the final value of the corresponding output are given respectively by

$$\mathscr{L}\,[z(t)]=\frac{1}{s}\,F(s)\,, \tag{22}$$

$$\lim_{t\to\infty} z(t)=\lim_{s\to 0} s\mathscr{L}\,[z(t)]=F(0)=\beta_n/\alpha_n\,. \tag{23}$$

In consequence, the Laplace transform of the purely transient part of the output signal takes the form:

$$\mathscr{L}\,[z(t)]-z(\infty)\,\mathbf{1}(t)=\frac{1}{s}\,[F(s)-F(0)]=$$

$$=\frac{[1\; s\;\ldots\; s^{n-2}\; s^{n-1}]}{s^n+\alpha_1\, s^{n-1}+\ldots+\alpha_{n-1}\, s+\alpha_n}\begin{bmatrix}\beta_{n-1}-F(0)\,\alpha_{n-1}\\ \beta_{n-2}-F(0)\,\alpha_{n-2}\\ \vdots\\ \beta_1-F(0)\,\alpha_1\\ \beta_0-F(0).\end{bmatrix}. \tag{24}$$

Similarly, we can formulate expressions for Laplace transforms of the derivatives of the step response of the system. Confining ourselves, for example, to the first two derivatives of $z(t)$, we get

$$\mathscr{L}\,[\dot z(t)]=s\mathscr{L}\,[z(t)]-z(0)=s\mathscr{L}\,[z(t)]-\lim_{s\to\infty} s\mathscr{L}\,[z(t)]=F(s)-\beta_0=$$

$$=\frac{[1 s\;\ldots\; s^{n-2}\; s^{n-1}]}{s^n+\alpha_1\, s^{n-1}+\ldots+\alpha_{n-1}\, s+\alpha_n}\begin{bmatrix}\beta_n-\beta_0\,\alpha_n\\ \beta_{n-1}-\beta_0\,\alpha_{n-1}\\ \vdots\\ \beta_2-\beta_0\,\alpha_2\\ \beta_1-\beta_0\,\alpha_1\end{bmatrix} \tag{25}$$

and

$$\mathscr{L}\,[\ddot z(t)]=s\mathscr{L}\,[\dot z(t)]-\dot z(0)=s\mathscr{L}\,[\dot z(t)]-\lim_{s\to\infty} s\mathscr{L}\,[\dot z(t)]=s\,[F(s)-\beta_0]-(\beta_1-\beta_0\,\alpha_1)=$$

$$=\frac{[1\; s\;\ldots\; s^{n-2}\; s^{n-1}]}{s^n+\alpha_1\, s^{n-1}+\ldots+\alpha_{n-1}\, s+\alpha_n}\begin{bmatrix}-\beta_1\,\alpha_n+\beta_0\,\alpha_1\,\alpha_n\\ \beta_n-\beta_1\,\alpha_{n-1}+\beta_0(\alpha_1\,\alpha_{n-1}-\alpha_n)\\ \vdots\\ \beta_3-\beta_1\,\alpha_2+\beta_0\,(\alpha_1\,\alpha_2-\alpha_3)\\ \beta_2-\beta_1\,\alpha_1+\beta_0\,(\alpha_1^2-\alpha_2)\end{bmatrix} \tag{26}$$

Since two functions having identical Laplace transforms must be identical, then equating one by one expressions given by Eqs. (24), (25) and (26) to the right-hand side of Eq. (20), we shall obtain the output $y(t)$ of the autonomous system in the form of $z(t)-z(\infty)\,\mathbf{1}(t)$, $\dot z(t)$ and $\ddot z(t)$ respectively. To achieve such situation, it is, of course, necessary to satisfy the following identity:

$$\det\,(s\boldsymbol{I}-\boldsymbol{A})\equiv s^n+\alpha_1\, s^{n-1}+\ldots+\alpha_{n-1}\, s+\alpha_n\,. \tag{27}$$

It gives us the correlation between the coefficients $a_i$ of the system matrix $A$ and the coefficients $\alpha_j$ of the transfer function $F(s)$. To write down this correlation in explicit form, let us notice that according to Eq. (15), we have

$$\det(sI-A)=(s+a_1)\,\Delta_n+a_2\,\Delta_{n-1} \tag{28}$$

or substituting for $\Delta_n$ and $\Delta_{n-1}$ their values given by Eq. (19), identity (27) may be rewritten in the form:

$$\det(sI-A)=\begin{bmatrix} 1 \\ s \\ \vdots \\ s^{n-7} \\ s^{n-6} \\ s^{n-5} \\ s^{n-4} \\ s^{n-3} \\ s^{n-2} \\ s^{n-1} \\ s^{n} \end{bmatrix}' \begin{bmatrix} \vdots \\ \vdots \\ \vdots \\ a_1 \sum\limits_{l=3}^{n-4} a_l \sum\limits_{k=l+2}^{n-2} a_k \sum\limits_{i=k+2}^{n} a_i \\ \sum\limits_{l=2}^{n-4} a_l \sum\limits_{k=l+2}^{n-2} a_k \sum\limits_{i=k+2}^{n} a_i \\ a_1 \sum\limits_{k=3}^{n-2} a_k \sum\limits_{i=k+2}^{n} a_i \\ \sum\limits_{k=2}^{n-2} a_k \sum\limits_{i=k+2}^{n} a_i \\ a_1 \sum\limits_{i=3}^{n} a_i \\ \sum\limits_{i=2}^{n} a_i \\ a_1 \\ 1 \end{bmatrix} \equiv \begin{bmatrix} 1 \\ s \\ \vdots \\ s^{n-7} \\ s^{n-6} \\ s^{n-5} \\ s^{n-4} \\ s^{n-3} \\ s^{n-2} \\ s^{n-1} \\ s^{n} \end{bmatrix}' \begin{bmatrix} \alpha_n \\ \alpha_{n-1} \\ \vdots \\ \alpha_7 \\ \alpha_6 \\ \alpha_5 \\ \alpha_4 \\ \alpha_3 \\ \alpha_2 \\ \alpha_1 \\ 1 \end{bmatrix}. \tag{29}$$

By comparing the coefficients related with the same power of $s$ at both sides of the identity symbol, the last expression gives us directly $\alpha_j$ as a function of $a_i$. To obtain, vice versa, $a_i$ as a function of $\alpha_j$, we are making the same comparison keeping, however, the following order: from the last two rows of Eq. (29), i.e. the rows related with $s^n$ and $s^{n-1}$, we have just $a_1=\alpha_1$; going up and taking next two rows we can determine $a_2$; going again up for the next two rows we can find $a_3$, and in the following steps we can determine all the remaining coefficients up to $a_n$. It is easy to show [3]) that the result of this process can be written with the utilization of Hurwitz determinants in the following final form:

$$a_1=\Delta_1,\ a_2=\Delta_2/\Delta_1,\ a_3=\Delta_3/\Delta_2\,\Delta_1,\ \dots\ a_r=\frac{\Delta_r\,\Delta_{r-3}}{\Delta_{r-1}\,\Delta_{r-2}},\ \dots \text{ for } r=4,\dots,n, \tag{30}$$

[3]) Cf. Ref. [2] p. 473, Theorem VI.

where $\Delta_r$ is, as usually, the symbol of Hurwitz determinant of order $r$:

$$\Delta_r = \begin{vmatrix} \alpha_1 & 1 & 0 & \dots & \alpha_{2-r} \\ \alpha_3 & \alpha_2 & \alpha_1 & \dots & \alpha_{4-r} \\ \alpha_5 & \alpha_4 & \alpha_3 & \dots & \alpha_{6-r} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \alpha_{2r-1} & \alpha_{2r-2} & \alpha_{2r-3} & \dots & \alpha_r \end{vmatrix},$$

$$\alpha_s = 0 \text{ for } \begin{cases} s<0 \\ s>n \end{cases}; \quad \alpha_s = 1 \text{ for } s=0. \tag{31}$$

Assuming that condition (30) is satisfied, the problem of comparing the right-hand side of Eq. (20) with one of the Laplace transforms given by Eqs. (24), (25) or (26) reduces now to the determination of the initial value state vector $\boldsymbol{x}(0)$ by premultiplying at first the column vectors of Eqs. (24)–(26) by the inverse of the matrix $T'$ and then by equating one by one the result to the column vector of Eq. (20). Let us notice that, due to the simple, triangular form of the matrix $T'$, there is no special problem to find its inverse; most simply it can be determined by means of a series of elementary transformations of matrices — the result is:

$$(T')^{-1} =$$

$$= \left[\begin{array}{ccc|c|ccccccc}
1 & & -a_n & & & & & & & & \\
 & 1 & & & & & & & & & \\
 & & 1 & & & & & & & & \\
\hline
 & & & \ddots & & & & & & & \\
\hline
 & & & & 1 & & -\sum\limits_{i=7}^{n} a_i & & \sum\limits_{k=7}^{n} a_k \sum\limits_{i=k-1}^{n} a_i & & -\sum\limits_{l=7}^{n} a_l \sum\limits_{k=l-1}^{n} a_k \sum\limits_{i=k-1}^{n} a_i \\
 & & & & & 1 & & -\sum\limits_{i=6}^{n} a_i & & \sum\limits_{k=6}^{n} a_k \sum\limits_{i=k-1}^{n} a_i & \\
 & & & & & & 1 & & -\sum\limits_{i=5}^{n} a_i & & \sum\limits_{k=5}^{n} a_k \sum\limits_{i=k-1}^{n} a_i \\
 & & & & & & & 1 & & -\sum\limits_{i=4}^{n} a_i & \\
 & & & & & & & & 1 & & -\sum\limits_{i=3}^{n} a_i \\
 & & & & & & & & & 1 & \\
 & & & & & & & & & & 1
\end{array}\right] \tag{32}$$

Considering, as an example, the case of $n=5$ and denoting the initial value state vector $\boldsymbol{x}(0)$ which leads to the output signal $y(t)$ equal either to $z(t)-z(\infty)\,\mathbf{1}(t)$ or to $\dot{z}(t)$ or to $\ddot{z}(t)$ by $\boldsymbol{x}_0(0)$, $\boldsymbol{x}_1(0)$ and $\boldsymbol{x}_2(0)$ respectively, we get the following, practically important, final result:

$$
\mathbf{x}_0(0) = \begin{bmatrix}
\dfrac{1}{a_5\, a_4\, a_3\, a_2}\{-a_4\, a_2\, F(0) + a_5(a_5+a_4)\,\beta_0 - a_5\,\beta_2 + \beta_4\} \\
\dfrac{1}{a_4\, a_3\, a_2}\{\ a_3\, a_1\, F(0) + (a_5+a_4)\,\beta_1 - \beta_3\} \\
\dfrac{1}{a_3\, a_2}\{-a_2\, F(0) - (a_5+a_4+a_3)\,\beta_0 + \beta_2\} \\
\dfrac{1}{a_2}\{\ a_1\, F(0) - \beta_1\} \\
-F(0) + \beta_0
\end{bmatrix}, \qquad (33)
$$

$$
\mathbf{x}_1(0) = \begin{bmatrix}
\dfrac{1}{a_5\, a_4\, a_3\, a_2}\{\ a_5(a_5+a_4)\,\beta_1 - a_5\,\beta_3 + \beta_5\} \\
\dfrac{1}{a_4\, a_3\, a_2}\{-[(a_5+a_4)^2 + a_4\, a_3]\,\beta_0 + (a_5+a_4)\,\beta_2 - \beta_4\} \\
\dfrac{1}{a_3\, a_2}\{-(a_5+a_4+a_3)\,\beta_1 + \beta_3\} \\
\dfrac{1}{a_2}\{\ (a_5+a_4+a_3+a_2)\,\beta_0 - \beta_2\} \\
-a_1\,\beta_0 + \beta_1
\end{bmatrix}, \qquad (34)
$$

$$
x_2(0)=\begin{bmatrix}
\frac{1}{a_4\,a_3\,a_2}\{\quad -[(a_5+a_4)^2+a_4\,a_3]\,\beta_0 \quad +(a_5+a_4)\,\beta_2 \quad -\beta_4\} \\
\frac{1}{a_4\,a_3\,a_2}\{\quad -[(a_5+a_4)^2+a_4\,a_3]\,\beta_1 \quad +(a_5+a_4)\,\beta_3 \quad -\beta_5\} \\
\frac{1}{a_3\,a_2}\{[(a_5+a_4)\,(a_5+a_4+a_3)+a_3\,(a_4+a_3+a_2)]\,\beta_0 \quad -(a_5+a_4+a_3)\,\beta_2 \quad +\beta_4\} \\
\frac{1}{a_2}\{\quad -a_2\,a_1\,\beta_0 \quad +(a_5+a_4+a_3+a_2)\,\beta_1 \quad -\beta_3\} \\
-(a_5+a_4+a_3+a_2-a_1^2)\,\beta_0 \quad -a_1\,\beta_1\beta_2+
\end{bmatrix}
\tag{35}
$$

It is essential to point out here that formulae (33)–(35) are valid not only for $n=5$ but as well for any positive integer $n<5$; in such a case, as it is apparent from Eq. (20), the components of n-dimensional vectors $\boldsymbol{x}_c(0)$ $(c=0, 1, 2)$ are the same as the last $n$ components of the five-dimensional vectors given by Eqs. (33)–(35) with the obvious modification that the coefficients $\alpha_i$ and $\beta_i$ corresponding to $i>n$ should be replaced by zeros.

Finally, let us see that the nonsingular matrix $\boldsymbol{T}$ defined by Eq. (19) has a clear mathematical meaning: it is the transformation matrix which transforms similarly the system matrix $\boldsymbol{A}$ given by Eq. (2) into its standardized form $\boldsymbol{A}_\alpha$ given by

$$\boldsymbol{A}_\alpha=\begin{bmatrix} & 1 & & & & \\ & & 1 & & & \\ & & & \ddots & & \\ & & & & 1 & \\ & & & & & 1 \\ -\alpha_n & -\alpha_{n-1} & -\alpha_{n-2} & \cdots & -\alpha_2 & -\alpha_1 \end{bmatrix} \tag{36}$$

i.e. $\boldsymbol{T}$ occurs in the formula

$$\boldsymbol{A}_\alpha=\boldsymbol{T}^{-1}\boldsymbol{A}\boldsymbol{T}. \tag{37}$$

Really, at first let us notice that the characteristic polynomial of the matrix $\boldsymbol{A}_\alpha$ is the same as the right-hand side of the identity (27); it means that the necessary condition of similarity of matrices, viz. identity of their characteristic polynomials:

$$\det(s\boldsymbol{I}-\boldsymbol{A})=\det(s\boldsymbol{I}-\boldsymbol{A}_\alpha) \tag{38}$$

and, in consequence, identity of their characteristic values $s_l$ $(l=1, \ldots, n)$ is here satisfied.

Next, les us notice that the vectors

$$\boldsymbol{a}\overset{\text{def}}{=}\begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_{n-1} \\ A_n \end{bmatrix}, \qquad \boldsymbol{\alpha}\overset{\text{def}}{=}\begin{bmatrix} 1 \\ s \\ \vdots \\ s^{n-2} \\ s^{n-1} \end{bmatrix} \tag{39}$$

calculated for $s=s_l$ take the form of the characteristic vectors of matrices $\boldsymbol{A}$ and $\boldsymbol{A}_\alpha$ respectively, i.e.

$$(s\boldsymbol{I}-\boldsymbol{A})\,\boldsymbol{a}|_{s=s_l}=\boldsymbol{0}, \tag{40}$$

$$(s\boldsymbol{I}-\boldsymbol{A}_\alpha)\,\boldsymbol{\alpha}|_{s=s_l}=\boldsymbol{0}. \tag{41}$$

For any $s$, however, $\boldsymbol{a}$ is related to $\boldsymbol{\alpha}$ by Eq. (19), which in the present, more compact, notation takes the form

$$\boldsymbol{a}=\boldsymbol{T}\boldsymbol{\alpha} \tag{42}$$

Thus, replacing $\boldsymbol{a}$ in Eq. (40) by $\boldsymbol{T}\boldsymbol{\alpha}$ and premultiplying both sides of that equation by $\boldsymbol{T}^{-1}$, we get

$$(s\boldsymbol{I}-\boldsymbol{T}^{-1}\boldsymbol{A}\boldsymbol{T})\,\boldsymbol{\alpha}|_{s=s_l}=\boldsymbol{0}. \tag{43}$$

Comparison of the last equation with Eq. (41) verifies the statement of Eq. (37).

Closing the section, let us concentrate our attention on the form of the performance index itself.

On the basis of result (13) and the fact that $\boldsymbol{x}_0(0)$, $\boldsymbol{x}_1(0)$ and $\boldsymbol{x}_2(0)$ given by Eqs. (33)–(35) correspond to $y(t)$ identified with $z(t)-z(\infty)\mathbf{1}(t)$, $\dot{z}(t)$ and $\ddot{z}(t)$ respectively, the performance index $J_k$ defined by integral (6) reduces to the following, canonical quadratic form:

$$J_k = \begin{bmatrix} \boldsymbol{x}_0(0) \\ \boldsymbol{x}_1(0) \\ \boldsymbol{x}_2(0) \\ \vdots \\ \boldsymbol{x}_k(0) \end{bmatrix}' \mathbf{diag} \begin{vmatrix} \boldsymbol{P} \\ \tau_1^2 \boldsymbol{P} \\ \tau_2^4 \boldsymbol{P} \\ \vdots \\ \tau_k^{2k} \boldsymbol{P} \end{vmatrix} \begin{bmatrix} \boldsymbol{x}_0(0) \\ \boldsymbol{x}_1(0) \\ \boldsymbol{x}_2(0) \\ \vdots \\ \boldsymbol{x}_k(0) \end{bmatrix}, \tag{44}$$

where $\boldsymbol{x}_k(0)$ is the initial value state vector corresponding to $y(t)$ identified with $\overset{(k)}{z}(t)$ and which can be calculated by exactly the same procedure as the vectors $\boldsymbol{x}_1(0)$ and $\boldsymbol{x}_2(0)$.

As regards an integer $k$, it can take any value from the sequence: 0, 1, 2, ..., $n-1$. In applied problems of parametric optimization its most practical value is, however, $k=1$, sometimes $k=2$; other values are used rather occasionally — the simplest $k=0$ usually yields systems with too small stability margin.

As regards the weighting factors $\tau_1, ..., \tau_k$ there is, as well, no rigorous method for their proper choice. To reduce too big arbitrariness related with their estimate, it is quite useful to leave in Eq. (44) just one arbitrary parameter, for example $\tau_k$, and to express all remaining by that one. A possibility of such a choice is given by the formula:

$$\left.\begin{aligned} \tau_1^2 &= \binom{k}{1} \tau_k^2 \\ \tau_2^4 &= \binom{k}{2} \tau_k^4 \\ &\vdots \\ \tau_{k-1}^{2(k-1)} &= \binom{k}{k-1} \tau_k^{2(k-1)} \end{aligned}\right\}. \tag{45}$$

Since in all problems of parametric optimization, the performance index should be minimized, it is natural to look for the function $z(t)$ for which the performance index as defined by Eq. (6) with its weighting factors chosen according to Eq. (45), attains a minimum. A solution of this problem leads to the well-known Euler's equation [4]) which in the case considered takes the form:

$$[(1-\tau_k D)(1+\tau_k D)]^k z(t) = z(\infty) \tag{46}$$

[4]) $\dfrac{\partial L}{\partial z} - D\left(\dfrac{\partial L}{\partial \dot{z}}\right) + D^2\left(\dfrac{\partial L}{\partial \ddot{z}}\right) + ... + (-D)^k \left(\dfrac{\partial L}{\partial \overset{(k)}{z}}\right) = 0;$

$L \overset{\text{def}}{=} L[z(t), \dot{z}(t), ..., \overset{(k)}{z}(t)]$ being integrant of the functional of Eq. (6) and $D \overset{\text{def}}{=} d/dt$.

and which, for hypothetically stable system, possesses, in turn, the following general solution:

$$z(t)-z(\infty)=\exp\left(-\frac{t}{\tau_k}\right)(c_1+c_2\,t+\ldots+c_k\,t^{k-1})\,. \tag{47}$$

Thus, for the weighting factors chosen according to Eq. (45), an ideal form of the purely transient part of the step response of the system, i.e. a form which yields a minimum of the integral (6), is for $k=0$ exactly equal to zero and for $k$ being any positive integer is of the so-called critical type characterized in its exponential part by just one time-constant equal to the weighting factor $\tau_k$. As regards $\tau_k$ itself, its value should be related in some form to the parameters of the system. One of the simplest possible relation of this type is given by expression

$$\tau_k=1/\alpha_1\,. \tag{48}$$

It should be noticed that according to the well-known meaning of the parameter $\alpha_1$, Eq. (48) expresses the equality between the inverse of $\tau_k$ and the sum of inverses of all time-constants of the system — a feature which, in some sense, guarantees the correctness of formula (48).

Finally, utilizing Eqs. (44), (45) and (48) and putting $\alpha_1=a_1$, we get for the performance index $J_k$ the following expression of a quite big practical value:

$$J_k=\begin{bmatrix} \boldsymbol{x}_0(0) \\ \boldsymbol{x}_1(0) \\ \boldsymbol{x}_2(0) \\ \vdots \\ \boldsymbol{x}_{k-1}(0) \\ \boldsymbol{x}_k(0) \end{bmatrix}' \textbf{diag}\begin{bmatrix} \boldsymbol{P} \\ \binom{k}{1}\boldsymbol{P}/a_1^2 \\ \binom{k}{2}\boldsymbol{P}/a_1^4 \\ \vdots \\ \binom{k}{k-1}\boldsymbol{P}/a_1^{2(k-1)} \\ \boldsymbol{P}/a_1^{2k} \end{bmatrix} \begin{bmatrix} \boldsymbol{x}_0(0) \\ \boldsymbol{x}_1(0) \\ \boldsymbol{x}_2(0) \\ \vdots \\ \boldsymbol{x}_{k-1}(0) \\ \boldsymbol{x}_k(0) \end{bmatrix} \tag{49}$$

## IV. Parametric optimization — Examples

1. For the simple *RLC* network shown in Fig. 1, let us determine for the fixed values of $L$ and $C$, the optimal value of the resistance $R$, identifying the input and the output of the system with voltages $e_i(t)$ and $e_0(t)$ respectively, and taking as the criterion of optimality a) $\min\limits_R J_0$, b) $\min\limits_R J_1$, with $J_k$ defined by Eq. (49).

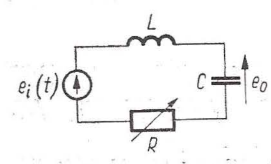


Fig. 1. Electric network considered in Example 1

The transfer function of the system written in standardized form is:

$$F(s)=\frac{\omega_0^2}{s^2+2\xi\omega_0 s+\omega_0^2}, \tag{50}$$

where

$$\omega_0=1/\sqrt{LC}, \quad \xi=\frac{R}{2}\sqrt{\frac{C}{L}}.$$

Thus, in the present notation, the role of $R$ is being played by the damping coefficient $\xi$.

For the given system, we have:

$$F(0)=1, \quad \beta_0=\beta_1=0, \quad \beta_2=\omega_0^2, \quad a_1=2\xi\omega_0, \quad a_2=\omega_0^2.$$

Applying Eqs. (10), (33), (34) and (49), we get

$$\boldsymbol{P}=\mathbf{diag}(\omega_0^2, 1)/(4\xi\omega_0), \quad \boldsymbol{x}_0(0)=\begin{bmatrix} \dfrac{2\xi}{\omega_0} \\ -1 \end{bmatrix}, \quad \boldsymbol{x}_1(0)=\begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

and

$$J_0=\frac{1}{\omega_0}\left(\xi+\frac{1}{4\xi}\right), \quad J_1=\frac{1}{\omega_0}\left(\xi+\frac{1}{4\xi}+\frac{1}{16\xi^3}\right).$$

Thus,

a) $\min\limits_{\xi} J_0 \to \dfrac{dJ_0}{d\xi}\equiv 0 \to \xi=0.5,$

b) $\min\limits_{\xi} J_1 \to \dfrac{dJ_1}{d\xi}\equiv 0 \to 16\xi^4-4\xi^2-3=0, \quad \xi=\dfrac{1}{2}\sqrt{\dfrac{1+\sqrt{13}}{2}}\approx 0.76.$

As is well known, the value $\xi=0.5$ corresponds to a system which is too oscillatory; on the contrary, $\xi=0.76$ can be considered as a very reasonable result. Let us also notice that according to the ratio

$$\frac{J_0|_{\xi=0.76}}{J_0|_{\xi=0.5}}=1.09,$$

the damping coefficient $\xi$, which is optimal in the sense of $J_1$ criterion, introduced into $J_0$, gives the result which is quite near its minimal value.

2. For the simple servomechanism of class two [5]) shown in block diagram form in Fig. 2 with its open-loop transfer function given by

$$G(s)=\frac{K(1+CTs)}{s^2(1+Ts)}, \quad T>0, K>0, C>1, \tag{51}$$

which corresponds to the double integral unit in series with the lead type equalizer, let us determine, for the fixed values of $K$ and $C$, the optimal value of the time-

[5]) Cf. Ref. [3] p. 316–318, Examples 7.15 and 7.16.

constant $T$, identifying the input and the output of the system with $r(t)$ and $c(t)$ respectively, and taking as the criterion of optimality a) $\min_T J_0$, b) $\min_T J_1$, with $J_k$ defined by Eq. (49).

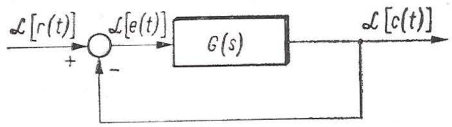


Fig. 2. Block diagram of the servomechanism considered in Example 2
$r(t)$ — reference input, $c(t)$ — controlled variable, $e(t)$ — error signal

The transfer function of the closed-loop system is

$$F(s)=\frac{G(s)}{1+G(s)}=\frac{CKs+\dfrac{K}{T}}{s^3+\dfrac{1}{T}s^2+CKs+\dfrac{K}{T}}, \tag{52}$$

for which we have either directly or according to Eq. (30) the following coefficients:

$$F(0)=1, \quad \beta_0=\beta_1=0, \quad \beta_2=CK, \quad \beta_3=K/T,$$
$$a_1=1/T, \quad a_2=(C-1)K, \quad a_3=K.$$

Applying Eqs. (10), (33), (34) and (49), we get

$$\boldsymbol{P}=\frac{T}{2}\,\mathbf{diag}\,((C-1)K^2, (C-1)K, 1)$$

$$\boldsymbol{x}_0(0)=\begin{bmatrix}\dfrac{1}{(C-1)K}\\[2ex] \dfrac{1}{(C-1)KT}\\[2ex] -1\end{bmatrix}, \qquad \boldsymbol{x}_1(0)=\begin{bmatrix}\dfrac{1}{(C-1)KT}\\[2ex] -\dfrac{C}{C-1}\\[2ex] 0\end{bmatrix}$$ [6])

and

$$J_0=\frac{1}{2(C-1)K}\left[CKT+\frac{1}{T}\right],$$

$$J_1=\frac{1}{2(C-1)K}\left[(C+1)KT+\frac{1}{T}+(CK)^2T^3\right],$$

[6]) Notice that identifying the input and the output of the system with $r(t)$ and $e(t)$ respectively (instead of $r(t)$ and $c(t)$), i.e. taking as the transfer function of the closed-loop system the ratio:

$$F(s)=\frac{1}{1+G(s)}=\frac{s^3+\dfrac{1}{T}s^2}{s^3+\dfrac{1}{T}s^2+CKs+\dfrac{K}{T}},$$

we shall obtain for the vectors $\boldsymbol{x}_0(0)$ and $\boldsymbol{x}_1(0)$ expressions differing from those given in the text just in signs $\left(\text{since: } \dfrac{1}{1+G(s)}=1-\dfrac{G(s)}{1+G(s)}\right)$, and in consequence the performance index $J_k$ calculated from both formulae will be exactly the same.

Thus

a) $\min\limits_T J_0 \to CKT^2 = 1\,,$

b) $\min\limits_T J_1 \to 3(CKT^2)^2 + \left(1 + \dfrac{1}{C}\right) CKT^2 - 1 = 0$

$$CKT^2 = \frac{\sqrt{13C^2+2C+1}-(C+1)}{6C} \overset{\text{def}}{=} \frac{1}{\gamma(C)}\,, \tag{53}$$

Drawing, for the determined optimal values of $T$, the asymptotic Bode diagrams of the open-loop system we get the situation as shown in Fig. 3. Let us notice that

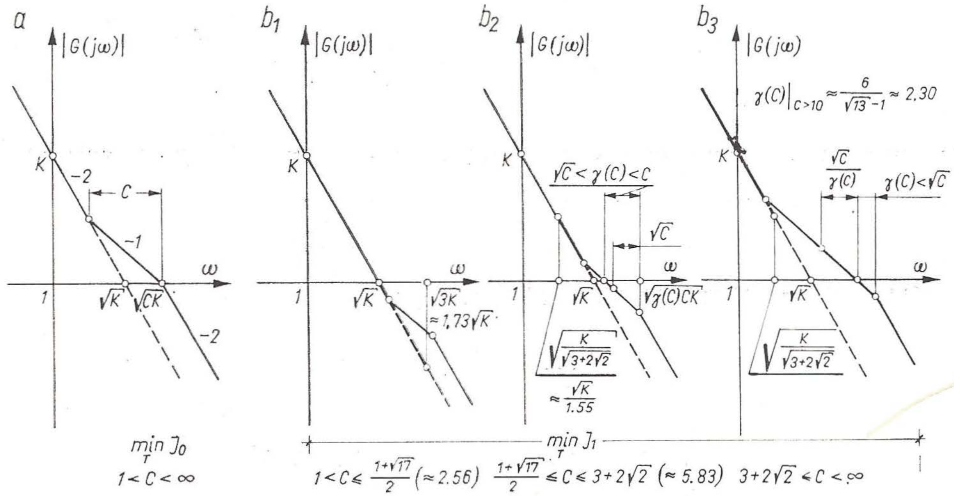


Fig. 3. Asymptotic Bode diagrams corresponding to optimal values of $T$

applying $J_1$ criterion, it is possible to choose such value of $C$ that the corresponding Bode plot will be symmetric with respect to the point of unity gain. According to Figs. $3b_2$ and $3b_3$, this kind of symmetry will take place if $C$ satisfies the identity

$$\gamma(C) \equiv \sqrt{C}\,. \tag{54}$$

The last condition together with the definition of $\gamma(C)$ given by Eq. (53) leads to

$$\begin{gathered} C^2 - 6C + 1 = 0, \\ C = 3 + 2\sqrt{2} \approx 5.83, \\ CKT^2 = \frac{1}{\sqrt{3+2\sqrt{2}}} \approx \frac{1}{2.41}, \end{gathered} \tag{55}$$

and, in consequence, to the following value of the time-constant

$$T = \frac{1}{(3+2\sqrt{2})^{3/4}\sqrt{K}} \approx \frac{1}{3.73\sqrt{K}}\,. \tag{56}$$

Let us now recall that a Bode plot which is symmetric with respect to the point of unity gain and which is characterized by any $C$ from the range $5<C<20$, exhibits a very typical situation for a simple servomechanism of the type considered in the Example. Thus $J_1$ criterion combined with the condition (54) gives the result which do agree very well with the common practice. On the contrary, $J_0$-criterion, as it is apparent from the graph shown in Fig. 3a, is in this sense quite ineffectual.

Finally, notice that according to expression

$$\frac{J_0\Big|_{\substack{C=5{,}83\\ CKT^2=1/2.41}}}{J_0\Big|_{\substack{C=5.83\\ CKT^2=1}}}=\frac{\left(\dfrac{3.41}{\sqrt{2.41}}\right)}{2}=1.10,$$

the performance index $J_0$ calculated for $C$ and $T$ chosen according to Eqs. (55) and (56) is increased with respect to its minimal value in a rather small amount equal to 10%.

3. For the simple control system [7]) shown in block diagram form in Fig. 4 with the transfer function of the plant given by

$$G(s)=\frac{C}{s^2+As+B}, \qquad A\geqslant 2\sqrt{B},\, B>0,\, C>0, \tag{57}$$

i.e. corresponding to the two time-constants unit and the transfer function of the feedback element

$$H(s)=K/s, \qquad K>0, \tag{58}$$

corresponding to the integral type controller, let us determine the optimal value of its parameter $K$, identifying the input and the output of the system with $d(t)$ and $c(t)$ respectively, and taking as the criterion of optimality a) $\min\limits_K J_0$, b) $\min\limits_K J_1$, c) $\min\limits_K J_2$, with $J_k$ defined by Eq. (49).

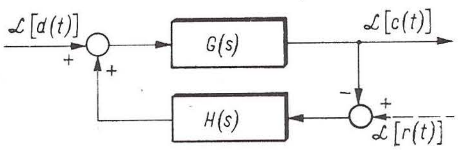


Fig. 4. Block diagram of the control system considered in Example 3
$d(t)$ — disturbance signal, $c(t)$ — controlled variable, $r(t)$ — reference input

The transfer function of the closed-loop system is

$$F(s)=\frac{G(s)}{1+G(s)\,H(s)}=\frac{Cs}{s^3+As^2+Bs+CK}, \tag{59}$$

[7]) Cf. Ref. [3] p. 316, Example 7.14.

for which we obtain either directly or with the aid of Eq. (30) the following coefficients:

$$F(0)=0, \quad \beta_0=\beta_1=\beta_3=0, \quad \beta_2=C,$$

$$a_1=A, \qquad a_2=\frac{AB-CK}{A}, \qquad a_3=\frac{CK}{A}.$$

Thus, according to condition (12), which ensures the asymptotic stability of the system, the coefficients of the plant and of the controller must satisfy the additional requirement:

$$AB>CK. \tag{60}$$

Applying now Eqs. (10), (33)–(35) and (49), we get

$$\boldsymbol{P}=\frac{1}{2A}\,\mathbf{diag}\left(\frac{(AB-CK)CK}{A^2}, \frac{AB-CK}{A}, 1\right),$$

$$\boldsymbol{x}_0(0)=\begin{bmatrix} \dfrac{A^2C}{(AB-CK)CK} \\ 0 \\ 0 \end{bmatrix}, \quad \boldsymbol{x}_1(0)=\begin{bmatrix} 0 \\ -\dfrac{AC}{AB-CK} \\ 0 \end{bmatrix}, \quad \boldsymbol{x}_2(0)=\begin{bmatrix} -\dfrac{AC}{AB-CK} \\ 0 \\ C \end{bmatrix}$$

and

$$J_0=\frac{AC^2}{2}\,\frac{1}{(AB-CK)CK}, \tag{61}$$

$$J_1=\frac{C^2}{2A^2}\,\frac{A^3+CK}{AB-CK)CK},$$

$$J_2=\frac{C^2}{2A^4}\,\frac{A^5+(2A^2+B)CK}{(AB-CK)CK}.$$

Thus

a) $\min\limits_K J_0 \rightarrow CK=\dfrac{AB}{2},$ (62)

b) $\min\limits_K J_1 \rightarrow (CK)^2+2A^3\,CK-A^4\,B=0$

$$CK=A^2(\sqrt{A^2+B}-A)=\frac{AB}{2}\left(1-\frac{B}{2A^2+B+2A\sqrt{A^2+B}}\right), \tag{63}$$

c) $\min\limits_K J_2 \rightarrow (2A^2+B)\,(CK)^2+2A^5\,CK-A^6\,B=0$

$$CK=\frac{A^3B}{2A^2+B}=\frac{AB}{2}\left(1-\frac{B}{2A^2+B}\right). \tag{64}$$

The value of $CK$ determined by any of Eqs. (62) [8])–(64) can really be considered as its optimal value since for each of them the condition (60) is satisfied and the corresponding $J_k$ takes a minimum. Let us also notice that optimal $CK$ is the biggest for $J_0$-, smaller for $J_1$- and still a little smaller for $J_2$-criterion. This decrease of $CK$ results, of course, in some improvement in the stability of the system. It should be, however, emphasized that the differences between the $CK$ parameters determined from various criterions are numerically quite small. Taking, for illustration, the most pronounced case of two equal time-constants of the plant, i.e. putting $A=2\sqrt{B}$, we get

$$\frac{CK|_{\min J_1}-CK|_{\min J_0}}{CK|_{\min J_0}}=\left.\frac{B}{2A^2+B+2A\sqrt{A^2+B}}\right|_{A=2\sqrt{B}}=\frac{1}{9+4\sqrt{5}}\approx 0.06\,,$$

$$\frac{CK|_{\min J_2}-CK|_{\min J_0}}{CK|_{\min J_0}}=\left.\frac{B}{2A^2+B}\right|_{A=2\sqrt{B}}=\frac{1}{9}\approx 0.11\,.$$

In consequence, as is seen from Eq. (61) or more clearly from formulae:

$$\frac{J_0\big|_{CK=A^2(\sqrt{A^2+B}-A)}}{J_0\big|_{CK=AB/2}}=\frac{1}{1-\left(\dfrac{B}{2A^2+B+2A\sqrt{A^2+B}}\right)^2}\,,$$

$$\frac{J_0\big|^{CK=A^3B/(2A^2+B)}}{J_0\big|_{CK=AB/2}}=\frac{1}{1-\left(\dfrac{B}{2A^2+B}\right)^2}\,,$$

the performance index $J_0$ calculated for $CK$ optimal in the sense of $J_1$- or $J_2$-criterion remains nearly the same as its minimal value.

4. For the system described by the transfer function of the following general form:

$$F(s)=\frac{1}{s^n+\alpha_1 s^{n-1}+\ldots+\alpha_{n-1}s+1}\,, \tag{65}$$

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[8]) It may be interesting to remark that if the transfer function of the system is defined by

$$F(s)=\frac{G(s)\,H(s)}{1+G(s)\,H(s)}=\frac{CK}{s^3+As^2+Bs+CK}\,,$$

i.e. if its input is identified with the reference signal (instead of disturbance), then the $CK$ parameter optimal in the sense of $J_0$ criterion will take a value

$$CK=\frac{AB}{2}\left(1-\frac{A-\sqrt{B}}{A+\sqrt{B}}\right)$$

smaller for any $A\geqslant 2\sqrt{B}$ than $CK$ expressed by Eq. (62), and for $A\gg 2\sqrt{B}$ reaching a "saturation" level:

$$CK\Big|_{A\gg 2\sqrt{B}}=B^{3/2}\,.$$

and for $n=3$, 4 and 5 respectively [9]), let us determine the optimal values of the parameters $\alpha_1, \ldots, \alpha_{n-1}$, taking as the criterion of optimality a) $\min\limits_{\alpha_1, \ldots \alpha_{n-1}} J_0$, b) $\min\limits_{\alpha_1, \ldots, \alpha_{n-1}} J_1$ with $J_k$ defined by Eq. (49) (the so-called "optimum" transfer functions' problem).

As is easy to verify, the calculations are essentially simplified by expressing the performance index $J_k$ as a function of the parameters $a_1, \ldots, a_{n-1}$ only and, in consequence, by evaluating the optimal values of the final parameters $\alpha_1, \ldots, \alpha_{n-1}$ by determining at first the optimal values of the auxiliary parameters $a_1, \ldots, a_{n-1}$ and then by using the formula (29).

For $n=3$, according to Eqs. (10), (33), (34) and (49) with $a_3$ replaced by $\dfrac{1}{a_1}$ ($\alpha_3=1$, Eq. (29)), we have

$$J_k=\frac{1}{2}\left(\frac{1}{a_1}+\frac{a_1}{a_2}+a_2+\frac{k}{a_1^2 a_2}\right), \qquad k=0 \text{ or } 1 .$$

Thus,

$$\text{a)}\quad \min_{a_1, a_2} J_0 \rightarrow \frac{\partial J_0}{\partial a_1} \equiv 0 , \qquad \frac{\partial J_0}{\partial a_2} \equiv 0 ,$$

$$\left.\begin{matrix} a_1 \\ \\ a_2 \end{matrix}\right\} =1 \qquad \begin{matrix} \alpha_1=a_1=1 , \\ \\ \alpha_2=a_2+\dfrac{1}{a_1}=2 , \end{matrix}$$

$$\text{b)}\quad \min_{a_1, a_2} J_1 \rightarrow \frac{\partial J_1}{\partial a_1} \equiv 0 , \qquad \frac{\partial J_1}{\partial a_2} \equiv 0 ,$$

$$a_1^6-5a_1^3+3=0, \qquad a_2=\frac{a_1^3-2}{a_1},$$

$$a_1=\sqrt[3]{\frac{5+\sqrt{13}}{2}} \approx 1.63 , \qquad \alpha_1=1.63 ,$$

$$a_2=\frac{1+\sqrt{13}}{2}\sqrt[3]{\frac{2}{5+\sqrt{13}}} \approx 1.41 , \qquad \alpha_2=2.03 ,$$

and

$$\frac{J_0\big|_{\substack{a_1=1.63 \\ a_2=1.41}}}{J_0\big|_{a_1=a_2=1}}=1.06 .$$

Similarly, for $n=4$, we have $\left(a_4=\dfrac{1}{a_2}, \text{ Eq. (29)}\right)$

$$J_k=\frac{1}{2}\left(\frac{1}{a_1}+\frac{a_1}{a_2}+\frac{a_2}{a_1 a_3}+a_1 a_3+\frac{k}{a_1^3 a_3}\right), \qquad k=0 \text{ or } 1 :$$

[9]) The case $n=2$ was treated above, in Example 1.

a) $\min\limits_{a_1, a_2, a_3} J_0 \to \left.\begin{matrix} a_1 \\ \\ a_2 \\ \\ a_3 \end{matrix}\right\} = 1 \qquad \begin{aligned} &\alpha_1 = a_1 = 1\,, \\ &\alpha_2 = a_2 + a_3 + \frac{1}{a_2} = 3\,, \\ &\alpha_3 = a_1\left(a_3 + \frac{1}{a_2}\right) = 2\,; \end{aligned}$

b) $\min\limits_{a_1, a_2, a_3} J_1 \to a_2^4 - a_2^2 - 3 = 0\,, \qquad a_1^2 = \dfrac{a_2^2+2}{a_2}, \qquad a_3 = \dfrac{a_2^3}{a_2^2+2},$

$$a_1 = \sqrt{\frac{5+\sqrt{13}}{2}\sqrt{\frac{2}{1+\sqrt{13}}}} \approx 1.68\,, \qquad \alpha_1 = 1.68\,,$$

$$a_2 = \sqrt{\frac{1+\sqrt{13}}{2}} \approx 1.52\,, \qquad \alpha_2 = 3.08\,,$$

$$a_3 = \sqrt{\frac{2}{5+\sqrt{13}}\sqrt{\left(\frac{1+\sqrt{13}}{2}\right)^5}} \approx 0.90\,, \qquad \alpha_3 = 2.61\,,$$

$$\frac{J_0\Big|_{\substack{a_1=1.68\\ a_2=1.52\\ a_3=0.90}}}{J_0\Big|_{a_1=a_2=a_3=1}} = 1.05\,.$$

Finally, for $n=5$, we get $\left(a_5 = \dfrac{1}{a_1\, a_3}, \text{ Eq. (29)}\right)$.

$$J_k = \frac{1}{2}\left(\frac{1}{a_1} + \frac{a_1}{a_2} + \frac{a_2}{a_1\, a_3} + \frac{a_1\, a_3}{a_2\, a_4} + a_2\, a_4 + \frac{k}{a_1^2\, a_2\, a_4}\right), \qquad k=0 \text{ or } 1:$$

a) $\min\limits_{a_1, a_2, a_3, a_4} J_0 \to \left.\begin{matrix} a_1 \\ \\ a_2 \\ \\ a_3 \\ \\ a_4 \end{matrix}\right\} = 1 \qquad \begin{aligned} &\alpha_1 = a_1 = 1\,, \\ &\alpha_2 = a_2 + a_3 + a_4 + \frac{1}{a_1\, a_3} = 4, \\ &\alpha_3 = a_1\left(a_3 + a_4 + \frac{1}{a_1\, a_3}\right) = 3\,, \\ &\alpha_4 = a_2\, a_4 + \frac{a_2 + a_3}{a_1\, a_3} = 3\,; \end{aligned}$

b) $\min\limits_{a_1, a_2, a_3\, a_4} J_1 \to (a_1^3\, a_4)^2 - 5\,(a_1^3\, a_4) + 3 = 0\,,$

$$a_1^5 = \frac{(a_1^3\, a_4)^3}{(a_1^3\, a_4)+1}, \qquad a_2 = \frac{(a_1^3\, a_4) - 2}{(a_1^3\, a_4)}\, a_1^2\,,$$

$$a_3 = a_4 = \frac{(a_1^3\, a_4)}{a_1^3},$$

$$a_1 = \left(2\frac{40+11\sqrt{13}}{7+\sqrt{13}}\right)^{1/5} \approx 1.72\,, \qquad \alpha_1 = 1.72\,,$$

$$a_2 = \frac{1+\sqrt{13}}{5+\sqrt{13}}\left(2\frac{40+11\sqrt{13}}{7+\sqrt{13}}\right)^{2/5} \approx 1.58\,, \qquad \alpha_2 = 3.96\,,$$

$$\left.\begin{matrix} a_3 \\ \\ a_4 \end{matrix}\right\} = \frac{5+\sqrt{13}}{2}\left(\frac{1}{2}\,\frac{7+\sqrt{13}}{40+11\sqrt{13}}\right)^{3/5} \approx 0.85 \qquad \begin{matrix} \alpha_3 = 4.09\,, \\ \\ \alpha_4 = 2.99\,, \end{matrix}$$

$$\frac{J_0\Big|_{\substack{a_1=1.72 \\ a_2=1.58 \\ a_3=a_4=0.85}}}{J_0\Big|_{a_1=a_2=a_3=a_4=1}} = 1.04\,.$$

Summarizing the transfer function of the type discussed in the Example (Eq. (65)) has, for $n=2, \ldots, 5$, the following optimal denominators:

a) in the sense of $J_0$-criterion:

$$\begin{aligned} s^2+s+1 \\ s^3+s^2+2s+1 \\ s^4+s^3+3s^2+2s+1 \\ s^5+s^4+4s^3+3s^2+3s+1 \end{aligned} \tag{66}$$

b) in the sense of $J_1$-criterion:

$$\begin{aligned} s^2+1.52s+1 \\ s^3+1.63s^2+2.03s+1 \\ s^4+1.68s^3+3.08s^2+2.61s+1 \\ s^5+1.72s^4+3.96s^3+4.09s^2+2.99s+1 \end{aligned} \tag{67}$$

From the engineering point of view, the results given by (67) should be considered as more practical than those of (66)[10]. Systems with denominators in the form of (67) satisfy in fact quite well the conflicting demands of the stability on one side and the speed of response on the other, whereas systems based on formula (66) exhibit a rather unsatisfactory stability margin.

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[10]) Cf. Ref. [2] p. 475, Table II.

## Uwagi o kwadratowo-całkowej ocenie procesu przejściowego

Rozważono ocenę całki (6) traktowanej jako wskaźnik jakości układu liniowego, asymptotycznie stabilnego, opisanego równaniem (5), z wymuszeniem skokowym na wejściu. Podejście jest ogólne, możliwe do stosowania do układów dowolnego rzędu i polegające na wykorzystaniu macierzy (2) i (10) oraz zależności funkcji Lapunowa od kwadratowego wskaźnika jakości. Rozwiązanie problemu jest związane w sposób istotny z wyznaczeniem macierzy pomocniczej (32). Z jej pomocą są sformułowane szczegółowe wzory robocze (33)–(35) obowiązujące dla układów rzędu $n \leqslant 5$ i dla wskaźników jakości z $k \leqslant 2$ (równanie (6)). Ich uogólnienie na dowolne wartości $n$ oraz $k$ nie przedstawia w istocie większych trudności ze względu na ogólny charakter macierzy (32). Szczególny nacisk położono na wybór współczynników wagowych występujących we wzorach (6) i (44), który w wyniku doprowadził do zaproponowanej postaci wskaźnika jakości wyrażonego wzorem (49). Zastosowanie tego rezultatu zilustrowano czterema prostymi przykładami optymalizacji parametrycznej układów dynamicznych: układu elektrycznego, serwomechanizmu, regulatora przemysłowego oraz układów scharakteryzowanych transmitancjami typu ,,optymalnego"

## Замечания по вопросу квадратично-интегральной оценки переходного процесса

В работе рассматривается оценка интеграла (6), играющего роль показателя качества линейной, асимптотически устойчивой системы, описанной посредством уравнения (5), со скачкообразным возмущением на входе. Используется общий подход, позволяющий применять его к системам произвольного порядка, базирующий на матрицах (2) и (10), а также зависимостях между функцией Ляпунова к квадратичным показателем качества. Решение задачи в основном сводится к определению дополнительной матрицы (32). С её помощью формулируются отдельные рабочие формулы (33)–(35), правомерные для систем порядка $n \leqslant 5$ и для показателей качества с $k \leqslant 2$ (уравнение 6). Обобщение их для произвольных $n$ и $k$ является в действительности непосредственным, благодаря общему виду матрицы (32). Особое внимание уделено выбору весовых коэффициентов, имеющих место в формулах (6) и (44), проводя в конечном счете к предлагаемому виду показателя качества, выражаемого посредством уравнения (49). Применение этого вида иллюстрируется четыръмя простыми примерами параметрической оптимизации динамических систем: электрической, сервомеханизма и промышленного регулятора а также систем, гарактеризуемыг передаточными фрнкциями „оптимального" типа.