Control and Cybernetics
VOL, 5 (1976) No. 3

Control of retarded systems with function space constraints
Part 1. Necessary optimality conditions

by

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The system of the form $\dot{x}(t)=f(x(t), x(t-h_1), ..., x(t-h_s), u(t), u(t-h_1), ..., u(t-h_s), t)$, $t \in [0, T]$, is considered and a class of optimal control problems with integral performance index and function space constraints is examined. It is assumed that the final time T and the lags h_i are commensurable; the condition which can always be fulfilled in practical applications. It is shown that by applying an equivalent non-delayed system with some additional two-point boundary condition the problems can be restated in a form of control problems without delay and therefore can be treated with the help of existing general and strong results of the type of maximum principle. Necessary optimality conditions are derived effectively for the following special problems: the final state $x_T(\cdot)$ is constrained to lie in a given ball in function space C of continuous functions or, in two other cases, the spaces L^2 and Sobolev's $W^2_{(1)}$ are used. Also the case of fixed final state $x_T(\cdot)$ is considered and various possible generalizations are indicated. The obtained results are of the form of maximum principle with absolutely continuous adjoint variable and pointwise maximum condition, and therefore they are, in the class considered, stronger then the others known.

1. Introduction

The purpose of this paper is to establish necessary conditions for optimality for problems with non-linear retarded systems of a type

$$\dot{x}(t)=f\big(x(t), x(t-h_1), ..., x(t-h_s), u(t-h_s), ..., u(t-h_s), t\big), x(t)\in R^n, u(t)\in R^r, \quad (1.1)$$

and a functional

$$J\big(x(\cdot), u(\cdot)\big)=\int_0^T f_0\big(x(t), u(t), t\big)\, dt \quad (1.2)$$

to be minimized. We assume that $u(t)\in U$ — a given nonempty subset, the initial function $x(t)=\varphi(t)$, $t\in[-h, 0]$ is given and terminal complete state $x_T(\cdot)$ satisfies

$$\|x_T-\xi\|\leqslant\varepsilon \quad (1.3)$$

where $x_T(\theta)=x(T+\Theta)$ for $\Theta \in [-h_s, 0]$, $\xi: [T-h_s, T] \mapsto R^n$ is a given function, ε is a given positive number and $\|\cdot\|$ is a suitable norm in a function space.

Optimal control problems for systems less or more general than (1.1) were considered by many authors and various boundary conditions were assumed. We refer reader to Banks and Manitius [1] for thorough survey of the results up to 1973. Recall only some fundamental and strictly related to a given problem references. The extension of Pontriagin's maximum principle to systems of type (1.1) with fixed or constrained in R^n trajectory end point $x(T)$ is due to Kharatishvili [2], [3]. The only modification, comparing with usual maximum principle is that the adjoint equation is a linear differential-difference equation of advanced type (here $s=1$)

$$\dot{\psi}(t) = -\frac{\partial H}{\partial x}\bigg|_t - \frac{\partial H}{\partial y}\bigg|_{t+h_1} \tag{1.4}$$

where as usual

$$H(x, y, u, \psi, t) = \psi_0 f_0(x, u, t) + \psi^* f(x, y, u, t)$$

and $y(t)$ stand for $x(t-h_1)$. The results of Kharatishvili were generalized to variable lags and restricted phase coordinates in Banks [4] and Huang [5]. The case of delays in control only was examined by Wierzbicki [18].

The problem considered in this paper is characterized by function space (not R^n) terminal constraints (1.3). The motivation for such problem statement is that the true final state is represented by $x_T(\cdot)$. Therefore if the desired behaviour of a system is required for $t>T$ (e.g. small deviations from equilibrium state) the right final state $x_T(\cdot)$ should be reached.

Banks and Kent [6], [7] proved existence of an optimal control for a general class of functional differential systems of neutral type and the constraints $x_{t_1}(\cdot) \in \mathscr{T}_1$, $x_{t_2}(\cdot) \in \mathscr{T}_2$ on initial and terminal states, where $\mathscr{T}_1, \mathscr{T}_2$ are gixen subsets in function space. The necessary conditions for optimality for the case when x_{t_1}, x_{t_2} are fixed were also obtained. These conditions have a form of maximum principle in function space of admissible controls (integral form of maximum principle) and not specified to a finite-dimensional set U of control values as in Kharatishvili classical result [2]. The adjoint equation in [6] is in the form of integral Volterra equation and therefore the adjoint variables are left continuous only (not absolutely continuous as in [2]).

Since Banks and Kent handled the equality constraints $x(t)=\xi(t) \in R^n$, $t \in [t_2-h_1, t_2]$, t_2 — final time, as a set of $2n$ conflicting inequality constraints (this was caused by the use of Neustadt [8] abstract variational principle) they could not prove the nontriviality of the maximum principle.

The normal form ($\psi_0 \neq 0$) of a local maximum principle, and hence nontriviality, were obtained by Jacobs and Kao [9] who applied abstract Lagrange multiplier rule to systems of type (1.1) but under sever assumptions that the control is unconstrained and the system is completely function space controllable. For linear differential-difference systems with quadratic cost functional the nontriviality was established by Banks and Jacobs [10]. These results were generalized recently by Kurcyusz [11] who used Dubovitski—Milyutin formalism [12] for systems with variable lags, affine with respect to control u and with a set of admissible controls closed, convex of nonempty interior in function space. The result of [11] can be characterized as follows.

The major role play the attainable subspace of complete final states for a linearized system and the property of the closure of this subspace in a given topology of function space. If the attainable subspace is not a proper subspace dense in the entire space of final states then the nontrivial maximum principle holds. If the attainable subspace is closed then normal adjoint variables exist. The explicite conditions for closedness of the attainable subspace in Sobolev space $W^q_{(1)}$ were derived in [13].

In this paper we give nontrivial necessary conditions for optimality for the extremum problem defined by formulas (1.1), (1.2), (1.3). Due to different setting of terminal constraints than in earlier works we are able to reformulate the problem to a classical problem for ordinary differential system without delay, with mixed two-point boundary conditions of the type $X(x(0), x(T))=0$ and additional inequality constraints for phase coordinates if the norm in (1.3) is the supremum norm in the space of continuous functions. If the norm is taken in L^2 or in $W^2_{(1)}$ the inequality constraints can be formulated as final state constraints in Euclidean space.

It should be pointed out that in contradistinction to other results concerning problems with function space constraints our maximum principle has a form rather similar to Kharatishvili [2] result. Also adjoint variables satisfy differential (not integral) equation and are therefore absolutely continuous.

2. Maximum principle for ordinary differential systems with mixed boundary conditions and phase variable constraints

In this section we recall necessary conditions for optimality of a controlled dynamical system described by ordinary differential equation in R^n with constraints both on the ends of trajectory and meanwhile values of the state. There exist several formulations of the problem [14, 15, 16, 17]. For our purpose in the sequal the most suitable is the formulation of L. W. Neustadt [14, 17] and this can be described as follows.

Let the system behaviour be described by the equation $\dot{x}(t)=f(x(t), u(t), t)$, $t\in[t_1, t_2]$ — a given interval (2.1) $x(t)\in G\subset R^n$, $u(t)\in U\subset R^r$, G, U are given nonvoid subsets, G is open.

The function $u: [t_1, t_2]\mapsto U$ is assumed from $L^{\infty}(t_1, t_2; U)$. The function $f: G\times U\times[t_1, t_2]\mapsto R^n$ is continuous and of class $C^{(1)}$ with respect to the first argument.

Define functions

$g: G\times[t_1, t_2]\mapsto R^1$,

$X_0: G\times G\mapsto R^1$,

$X_i: G\times G\mapsto R^{m_i}$, $i=1, 2$.

Let the functions X_0, X_1, X_2 be of class $C^{(1)}$ and g of class $C^{(2)}$. The problem of optimal control is stated as follows.

$$\text{P(2.1)}\quad \begin{cases} \text{minimize } X_0\left(x(t_1), x(t_2)\right) \text{ under constraints:} \\ \text{(i) there exist } u\in L^\infty(t_1, t_2; U) \text{ such that (2.1) is satisfied,} \\ \text{(ii) } X_1\left(x(t_1), x(t_2)\right)=0, \\ \text{(iii) } X_{2i}\left(x(t_1), x(t_2)\right)\leqslant 0,\ i=1, \dots, m_2, \\ \text{(iv) } g_i\left(x(t), t\right)\leqslant 0\ \forall t\in[t_1, t_2]\ \forall i=1, \dots, l. \end{cases}$$

Necessary optimality conditions can be described by the following maximum principle provided all the assumptions above are satisfied.

THEOREM 2.1. Let u^o, x^o be optimal control and optimal trajectory respectively.

Then there exist a number $\alpha_0\leqslant 0$, vectors $\alpha_i\in R^{m_i}$, $i=1, 2$; $\alpha_{2i}\leqslant 0$, $i=1, \dots, m_2$ and functions $\lambda: [t_1, t_2]\mapsto R^l$, $\psi: [t_1, t_2]\mapsto R^n$ such that the following conditions hold.

(a) λ_i, $i=1, \dots, l$, is of bounded variation, continuous from the right, non-increasing on; $t_1, t_2]$ and constant on subintervals on which $g_i\left(x^o(t), t\right)<0$, $\lambda(t_2)=0$;

(b) $|\alpha_0|+|\alpha_1|+|\alpha_2|+|\lambda(t_1)|>0$, $\alpha_{2i}X_{2i}\left(x^o(t_1), x^o(t_2)\right)=0\ \forall i=1, \dots, m_2$;

(c) the function ψ is absolutely continuous and almost everywhere in $[t_1, t_2]$ satisfies[1])

$$\dot\psi^*(t)=-\frac{\partial}{\partial x}H\left(\psi(t), x^o(t), u^o(t), t\right),$$

where

$$H\left(\psi(t), x(t), u(t), t\right)\overset{\text{df}}{=}[\psi^*(t)-\lambda^*(t)\,g_x\left(x(t), t\right)]\,f\left(x(t), u(t), t\right)-\lambda^*(t)\,g_t\left(x(t), t\right);$$

(d) the ends of ψ are subject to transversality conditions

$$\psi^*(t_1)=-\alpha_0 X'_{0x}-\alpha_1^* X'_{1x}-\alpha_2^* X'_{2x}+\lambda^*(t_1)\,g_x\left(x^o(t_1), t_1\right),$$
$$\psi^*(t_2)=\alpha_0 X''_{0x}+\alpha_1^* X''_{1x}+\alpha_2^* X''_{2x},$$

where$'$ and$''$ are explained as follows

$$X'_x(x^1, x^2)=\frac{\partial X(x^1, x^2)}{\partial x^1};\qquad X''_x(x^1, x^2)=\frac{\partial X(x^1, x^2)}{\partial x^2};$$

(e) maximum condition holds for hamiltonian H a.e. on $[t_1, t_2]$

$$H\left(\psi(t), x^o(t), u^o(t), t\right)=\max_{u\in U}H\left(\psi(t), x^o(t), u, t\right).$$

In the condition (e) one may use the modified hamiltonian $\bar H=[\psi^*-\lambda^* g_x]\,f$ as well since the remained term $-\lambda^*(t)\,g_t\left(x(t), t\right)$ does not depend explicitly on u.

[1]) By a vector in R^n we mean a column vector and $*$ denotes transposition, f_x denotes a derivative of a vector valued function f with respect to a vector x. This is a matrix with $\partial f_i/\partial x_j$ as (i, j)-th element.

REMARK 2.1. The Theorem 2.1 can be generalized to include more complicated forms of constraints as for instance the set of admissible control values U dependent on time t and state $x(t)$, equality constraints satisfied for all $t \in [_{k1}, t_2]$, equality and inequality constraints imposed on intermediate state values $x(\tau_i)$, $\tau_i \in [t_1, t_2]$.

We shall not use all these generalizations in our considerations, so that we refer reader to the references [16, 17].

3. Reformulation of the control problem with delay

Let us rewrite the problem to be considered.

$$
\text{P(3.1)} \quad \begin{cases} \text{minimize } \int\limits_0^T f_0\big(x(t), u(t), t\big)\, dt \\ \text{under constraints:} \\ \dot{x}(t) = f\big(x(t), x(t-h_1), \dots, x(t-h_s), u(t), u(t-h_1), \dots, u(t-h_s), t\big) \text{ a.e. on } [0, T], \\ x(t) = \varphi(t) \text{ and } u(t) = \omega(t) \text{ for } t \in [-h_s, 0], \\ \|x_T(\cdot) - \xi(\cdot)\| \leqslant \varepsilon \\ u \in \mathcal{U} \overset{\text{df}}{=} L^\infty(0, T; U) \text{ and } R^r \supset U \text{ — a given nonempty subset.} \end{cases}
$$

Complete the remaining assumptions.

H 3.1. The numbers $\varepsilon > 0$, $T > h_s > \dots > h_1 > 0$ are given as well as the functions $\varphi\colon [-h_s, 0] \mapsto R^n$, $\omega\colon [-h_s, 0] \mapsto R^r$, $\xi\colon [T-h_s, T] \mapsto R^n$. It is assumed that φ and ω are continuous. ξ is always assumed to be continuous but if the supremum $(W^2_{(1)})$ norm is applied to final state constraints then we require additionaly the function ξ to be of class $C^{(2)}$ $(W^2_{(1)})$.

H 3.2. The functions f_0, f are continuous on the following domains

$$f_0\colon G \times U \times [0, T] \mapsto R,$$

$$f\colon G^{s+1} \times U^{s+1} \times [0, T] \mapsto R^n$$

where G is a nonempty open subset in R^n. Additionally assume $f_0(x, u, t)$ is of class $C^{(1)}$ with respect to x and $f(x, y_1, \dots, y_s, u, w_1, \dots, w_s, t)$ is of class $C^{(1)}$ with respect to $x, y_1, \dots, y_s$.

H 3.3. One of the three following norms will be taken to measure the distance between the final complete state $x_T(\Theta) \overset{\text{df}}{=} x(T+\Theta)$, $\Theta \in [-h_s, 0]$ and a given function ξ. Here $z\colon [0, h_s] \mapsto R^n$ is, in each case, of suitable class.

$$\|z\|_1 = \sup_{\Theta \in [0, h_s]} |z(\Theta)| \text{ and } |z|^2 = \sum_{i=1}^{n} (z_i)^2,$$

$$\|z\|_2 = \left(\int\limits_0^{h_s} z^*(\Theta)\, z(\Theta)\, d\Theta\right)^{1/2},$$

$$\|z\|_3 = \Big(z^*(h_s)\, z(h_s) + \int\limits_0^{h_s} \dot{z}^*(\Theta)\, \dot{z}(\Theta)\, d\Theta\Big)^{1/2}.$$

Consider first the case $s=1$ and T, h_1 commensurable that is $T=kh$, $h_1=k_1 h$ for some $h>0$ and some integers $k>k_1>0$. The case of many commensurable delays can be treated similarly.

We shall show that the problem P(3.1) can be reformulated and then the Theorem 2.1 does apply.

Introduce the following notation:

$$z_i(t)=x\big(t+(i-1)\,h\big);\; v_i(t)=u\big(t+(i-1)\,h\big). \tag{3.1}$$

Define kn-vector $z(t)$ and kr-vector $v(t)$.

$$z^*(t)=\big(z_1^*(t), \ldots, z_k^*(t)\big),\; v^*(t)=\big(v_1^*(t), \ldots, v_k^*(t)\big). \tag{3.2}$$

After dividing the interval $[0, kh]$ into k subintervals of the length h the system equation can be written in the form

$$\dot{z}_i(t)=f\big(z_i(t), z_{i-k_1}(t), v_i(t), v_{i-k_1}(t), t+(i-1)\,h\big),\; i=1, \ldots, k;\; t\in[0, h] \tag{3.3}$$

where suitable values of initial functions φ and ω are taken for z_j, v_j respectively if $j<1$.

Also the following continuity conditions are satisfied $z_i(h)=z_{i+1}(0)$ for $i=1, \ldots, k-1$,

$$z_1(0)=\varphi(0). \tag{3.4}$$

Conversely, if $z(\cdot)$, $v(\cdot)$ satisfy (3.3), (3.4) then, after reverse transformations (3.2), (3.1), we get that the functions $x(\cdot)$, $u(\cdot)$ satisfy the retarded system equation. Denote further for compactness

$$f\big(z_i(t), z_{i-k_1}(t), v_i(t), v_{i-k_1}(t), t\big)=F_i\big(z(t), v(t), t\big) \qquad i=1, 2, \ldots, k, \tag{3.5}$$

$$F\big(z, (t), v(t), t\big)=\begin{Bmatrix} F_1\big(z(t), v(t), t\big) \\ \vdots \\ F_k\big(z(t), v(t), t\big) \end{Bmatrix}, \tag{3.6}$$

$$\Phi_0=\begin{Bmatrix} \varphi(0) \\ 0 \\ \vdots \\ 0 \end{Bmatrix} \in R^{kn}, \tag{3.7}$$

$$J_k=\begin{Bmatrix} 0 & 0 & . & . & 0 & 0 \\ I & 0 & & & & . \\ 0 & I & & & & . \\ . & . & . & & & . \\ . & . & . & . & & . \\ 0 & 0 & . & . & I & 0 \end{Bmatrix}. \tag{3.8}$$

Now the equations (3.3), (3.4) can be rewritten as

$$\dot{z}(t)=F\big(z(t), v(t), t\big), \tag{3.9}$$

$$z(0)-\Phi_0-J_k\, z(h)=0. \tag{3.10}$$

Similarly after setting

$$F_0(z(t), v(t), t) = \sum_{i=1}^{k} f_0(z_i(t), v_i(t), t+(i-1)h) \tag{3.11}$$

the performance index J takes the form

$$J = \int_0^h F_0(z(t), v(t), t)\, dt. \tag{3.12}$$

With the help of the formulas above the problem P(3.1) may be restated in an equivalent form.

P(3.2) $\begin{cases} \text{minimize the functional (3.12)} \\ \text{under constraints (3.9), (3.10) and} \\ \|x_T(\cdot) - \xi(\cdot)\| \leqslant \varepsilon. \end{cases}$

This form will be used in the next sections for deriving necessary optimality conditions in cases when the system is to be steered to a final state into a ball in some function space.

4. Control to a ball in the space of continuous functions

Assume the supremum norm in the problem P(3.2). The inequality

$$\sup_{t \in [T-h_1, T]} |x(t) - \xi(t)| \leqslant \varepsilon$$

we rewrite as

$$|z_{k-i}(t) - \xi_{k-i}(t)|^2 \leqslant \varepsilon^2 \;\forall\, t \in [0, h]\; \forall i = 0, 1, \ldots, k_1 - 1,$$

where $\xi_{k-i}(t) = \xi(T - (i+1)h + t)$.

Thus we achieve the compatibility with the statement of the problem P(2.1) by setting

$$g_{k-i}(z(t), t) = |z_{k-i}(t) - \xi_{k-i}(t)|^2 - \varepsilon^2 \leqslant 0 \text{ for all } t \in [0, h] \quad \forall i = 0, 1, \ldots, k_1 - 1. \tag{4.1}$$

Applying Theorem 2.1 to problem

P(4.1) $\begin{cases} \text{minimize the functional (3.12)} \\ \text{under (3.9), (3.10), 4.1)} \end{cases}$

yields the following result.

LEMMA 4.1. Assume the functions ξ_i, $i = k - k_1 + 1, \ldots, k$ are of class $C^{(2)}$. Let $z^o(t), v^o(t)$ be the optimal solution to problem P(4.1). Then there exist a constant $\psi_0 \leqslant 0$ and functions $\lambda\colon [0, h] \to R^{k_1}$, $\psi\colon [0, h] \to R^{kn}$, not all equal zero identically, such that:

(i) For $i = k, k-1, \ldots, k-k_1+1$, λ_i, corresponding to ξ_i, is nonincreasing, continuous from the right, of bounded variation on $[0, h]$ and constant on subintervals on which $|z_i(t)| < \varepsilon$, $\lambda(h) = 0$;

(ii) ψ is absolutely continuous and satisfies a.e. on $[0, h]$ the equation

$$\dot{\psi}^*(t) = -\psi_0 F_{0z}(^o) - \psi^*(t) F_z(^o) + \sqrt{2} \sum_{i=k-k_1+1}^{k} 2\lambda_i(t) \{[0, \dots, 0, -\dot{\xi}_i(t) + F_i^*(^o), 0, \dots, 0] + (z_i^o(t) - \xi_i(t))^* [0, \dots, 0, F_{iz_{i-k_1}}(^o), 0, \dots, 0, F_{iz_i}(^o), 0, \dots, 0]\} \quad (4.2)$$

where for instance $F_{(i-k_1)z_i}(^o)$ denotes that optimal values $z^o(t), v^o(t)$ were substituted as arguments of the derivative of F_{i-k_1} with respect to z_i.

(iii)

$$\psi^*(h) = \psi^*(0) J_k - \sum_{i=k-k_1+1}^{k} 2\lambda_i(0) [0, \dots, 0, (z_i^o(0) - \xi_i(0))^*, 0, \dots, 0] J_k. \quad (4.3)$$

(iv)

$$\psi_0 F_0(^o) + \psi^*(t) F(^o) - \sum_{i=k-k_1+1}^{k} 2\lambda_i(t) (z_i^o(t) - \xi_i(t))^* F_i(^o) = \max_{v \in U^k} [\psi_0 F_0(z^o(t), v, t) + \psi^*(t) F(z^o(t), v, t) - \sum_{i=k-k_1+1}^{k} 2\lambda_i(t) (z_i^o(t) - \xi_i(t))^* F_i(z^o(t), v, t)] \quad (4.4)$$

almost everywhere on $[0, h]$.

Here $U^k = U \times U \times \dots \times U$ (k-times).

Proof. We clarify formulas (4.2), (4.3), (4.4) and prove nontriviality.

Compute from (4.1)

$$g_{iz}(z(t), t) = 2 [0, \dots, 0, (z_i(t) - \xi_i(t))^*, 0, \dots, 0], \quad i = k - k_1 + 1, \dots, k. \quad (4.5)$$

After standard substitution

$$z_0(t) = \int_0^t F_0(z(t), v(t))\, dt,$$

$$\tilde{z} = \begin{bmatrix} z_0 \\ z \end{bmatrix}, \quad \tilde{F} = \begin{bmatrix} F_0 \\ F \end{bmatrix},$$

we get a system

$$\dot{\tilde{z}}(t) = \tilde{F}(\tilde{z}(t), v(t), t)$$

and a functional to be minimized

$$X_0(\tilde{z}(0), \tilde{z}(h)) = \tilde{z}_0(h) - z_0(0))$$

under equality and inequality constraints (3.10), (4.1).

From Theorem 2.1 there exist nontrivial $\alpha_0 \leqslant 0$, $\alpha \in R^{kn}$ and a function λ as required in (i) such that for functions $\psi(t)$, $\psi_0(t)$ satisfying $\dot{\psi}_0(t) \equiv 0 = \frac{-\partial H}{\partial z_0}$ and $\dot{\psi}^*(t) = -\frac{\partial H}{\partial z}$, where

$$H(z, v, t) = \psi_0 F_0(z, v, t) + [\psi^*(t) - \lambda^*(t) g_z(z, t)] F(z, v, t) - \lambda^*(t) g_t(z, t).$$

the maximum condition (4.4) holds.

Compute the following derivatives

$$\frac{\partial}{\partial z}\lambda_i(t)\, g_{iz}\, F=2\lambda_i(t)\frac{\partial}{\partial z}(z_i-\xi_i)^* F_i=2\lambda_i(t)\,[0, \ldots, 0, (z_i-\xi_i)^* \times$$

$$\times F_{iz_{i-k_1}}, 0, \ldots, 0, (z_i-\xi_i)^* F_{iz_i}+F_i^*, 0, \ldots, 0]$$

$$\frac{\partial}{\partial z}\lambda_i(t)\, g_{it}=\lambda_i(t)\, g_{izt}=2\lambda_i(t)\,[0, \ldots, 0, -\dot{\xi}_i(t), 0, \ldots, 0].$$

Direct substitution to adjoint equation shows that (4.2) is satisfied. The transversality conditions give $\psi_0=\alpha_0$ and

$$\psi^*(0)=-\alpha^*+\sum_{i=k-k_1+1}^{k} 2\lambda_i(0)\,[0, \ldots, 0, (z_i^o(0)-\xi_i(0))^*, 0, \ldots, 0] \tag{4.6}$$

$$\psi^*(h)=-\alpha^* J_k,$$

from which condition (4.3) follows.

Finally we see from condition (b) of Theorem 2.1 that if $\alpha_0=0$, $\lambda(t)\equiv 0$ then $\alpha\neq 0$ and this implies that $\psi(t)$ cannot vanish identically. In fact, if $\psi(t)\equiv 0$ then from (4.6) — compare the form (3.8) of J_k — we get $\alpha=0$, a contradiction.

The optimality conditions of Lemma 4.1 may serve as a base for construction of computational algorithms solving problem P(3.1) numerically. However, the major disadvantage of this approach is the large dimension of the vector $z(t)$ which increases proportionally as the ratio T/h increases.

This disadvantage might be reduced completely if the adjoint equation (4.2) can be written in R^n. The aim of this paper is to show that it is possible to do that at least in the case of constraints which are sufficiently regular. For the problem P(3.1) the following theorems obtains.

THEOREM 4.1. Assume $T=kh$, $h_i=k_i h$ for some $h>0$ and some integers $k>k_s>\ldots>k_1>0$. Suppose the hypothesis H3.1. and H3.2 are satisfied. If $u^o(\cdot)$ and $x^o(\cdot)$ are the optimal control and the corresponding optimal trajectory the solutions to the problem P(3.1) with $\|\cdot\|_1$ in H3.3 then there exist nonzero triple $(p_0, p(\cdot), \mu(\cdot))$ where the real $p_0\leqslant 0$, and the functions

$$p\colon [0, T+h_s]\to R^n,$$

$$\mu\colon [0, T+h_s]\to R$$

satisfy the following conditions.

(i) $\mu(t)=\mu_{k-k_s+1}(t)+\ldots+\mu_k(t)$. For each $i=k-k_s+1, \ldots, k$ the function $\mu_i(t)$ is nonincreasing on $[ih-h, ih]$ and is equal to zero on $[0, ih-h)\cup[ih, T]$. μ is right continuous. On subintervals of $[ih-h, ih]$ on which $|x(t)-\xi(t)|<\varepsilon$ the function $\mu_i(t)$ is constant.

(ii) The function $p(\cdot)$ is absolutely continuous on each of the following intervals $[0, T-h_s]$, $[T-ih, T-ih+h]$, $i=1, \ldots, k_s$. At the points $T-ih$, $i=1, \ldots, k_s$ the following "jump conditions" are fulfilled

$$p(T-ih-0)=p(T-ih+0)+2\mu(T-ih)\,(x(T-ih)-\xi(T-ih)). \tag{4.7}$$

Almost everywhere on $[0, T]$ the function $p(\cdot)$ satisfy the difference-differential equation of advanced type

$$\dot{p}^*(t) = -\frac{\partial}{\partial x(t)} \tilde{H}(t) \tag{4.8}$$

with boundary condition

$$p(t) = 0 \text{ on } [T, T+h_s], \tag{4.9}$$

where

$$\tilde{H}(t) = H(t) + H(t+h_1) + \ldots + H(t+h_s), \tag{4.10}$$

$$H(t) = H\big(p(t), x^o(t), x^o(t-h_1), \ldots, x^o(t-h_s), u^o(t), \ldots, u^o(t-h_s), t\big) \tag{4.11}$$

and

$$H(p_1, x, y_1, \ldots, y_s, u, w_1, \ldots, w_s, t) = p_0 f_0(x, u, t) + \\ + \big(p - 2\mu(t)\,(x - \xi(t))\big)^* \big(\dot{\xi}(t) + f(x, y_1, \ldots, y_s, u, w_1, \ldots, w_s, t)\big). \tag{4.12}$$

(iii) For almost all $t \in [0, T]$

$$\tilde{H}\big(u^o(t), t\big) = \max_{u(t) \in U} \tilde{H}\big(u(t), t\big) \tag{4.13}$$

where we denote

$$\tilde{H}\big(u(t), t\big) = H\big(u(t), t\big) + H\big(u(t+h_1), t+h_1\big) + \ldots + H\big(u(t+h_s), t+h_s\big), \tag{4.14}$$

$$H\big(u(t), t\big) = H\big(p(t), x^o(t), \ldots, x^o(t-h_s), u(t), u(t-h_1), \ldots, u(t-h_s), t\big). \tag{4.15}$$

Proof. In this proof we restrict ourselves to the case $s=1$, $T=kh$, $h_1=k_1 h$. This will simplify many formulas and the case of many commensurable lags can be treated in the same way.

Since the problem under consideration is equivalent to the problem P(4.1) we may use Lemma 4.1 to develop conditions for $x^o(\cdot)$ and $u^o(\cdot)$. Firstly, it is easily seen that the existence and properties of the functions $\mu_i(\cdot)$, $i=k-k_1+1, \ldots, k$, follows immediately from Lemma 4.1 by setting $\mu^i((i-1)h+t) = \lambda_i(t)$. Further exploitation of the equivalence between the optimal pairs $z^o(\cdot), v^o(\cdot)$ in Lemma 4.1 and $x^o(\cdot)$, $u^o(\cdot)$ in this theorem yields the existence of the function $p\colon [0, T+h_1] \to \to R^n$ defined on the base of $\psi(\cdot)$ from Lemma 4.1.

$$p\big(t+(i-1)h\big) = \psi_i(t) \text{ for } t \in [0, h], \tag{4.16}$$

$\psi_i(t) \in R^n$ being the i-th subvector of $\psi(t) \in R^{kn}$.

In order to rewrite the equation (4.2) in n-dimensional form compute the derivatives F_{0z}, F_z, F_{iz} with the use of (3.1), (3.2), (3.5), (3.6). Letting $t \in [0, h]$ we get

$$F_{0z} = [F_{0z_1}, \ldots, F_{0z_k}],$$

$$F_{0z_i}(t) = f_{0x}\big(x(t+(i-1)h), u\big(t+(i-1)h\big), t+(i-1)h\big),$$

$$F_z(t)=\begin{bmatrix} f_x(t) & 0 & . & . & . & . & . & . & . & 0 \\ 0 & f_x(t+h) & . & . & . & . & . & . & . & 0 \\ . & . & . & . & . & . & . & . & . & . \\ 0 & . & f_x(t+(i-k_1)h) & & . & . & . & . & . & 0 \\ . & . & . & . & . & . & . & . & . & . \\ 0 & . & . & f_y(t+ih) & . & . & f_x(t+ih) & . & . & 0 \\ . & . & . & . & . & . & . & . & . & . \\ 0 & . & . & . & f_y(t+(k-1)h) & . & . & . & . & f_x(t+(k-1)h) \end{bmatrix} \quad (4.17)$$

where

$$f(t) \overset{\mathrm{df}}{=} f(x^o(t), y^o(t), u^o(t), u^o(t-k_1 h), t)$$

$$y(t) \overset{\mathrm{df}}{=} x(t-h_1),$$

$$F_{iz_i}(t)=f_x(t+(i-1)h), \quad (4.18)$$

$$F_{iz_{i-k_1}}(t)=f_y(t+(i-1)h). \quad (4.19)$$

This and (4.2), (4.16) enables one to conclude that $p(\cdot)$ is absolutely continuous on each subinterval $((i-1)h, ih)$, $i=1, \ldots, k$ and the following equations hold.

$$\begin{aligned}\dot{\psi}_i^*(t)= & -\psi_0 f_{0x}(t+(i-1)h)-\psi_i^*(t) f_x(t+(i-1)h)-\psi_{i+k_1}^*(t) f_y(t+ \\ & +(i-1+k_1)h)+2\lambda_i(t)\{[f(t+(i-1)h)-\dot{\xi}(t+(i-1)h)]^*+ \\ & +(x^o(t+(i-1)h)-\xi(t+(i-1)h))^* f_x(t+(i-1)h)\}+2\lambda_{i+k_1}(t)(x^o(t+ \\ & +(i-1+k_1)h)-\xi(t+(i-1+k_1)h))^* f_y(t+(i-1+k_1)h)\end{aligned} \quad (4.20)$$

and hence substituting $p(\cdot)$ and $\mu(\cdot)$ one gets for $t\in((i-1)h, ih)$, $i=1, \ldots, k$.

$$\begin{aligned}\dot{p}^*(t)= & -\psi_0 f_{0x}(t)-p^*(t) f_x(t)-p^*(t+h_1) f_y(t+h_1)+2\mu(t)[(f(t)- \\ & -\dot{\xi}(t))+(x^o(t)-\xi(t))^* f_x(t)]+2\mu(t+h_1)(x^o(t+h_1)- \\ & -\xi(t+h_1))^* f_y(t+h_1)\end{aligned} \quad (4.21)$$

Introducing hamiltonian H by (4.12), (4.1) and shifted hamiltonian $\tilde{H}$ by (4.10) and setting $p_0=\psi_0$ we get (4.8). Two-point boundary conditions (4.3) imply immediately (see the form (3.8) of J_k) that $p(T)=0$, $\psi_i(h)=\psi_{i+1}(0)$ that is $p(ih+0)=$ $=p(ih-0)$, $i=1, \ldots, k-k_1-1$ and also the jump condition (4.7). Thus $p(\cdot)$ is absolutely continuous on $[0, T-h_1]$. Equation (4.21) will remain valid for $t\in$ $\in[T-h_1, T]$ only if we set $p(t)=0$ for $t\in(T, T+h_1]$. Rewrite (4.4) in the form

$$\begin{aligned}\mathcal{H}(v_1^o(t), v_2^o(t), \ldots, v_k^o(t)) \overset{\mathrm{df}}{=} & \sum_{j=1}^{k}(p_0 f_0(^o, v_j^o(t), t+(j-1)h)+ \\ & +p^*(t+(j-1)h) f(^o, v_j^o(t), v_{j-k_1}^o(t), t+(j-1)h)+ \\ & +2\mu(t+(j-1)h)(x^o(t+(j-1)h)-\xi(t+(j-1)h))^* f(^o, v_j^o(t), v_{j-k_1}^o(t), \\ & t+(j-1)h),\end{aligned} \quad (4.22)$$

where ^{o} denotes that remaining arguments are optimal.

$$\mathcal{H}\left(v_1^o(t), v_2^o(t), \ldots, v_k^o(t)\right) = \max_{v_1, \ldots, v_k \in U} \mathcal{H}(v_1, \ldots, v_k).$$

Hence

$$\mathcal{H}\left(v_1^o(t), \ldots, v_k^o(t)\right) = \max_{v_i \in U} \mathcal{H}\left(v_1^o(t), \ldots, v_i, \ldots, v_k^o(t)\right) \tag{4.23}$$

and this implies that

$$p_0 f_0\left({}^o, u(t), t), t\right) + \left[p^*(t) - 2\mu(t)\left(x^o(t) - \xi(t)\right)^*\right] f\left({}^o, u(t), u^o(t-h_1), t\right) +$$
$$+ \left[p^*(t+h_1) - 2\mu(t+h_1)\left(x^o(t+h_1) - \xi(t+h_1)\right)^*\right] f\left({}^o, u^o(t+h_1), u(t), t+h_1\right)$$

takes its maximal value over $u(t) \in U$ when substituting $u(t) = u^o(t)$. This holds for almost all $t \in [(i-1)h, ih]$ and, since i — is arbitrary, for almost all $t \in [0, T]$.

Replacing in the above term the function f with the term $f + \dot{\xi}$ — this will not change the property of maximality over $u(t)$ — and applying subsequently to intervals $[(i-1)h, ih]$, $i = k, k-1, \ldots, 1$ one obtains the condition (4.13). Nontriviality of $\left(\psi_0, \lambda(\cdot), \psi(\cdot)\right)$ implies clearly nontriviality of $\left(p_0, \mu(\cdot), p(\cdot)\right)$. Thus the proof is complete.

From the point of view of computational applications the case of $T, h_1, \ldots, h_s$ commensurable is general enough to cover all real problems of the type considered. It seems however that by modifying slightly the results of [17] we can treat the problems with arbitrary positive lags. This modification requires the existence of the functions $\lambda_i(\cdot)$ corresponding to the constraints (4.1) such that $\lambda_i(h) = \lambda_{i+1}(0)$. By this property one may construct a function $\mu(\cdot)$ in Theorem 4.2 which is non-increasing on the whole interval $[T - h_s, T]$. Having $\mu(\cdot)$ with such properties approximating sequence of problems with commensurable lags may be (probably) derived.

5. Control to a ball in $W^2_{(1)}$ and L^2 spaces

We consider now the problem P(3.1) and its equivalent P(3.2) where the norm for final state is either $W^2_{(1)}$ or L^2 norm as specified in Hypothesis H3.3.

The result for these two cases are so similar that we can state them in one theorem without considerable complication of statement. We formulate the results for the problem with many delays in both state and control variable but the proof will be carried out, for the sake of simplicity, with one delay only.

THEOREM 5.1. Let the pair $\left(u^o(\cdot), x^o(\cdot)\right)$ be optimal solution to the following problem

$$\begin{cases} \text{minimize } \int_0^T f_0\left(x(t), u(t), t\right) dt \\ \text{under the constraints:} \\ (1.1) \\ \|x_T(\cdot) - \xi(\cdot)\| \leqslant \varepsilon \\ u(\cdot) \in L^\infty(0, T; U). \end{cases}$$

Here $x_T(\Theta)=x(T+\Theta)$, $\xi(T+\Theta)$ are defined for $\Theta\in[-h_s, 0]$. Assume, as in Theorem 4.1, the lags h_i and T commensurable. ξ is a given function of class L^2 (or $W^2_{(1)}$) and the norm in final inequality constraints is the L^2 (or $W^2_{(1)}$) norm. Let the remaining assumptions concerning the functions f_0, f and admissible controls be as in Theorem 4.2.

Then there exist a nonzero triple $(p_0, p(\cdot), p_a)$, where the real numbers $p_0\leqslant 0$, $p_a\leqslant 0$ and the functions

$$p\colon [0, T+h_s]\to R^n \tag{5.1}$$

$$p_a(t)=\begin{cases} p_a & \text{for } t\in[T-h_s, T]\\ 0 & \text{for } t\in[0, T-h_s)\cup(T, T+h_s]\end{cases}$$

satisfy the following conditions.

(i)

$$p_a\big(\|x_T(\cdot)-\xi(\cdot)\|-\varepsilon\big)=0. \tag{5.2}$$

(ii) The function $p(\cdot)$ is absolutely continuous on $[0, T]$ with final condition

$$p(T)=\begin{cases} 2p_a\big(x(T)-\xi(T)\big) & \text{for } W^2_{(1)}\text{— norm},\\ 0 & \text{for } L_2\text{ — norm}.\end{cases} \tag{5.3}$$

Almost everywhere on $[0, T]$ the following equation holds

$$\dot{p}(t)=-\frac{\partial}{\partial x(t)}\tilde{H}(t), \tag{5.4}$$

where $\tilde{H}$ is defined by (4.10), (4.11) but the hamiltonian H is now

$$H(p, x, y_1, \ldots, y_s, u, w_1, \ldots, w_s, t)=p_0 f_0(x, u, t)+$$

$$+p^* f(x, y_1, \ldots, y_s, u, w_1, \ldots, w_s, t)+p_a(t) f_a(x, y_1, \ldots, y_s, u, w_1, \ldots, w_s, t) \tag{5.5}$$

$$f_a(\cdot)=\begin{cases} |x-\xi(t)|^2 & \text{for } L^2\text{ — norm}\\ |f(x, y_1, \ldots, y_s, u, w_1, \ldots, w_s, t)-\dot{\xi}(t)|^2 & \text{for } W_1^{(2)}\text{-norm}.\end{cases} \tag{5.6}$$

The boundary condition for $p(\cdot)$ is

$$p(t)=0 \text{ on } (T, T+h_s]. \tag{5.7}$$

(iii) The maximum condition holds

$$\tilde{H}\big(u^o(t), t\big)=\max_{u(t)\in U}\tilde{H}\big(u(t), t\big) \tag{5.8}$$

where $\tilde{H}\big(u(t), t\big)$ is defined by (4.14), (4.15), (5.5).

Proof. For the sake of simplicity the proof will be given in case $s=1$ (one delay).

Since we assume $T=kh$, $h_1=k_1 h$ it is possible to go on with a new system (3.9), (3.10) with a performance index (3.12). The functions F_i, $i=1, \ldots, k$ are defined now as

$$F_i(z, v, t)=f(z_i, z_{i-k_1}, v_i, v_{i-k_1}, t). \tag{5.9}$$

Therefore one can formulate an equivalent problem P(3.2) where the final constraints are given by

$$\sum_{i=0}^{k_1-1} \|z_{k-i}(\cdot)-\xi_{k-i}(\cdot)\|^2 \leqslant \varepsilon^2 \tag{5.10}$$

or

$$|z_k(h)-\xi_k(h)|^2+\sum_{i=0}^{k_1-1} \|\dot{z}_{k-i}(\cdot)-\dot{\xi}_{k-i}(\cdot)\|^2 \leqslant \varepsilon^2. \tag{5.11}$$

in case of $W^2_{(1)}$ where $\|\cdot\|$ denotes L^2 norm.

Here $\xi_{k-i}(t)=\xi(T-ih+t)$.

Introduce new state variables z_0 and z_a satisfying the following equations

$$\dot{z}_0(t)=F_0\big(z(t), v(t), t\big), \tag{5.12}$$

$$\dot{z}_a(t)=F_a\big(z(t), v(t), t\big), \tag{5.13}$$

where

$$F_a(z, v, t)=\begin{cases} \sum_{i=0}^{k_1-1} |z_{k-i}-\xi_{k-i}(t)|^2 & \text{for } L^2\text{-space} \\ \sum_{i=0}^{k_1-1} |F_{k-i}(z, v, t)-\dot{\xi}_{k-i}(t)|^2 & \text{for } W^2_{(1)}\text{-space.} \end{cases} \tag{5.14}$$

These definitions enables one to rewrite the final constraints as

$$z_a(h)-z_a(0)+\sigma\|z_k(h)-\xi_k(h)\|^2-\varepsilon^2 \leqslant 0, \tag{5.15}$$

$$\sigma=0 \text{ for } L^2 \text{ and } \sigma=1 \text{ for } W^2_{(1)} \text{ case,}$$

and to restate equivalently the problem under consideration as

$$\begin{cases} \text{minimize } z_0(h)-z_0(0)=X_0\big(\tilde{z}(0), \tilde{z}(h)\big) \\ \text{under differential constraints (5.12), (5.13), (3.9)} \\ \text{and equality-nonequality constraints} \\ X\big(\tilde{z}(0), \tilde{z}(h)\big)=0, \\ X_a\big(\tilde{z}(0), \tilde{z}(h)\big)\leqslant 0. \end{cases}$$

Here we denote
$$\tilde{z}=\begin{bmatrix} z_0 \\ z \\ z_a \end{bmatrix}$$

and X, X_a are the left-hand sides of (3.10) and (5.15) respectively.

Applying now Theorem 2.1 one gets the existence of nontrivial

$$\tilde{\alpha}=(\alpha_0, \alpha, \alpha_a)\in R^{kn+2}, \quad \alpha_0\leqslant 0, \quad \alpha_a\leqslant 0$$

such that, as in Lemma 4.1, for $\tilde{\psi}\colon [0, h]\mapsto R^{kn+2}$ satisfying

$$\dot{\psi}_0=\dot{\psi}_a=0$$

$$\dot{\psi}^*(t)=-\psi_0 F_{0z}(^\circ)-\psi_a F_{az}(^\circ)-\psi^*(t) F_z(^\circ) \text{ a.e. on } [0, h] \tag{5.16}$$

and the boundary conditions

$$\alpha_a\left(z_a(h)-z_a(0)+\sigma\,|z_k(h)-\xi_k(h)|^2-\varepsilon^2\right)=0 \tag{5.17}$$

$$\tilde{\psi}(h)=\begin{bmatrix}\psi_0\\ \psi(0)\\ \psi_a\end{bmatrix}=\begin{bmatrix}\alpha_0\\ -\alpha\\ \alpha_a\end{bmatrix}$$

$$\tilde{\psi}(h)=\begin{bmatrix}\alpha_0\\ \\ -J_k^*\alpha\\ \\ 0\end{bmatrix}+\alpha_a\begin{bmatrix}0\\ \begin{bmatrix}0\\ \vdots\\ 0\\ 2\sigma\left(z_k(h)-\xi_k(h)\right)\end{bmatrix}\\ 1\end{bmatrix}$$

the maximum condition holds.

$$\psi_0\,F_0(^o)+\psi^*(t)\,F(^o)+\psi_a\,F_a(^o)=\max_{v\in U^k}\,[\psi_0\,F_0\left(z^o(t),v,t\right)+$$
$$+\psi^*(t)\,F\left(z^o(t),v,t\right)+\psi_a\,F_a\left(z^o(t),v,t\right)]. \tag{5.18}$$

Of course nontriviality of $\tilde{\alpha}$ implies nontriviality of $\tilde{\psi}(-\psi_0,\psi,\psi_a)$.

Furthermore $\psi_0=\alpha_0\leqslant 0$, $\psi_a=\alpha_a\leqslant 0$ and

$$\psi(h)=J_k^*\,\psi(0)+2\psi_a\,\sigma\begin{bmatrix}0\\ \vdots\\ 0\\ z_k(h)-\xi_k(h)\end{bmatrix}. \tag{5.19}$$

The latter and (5.16) by substitution of (4.16) yields the existence of nontrivial triple $p_0\overset{\text{df}}{=}\psi_0$ $p_a\overset{\text{df}}{=}\psi_a$ and $p(\cdot)$: $[0, T+h_1]\to R^n$ where we put by definition $p(t)=0$ for $t\in(T, T+h_1]$. In the same manner as in Theorem 4.1 we show that $p(\cdot)$ is absolutely continuous on $[0, T]$ and has to satisfy (5.4) with $p_a(t)$ defined in (5.1).

Condition (5.17) implies immediately (5.2) and from (5.19) one obtains (5.3).

The proof of maximum condition (5.8) is based on (5.18) and does not differ from analogous part of the proof of Theorem 4.1.

6. Generalizations

In order to achieve the clarity of the proofs and simplicity of the results we restricted our considerations in the previous sections to the basic problem P(3.1). However, it is easily seen that the Theorem 2.1 or the more general Theorems 12.1 and 13.1 of [17] enables one to generalize the results of this paper in several manners. We mean the possibility of the integrant in performance index to be depended on delayed state and control variables, the case of additional terminal constraints for the end of trajectory $x(T)$ or intermediate constraints for $x(t_i)$, where t_i, $i=1, \ldots, l$ are given instants in $[0, T]$, the case of equality and (or) inequality constraints for initial function, the more general target sets for the final complete state $x_T(\cdot)$, equality constraints for phase variable on the whole interval $[0, T]$ (or on a subset of it).

We shall not present the generalized results in closed forms of theorems since these can be easily deduced from Theorem 4.1 or 5.1 if we discuss earlier changes in optimality conditions caused by any of additional contraints mentioned above.

6.1. Equality and inequality pointwise (R^n) constraints

Assume $0=t_0<t_1<\dots<t_l=T$ arbitrarily. Consider the following additional constraints for $x(t_i)$ in P(3.1).

$$X^j(x(t_0), \dots, x(t_l))=0, \; j=1, \dots, d_1,$$
$$X^j(x(t_0), \dots, x(t_l))\leqslant 0, \; j=d_1+1, \dots, d.$$

This type of constraints applied to systems without delays (as in Theorem 2.1) gives, in case of optimality, the existence of a vector $\beta=(\beta^0, \beta^1, \dots, \beta^d)$ such that [17] (β, λ) is a nontrivial pair and transversality conditions consist of boundary conditions

$$\psi(t_0)=-\sum_{j=0}^{d}\beta^j\frac{\partial}{\partial x(t_0)}X^j+\lambda^*(t_0)\,g(x(t_0), t_0), \tag{6.1}$$

$$\psi(t_l)=\sum_{j=0}^{d}\beta^j\frac{\partial}{\partial x(t_l)}X^j, \tag{6.2}$$

and jumps

$$\psi(t_i+0)-\psi(t_i-0)=-\sum_{j=0}^{d}\beta^j\frac{\partial}{\partial x(t_i)}X^j. \tag{6.3}$$

Here $\beta^o, \beta^{d_1+1}, \dots, \beta^d\leqslant 0$.

The situation is similar for retarded systems. The jumps for adjoint variable $p(t)$ at $t=t_i$ are obtained from identical formula as well as the boundary conditions for $p(0)$ and $p(T)$. This is easily seen from a direct generalization of e.g. Lemma 4.1. Note that in the current formulation of the problem we assume that the initial function $\varphi(\cdot)$ is given with possible exception of $\varphi(0)=x(0)$ subject to constraints $X^j(x(0), \cdot)=0$ $(\leqslant 0)$. Such formulation may have minor physical justification since we are usually able to choose initial conditions of a system before it starts to work for instance in the course of designing procedure and then we have some freedom, if we have any, in choosing all the initial complete state. Mathematically, however, the problem is correctly stated. It is visible from (6.1), (6.2) that if the functions X^j does not depend essentially on both $x(0)$ and $x(T)$ that is any X^j depends either on $x(0)$ or on $x(T)$ then we get (if the phase variable constraints $g(x(t), t)\leqslant 0$ are absent) the well known orthogonality conditions for $p(0)$ and $p(T)$.

6.2. Constraints for initial function

If we have to find also an optimal initial function $\varphi(t)$ with values in a given nonempty subset $\mathcal{F} \subset R^n$ and $\varphi(0)=x(0) \in \mathcal{F}_0 \subset R^n$ or more generally $\varphi(t) \in \mathcal{F}(t) \subset R^n$ then the following approach is feasible. If the function φ can be considered as a member of $L^\infty(-h_s, 0; R^n)$ treat $\varphi(t)$, $t \in [-h_s, 0)$ and $\varphi(0)=x(0)$ separately. Set an additional control $v_0(t)=\varphi(t-h)$ in Lemma 4.1 (or k_s additional controls if $h_s=k_s h$). The conditions for optimal $x(0)$ are as described in subsection 6.1 above (conditions (6.1), (6.2)). The possibility of choosing the initial function results therefore only in maximum condition. On the interval $[0, h_i]$ the maximum condition holds also for optimal $\varphi^o(t-h_i)$ and admissible $\varphi(t-h_i) \in \mathcal{F}(t-h_i)$. The case of time dependent sets of admissible control values is examined, for non-delayed systems, in [16], [17]. Notice that the case of constraints for φ given analytically in the form of equality and (or) inequality constraints $Q_i(\varphi(t), t)=0$ (≤ 0) for a.a. $t \in [-h_s, 0]$ can be treated in the same manner. The functional constraints of the type $q_i: \varphi(\cdot) \to R$ may be also taken as functional constraints for control variable.

6.3. General final function space constraints

On the basis of Theorem 2.1 and of more general Theorems 12.1, 13.1 in [17] if needed we have no difficulties, except those of pure technical nature, in establishing the necessary optimality conditions for the following generalizations of the final constraints in P(3.1).

(a) General inequality constraints for $x_T(\cdot)$

$$Q_i(x(t), t) \leq 0 \text{ for all } t \in [T-h_s, T] \text{ and all } i=1, \ldots, \gamma_1. \tag{6.4}$$

In terms of the equivalent system (3.9), (3.10), (3.12) the problem can be characterized as one with phase variable constraints for subvectors $z_k(t)$, $z_{k-1}(t)$, ..., $z_{k-k_s+1}(t)$ of the form

$$Q_{ij}(z_j(t), t) \stackrel{\mathrm{df}}{=} Q_i\big(x(t+(j-1)h), (j-1)h+t\big), \ \forall t \in [0, h]$$
$$i=1, \ldots, \gamma_1; \ j=k, \ldots, k-k_s+1.$$

To each of the functions Q_{ij} there corresponds a multiplier $v_{ij}(\cdot)$ which is a scalar-valued function of bounded variation on $[0, h]$, nonincreasing, right continuous, constant on subintervals of $[0, h]$ on which $Q_{ij}(z_j^o(t), t)<0$ and with $v_{ij}(h)=0$. Thus the full analogy is observed between v_{ij} and λ_i in Lemma 4.1. Therefore, on the basis of Theorem 4.1, one can easily formulate an adequate result — the necessary conditions for optimality of the system (1.1) with constraints (6.4) included. It is also clear that the assumption that $Q_{ij}(\cdot)$ (not necessarily the functions $Q_i(\cdot)$) are of class $C^{(2)}$ quarantees the result in the form of pointwise maximum principle with the adjoint variable absolutely continuous on intervals $[0, T-h_s]$, $[T-ih, T-ih+h]$, $i=1, \ldots, k_s$.

(b) Functional type constraints

$$q_i\big(x_T(\cdot)\big)=0 \text{ for } i=1, \ldots, \gamma_2 \text{ (and } \leqslant 0 \text{ for } i=\gamma_2+1, \ldots, \gamma_3). \tag{6.5}$$

We assume here the following representation conditions for q_i. Namely $q_i\big(x_T(\cdot)\big)$ is supposed to be a value of $q_i\big(x_T(\cdot), t\big)$ for $t=T$ that is $q_i\big(x_T(\cdot), T\big)= =q_i\big(x_T(\cdot)\big)$ where $q_i\big(x_T(\cdot), t\big)$ may depend on $x(T)=x_T(0)$ but does not depend on $x(s), s\in(t, T)$. Moreover, it is assumed that the derivative $\frac{d}{dt}\, q_i\big(x_T(\cdot), t\big)$ exists for each absolutely continuous $x_T(\cdot)$ and is of the form

$$\frac{d}{dt}\, q_i\big(x_T(\cdot), t\big)=f_{ai}\big(x(t), \dot{x}(t), t\big),\ t\in[T-h_s, T], \tag{6.6}$$

$f_{ai}(x, \dot{x}, t)$ being continuous in all arguments and of class $C^{(1)}$ with respect to $\dot{x}$. These assumptions are motivated by our aim to reformulate the constraints (6.5) so that to get final constraints in Euclidean space for additional state variables. In fact, setting

$$q_{ij}\big(x_T(\cdot), t\big)\overset{\text{df}}{=\!=}q_i\big(x_T(\cdot), t+(j-1)\,h\big),$$

$$f_{aij}\big(z(t), t\big)=f_{ai}\big(x\big(t+(j-1)\,h\big), \dot{x}\big(t+(j-1)\,h\big), t+(j-1)\,h\big),$$

$$j=k, k-1, \ldots, k-k_s+1,$$

with $\dot{x}$ substituted from (1.1) and with $z(t)$ defined by (3.1) and (3.2),

$$F_{ai}\big(z(t), t\big)\overset{\text{df}}{=\!=}\sum_{j=k-k_1+1}^{k} f_{aij}\big(z(t), t\big),$$

$$\dot{z}_{ai}(t)\overset{\text{df}}{=\!=}F_{ai}\big(z(t), t\big),\ t\in[0, h], \tag{6.7}$$

it is clear that (6.5) may be written as

$$q_i\big(x_T(\cdot)\big)=z_{ai}(h)-z_{ai}(0)+q_i\big(x_T(\cdot), T-h_s\big)=0\ (\leqslant 0) \tag{6.8}$$

where, by assumption above $q_i\big(x_T(\cdot), T-h_s\big)$ may depend only on the values $x(T)$ and $x(T-h_s)$, of trajectory $x(\cdot)$. Thus we get the situation analogical to that in the proof of Theorem 5.1. For each $i=1, \ldots, \gamma_3$ there exists a multiplier p_{ai}, $p_{ai}\leqslant 0$ for $i=\gamma_2+1, \ldots, \gamma_3$. The hamiltonian is modified, similarly as in (5.5), by additional term $\sum_{i=1}^{\gamma_3} p_{ai} f_{ai}$. The final value of adjoint variable is given by

$$p(T)=\frac{\partial}{\partial x(T)}\sum_{i=1}^{\gamma_3} p_{ai}\, q_i\big(x_T(\cdot), T-h_s\big)$$

and the additional jump of $p(t)$ at $t=T-h_s$ is

$$p(T-h_s-0)-p(T-h_s+0)=\frac{\partial}{\partial x(T-h_s)}\sum_{i=1}^{\gamma_3} p_{ai}\, q_i\big(x_T(\cdot), T-h_s\big).$$

The remaining conditions for optimality will preserve theirs form.

(c) Fixed final state $x_T(\cdot)$. This case seems to be more difficult than the reported before in the following sense. It appears from one hand that under reasonable hypotheses the maximum principle can be obtained but its nontriviality cannot be proved. This was the case of Banks and Kent [6] where the equality constraints

$$x_i(t)=\xi_i(t)\ \forall t \in [T-h_s, T],\ i=1, \ldots, n \tag{6.9}$$

were considered as a set of $2n$ conflicting inequality constraints. From the other hand utilizing existing results in ptimal control theory for nondelayed systems we are able to derive nontrivial maximum principle but the hypotheses we have to assume are much stronger than for instance (H1), (H2) in Section 3 and restrict considerably the class of systems which can be treated. The assumptions concerning the case of equality constraints for state variable are, among others, the following [17]. The form of the constraints is

$$P\big(x(t), u(t), t\big)\equiv 0 \in R^1 \text{ for a.a. } t \in I_1 \subset [t_1, t_2], \tag{6.10}$$

I_1 — a measurable subset.

The function $p(\cdot)$ and the partial derivatives P_x, P_u are continuous in (x, y) uniformly with respect to t, $t \to P(x, u, t)$ is measurable and bounded. All these are not very restrictive but the following is, especially in case that the number l of equality constraints is large comparing with the number of controls $r\big(u(t) \in R^r\big)$.

Condition (C1) in [17].

$[P_u\big(x^o(t), u^o(t), t\big)\, P_u^*\big(x^o(t), u^o(t), t\big)]^{-1}$ exists for a.a. $t \in I_1$ and is of class L^∞ on I_1.

The equations (6.9) can be stated in the form (6.10) after simple manipulations. (6.9) is equivalent to

$$x_i(T)=\xi_i(T),\ i=1, \ldots, n,$$

$$\dot{\xi}_i(t)=\dot{x}_i(t)=f_i\big(x(t), x(t-h), u(t), t\big),\ t \in [T-h, T] \tag{6.11}$$

if taking for simplicity the case of one delay in state variable only.

The condition (C1) now obtains $[f_u f_u^*]^{-1}$ exists and is of class L^∞ in t.

This implies the number of controls $r \geqslant n\big(x(t) \in R^n\big)$. Therefore the results obtained via such approach will be of practical value only if the number of scalar equations (6.9) on $[T-h_s, T]$ is small. Actually we may consider a case which is not only a generalization of (c) but it also admits small number of constraints.

(d) Equality operator constraints

$$P\big(x_T(\cdot)\big)=0 \in C(T-h_s, T; R^l).$$

Here $P\big(x_T(\cdot)\big)(t) \overset{\text{df}}{=} P\big(x(t), t\big)$.

Of course, under suitable assumptions we get equivalent equations, similar to (6.11), which depend explicitly on $u(t)$ so that aproppriate theorems of [17] may be used,

$$P\big(x(T), T\big)=0, \tag{6.12}$$

$$P_x\big(x(t), t\big) f\big(x(t), x(t-h), u(t), t\big)+P_t\big(x(t), t\big)=0 \text{ for a.a. } t \in [T-h, T]. \tag{6.13}$$

The finite-dimensional equation (6.12) implies additional transversality conditions for $p(T)$. The constraints (6.13) imply the following modification of necessary conditions provided that the left hand side of (6.13) satisfies the conditions required in [17], Theorem 12.1. The multiplier $\nu \in L^{\infty}(T-h, T; R^{l})$ corresponding to (6.13) occurs in the term added to adjoint equation $-\nu(t)\frac{\partial}{\partial x}(P_x f+P_t)$ for $t \in [T-h, T]$. Also on this interval additional identity equation for $\nu(t) P_x f_u$ related to condition (v) in Theorem 12.1 of [17] is satisfied. In view of that theorem it can be shown in a straighforward way how to modify the maximum condition on $[T-h, T]$.

6.4. The generalization of the performance index and the system equation. Neutral systems

For the sake of simplicity the performance functional of integral form (1.2) with integrand depending only on nondelayed values state and control variable has been considered. It is however immediately seen from (3.11) that assuming

$$J=\int_{0}^{T} f_0\big(x(t), x(t-h_1), \dots, x(t-h_s), u(t), u(t-h_1), \dots, u(t-h_s), t\big)\, dt \tag{6.14}$$

the reformulated functional (3.12) will not change its form and the arguments of proof of Lemma 4.1 will apply.

In order to maintain the closed form of adjoint equation (4.8) the constant p_0 has to be replaced with the function $p_0(t)=p_0$ for $t \in [0, T]$ and zero for $t \in [T, T+h_s]$. As for nondelayed systems a term depending on final trajectory value $x(T)$ may be added to (6.14) and treated by standard methods. This term may depend as well on the complete final state $x_T(\cdot)$ and the procedure for obtaining necessary optimality conditions is the same as in case of functional type constraints for $x_T(\cdot)$ (see 6.3 (b)) that is showing that under some hypotheses the performance functional may take the form $X_0\big(x_0(0), x_0(T), x(0), x(T)\big)$ where $x_0(\cdot)$ is the suitable chosen new state variable.

It occurs also that the equivalent system of type (3.9), (3.10) can be constructed for neutral differential-difference equations with one delay.

$$\dot{x}(t)=f\big(x(t), x(t-h), \dot{x}(t-h), u(t), u(t-h), t\big),\ t \in [0, T], \tag{6.15}$$

or with many delays. It is seen that if considering (6.15) on $[(i-1)h, ih]$ one substitute the right hand side of (6.15) taken on $[(i-2)h, (i-1)h]$ and so on, unless reaching $[-h, 0]$. Thus a system of type (3.9), (3.10) can be derived. Therefore the necessary optimality conditions for neutral systems with any type of constraints discussed above can be obtained by the general method of this paper. The general limitation of this method is that it does apply to systems with finite number of delays. It does not work for systems with distributed delays, even sufficiently smooth as for instance

$$\dot{x}(t)=Ax(t)+\int_{-h}^{0} K(\Theta)\, x(t+\Theta)\, d\Theta+Bu(t).$$

Some results concerning the neutral systems depending linearly on delayed derivative are given in [6], [7] (see Section 1 of this paper for short report).

7. Examples

Let us illustrate the methods used extensively in this paper by the example following below.

Example 7.1. Consider one-dimensional retarded system

$$\dot{x}(t)=x(t-1)+u(t),\ t\in[0,2], \tag{7.1}$$

with initial condition

$$x(t)=\varphi(t)=1\quad \forall t\in[-1,0], \tag{7.2}$$

and the following quadratic functional

$$J=\frac{1}{2}\int_0^2 u^2(t)\,dt \tag{7.3}$$

to be minimized under final state constraints

$$\|x_T(\cdot)\|^2=(x(2))^2+\int_1^2(\dot{x}(t))^2\,dt\leqslant\varepsilon^2. \tag{7.4}$$

We shall assume for computations $\varepsilon=0.25$. The control values are unlimited ($U=R^1$). Let us apply Theorem 5.1. The hamiltonian (5.5) is

$$H=\frac{1}{2}p_0\,u^2+p(t)\,(x(t-1)+u(t))+p_a(t)\,(x(t-1)+u(t))^2.$$

Since H does not depend on delayed control we get from (5.8) that the optimal control $u(\cdot)$ satisfy

$$\frac{\partial\tilde{H}}{\partial u(t)}=\frac{\partial H}{\partial u(t)}=0,$$

which implies (we put $p_0=-1$ since $p_0=0$ does not fulfill (5.8))

$$u(t)=\frac{1}{1-2p_a(t)}\,[p(t)+2p_a(t)\,x(t-1)]. \tag{7.5}$$

Construct the adjoint equation (5.4)

$$\dot{p}(t)=-\frac{\partial}{\partial x(t)}\tilde{H}(t)=-\frac{\partial}{\partial x(t)}H(t)-\frac{\partial}{\partial x(t)}H(t+1)=$$
$$=-p(t+1)-2p_a(t+1)\,(x(t)+u(t+1)), \tag{7.6}$$

with boundary conditions (5.3), (5.7) for $p(\cdot)$ that is $p(t)=0$ for $t\in(2,3]$, $p(2)=$ $=2p_a\,x(2)$.

This and the definition (5.1) of $p_a(t)$ imply that $\dot{p}(t)=0$, $t\in[1,2]$ and hence

$$p(t)=2p_a x(2) \text{ for } t\in[1,2]. \tag{7.7}$$

Substituting (5.1) into (7.5) we get

$$u(t)=\frac{2p_a}{1-2p_a}\big(x(2)+x(t-1)\big),\ t\in[1,2]$$

$$u(t)=p(t),\ t\in[0,1]. \tag{7.8}$$

Further substitution of $u(t)$ into (7.1) and (7.6) yields for $t\in[0,1]$

$$\dot{x}(t)=1+p(t)$$

$$\dot{p}(t)=\frac{-2p_a}{1-2p_a}\big(x(2)+x(t)\big).$$

Hence the following second order differential equation for $x(\cdot)$ is obtained

$$\ddot{x}(t)=\dot{p}(t)=a^2\big(x(2)+x(t)\big),\ t\in[0,1].$$

where

$$a^2=\frac{-2p_a}{1-2p_a}. \tag{7.9}$$

Solving this with initial conditions $x(0)=1$, $\dot{x}(0)=c$, c is unknown, one obtains for $t\in[0,1]$.

$$x(t)=(1+x(2))\operatorname{ch}(at)+\frac{c}{a}\operatorname{sh}(at)-x(2),$$

and hence

$$p(t)=\dot{x}(t)-1=c\operatorname{ch}(at)+a\big(1+x(2)\big)\operatorname{sh}(at)-1. \tag{7.10}$$

Since, by Theorem 5.1, $p(\cdot)$ is absolutely continuous on $[0,2]$ we get by continuity condition $p(1-0)=p(1+0)$ and by (7.7), (7.10) that

$$c\operatorname{ch}a+a\big(1+x(2)\big)\operatorname{sh}a-1=2p_a x(2). \tag{7.11}$$

This is the first equation for unknown coefficients a, c, $x(2)$ (by (7.9) p_a is a function of a). The second equation is obtained in similar way by computing $x(t)$, $t\in[1,2]$, in backward direction, starting from $x(2)$

$$x(t)=3x(2)-tx(2)+\frac{1}{2p_a}\big[\big(1+x(2)\big)a\big(\operatorname{sh}a-\operatorname{sh}a(t-1)\big)+$$
$$+c\operatorname{ch}\big(a-\operatorname{ch}a(t-1)\big)\big], \tag{7.12}$$

and comparing $x(1-0)=x(1+0)$ which gives

$$\big(1+x(2)\big)\operatorname{ch}a+\frac{c}{a}\operatorname{sh}a-x(2)=2x(2)+\frac{1}{2p_a}\big[\big(1+x(2)\big)a\operatorname{sh}a+c(\operatorname{ch}a-1)\big]. \tag{7.13}$$

The last equation for coefficients follows from evaluating the norm (7.4) (we have from (5.2) $\|x_T(\cdot)\|=\varepsilon$ since otherwise $p_a=0$ and this implies $u(t)\equiv 0$ on [0, 2]; a control which does not give $\|x_T(\cdot)\|\leqslant 0.25$)

$$\varepsilon^2=2\left(x(2)\right)^2+\frac{x(2)}{p_a}\left[(1+x(2))\,a\,\mathrm{sh}\,a+c(\mathrm{ch}\,a-1)+\frac{a^2}{16p_a^2}\,a(1+\right.$$

$$\left.+x(2))^2\,(2a+\mathrm{sh}\,2a)+c^2\,(-2a+\mathrm{sh}\,2a)+2c\,\mathrm{ch}\,(2a-1)\right]. \quad (7.14)$$

Substituting p_a from (7.9) we may solve analytically (7.11), (7.13) as a system of two linear equations with respect to $x(2)$ and c. This result, when substituting to (7.14), gives one nonlinear equation with one unknown parameter a. Solving this numerically yields the following quantities.

$$a=1.0191032\approx 1.019,\ 2p_a=26.926012\approx 26.926$$

$$x(2)=-0.1651851\approx -0.165,\quad c=-2.8674713\approx -2.867.$$

Thus the optimal control, if exists, is defined as

$$-2.867\,\mathrm{ch}\,(1.019\,t)+0.851\,\mathrm{sh}\,(1.019\,t)-1,\ t\in[0,1]$$

$$-0.867\,\mathrm{ch}\,\big(1.019\,(t-1)\big)+2.922\,\mathrm{sh}\,\big(1.019\,(t-1)\big),\ t\in[1,2]. \quad (7.15)$$

The existence of the optimal control is assured by the general results in existence theory for linear problems with closed convex set of controls. In fact, if we come back to representation (3.9), (3.10) then the following problem is equivalent to (7.1)—(7.4)

$$\text{minimize } \|v(\cdot)\|\,L^2_{(0,\,1)},\ v(t)=\big(v_1(t),v_2(t)\big)$$

under constraints

$$\begin{cases}\dot z_1(t)=\varphi_0(t)+v_1(t),\\ \dot z_2(t)=z_1(t)+v_2(t),\ t\in[0,1],\\ z_2(0)=z_1(1),\end{cases} \quad (7.16)$$

$$\|z_2(\cdot)\|\,W^2_{(1)}(0,1)\leqslant\varepsilon,$$

where $\varphi_0(t)=1$ for $t\in[0,1]$.

This can be expressed in more abstract fashion as minimizing a continuous convex functional

$$F_0: L^2(0,1;R^3)\times W^2_{(1)}(0,1;R^2)\ni(\varphi,v,z)\to\|v\|\in R^1,$$

under the linear constraints $z=A(\varphi,v)$ on a closed convex set $V\subset L^2(0,1;R^3)\times\times W^2_{(1)}(0,1;R^2)$, where A is a linear bounded operator defined by solutions to equations (7.16) and

$$V=\{(\varphi,v,(z_1,z_2)):\ \varphi=\varphi_0,\ \|z_2\|\leqslant\varepsilon\}.$$

For such type of problems the Theorem V.3.5 of S. Rolewicz [19] applies immediately. Hence (7.15) is the unique optimal control.

The next example will show explicitly some of the possibilities in generalizing the basic problem P(3.1) which were described in Section 6.

Example 7.2. Let the evolution of a system be ruled by a differential-difference equation of neutral type.

$$\dot{x}(t)=f\big(x(t), \dot{x}(t-h), u(t)\big), \; t\in[0, 2h].$$

The initial function $x(t)=\varphi(t)$, $t\in[-h, 0]$ is allowed to be an arbitrary absolutely continuous function with derivative in $L^{\infty}(-h, 0, \mathscr{F})$, $\mathscr{F}$ — a nonempty subset of R^n, and with initial point $x(0)$ satisfying $X_0\big(x(0)\big)=0$. The control $u(\cdot)$ is assumed to be in $L^{\infty}(0, 2h; U)$, U — nonempty subset in R^m. The target set is defined by inequality

$$\int_h^{2h} k\big(x(t), u(t)\big)\, dt\leqslant b,$$

where b is a given real number.

The problem is to find optimal initial function $\varphi(\cdot)$ and optimal control $u(\cdot)$ such that all the constraints given above are satisfied and the functional

$$J(\varphi, u)=\bar{X}\big(x(2h)\big)$$

achieves minimum.

After applying transformation (3.1) and setting $w(t)=\dot{\varphi}(t-h)$ we get the following equivalent statement of the problem.

$$\begin{cases} \text{Minimize } \bar{X}\big(z_2(h)\big) \\ \text{under the constraints} \\ \dot{z}_0(t)=w(t), \\ \dot{z}_1(t)=f_1\big(z_1(t), w(t), v_1(t)\big), \\ \dot{z}_2(t)=f_2\big(z_1(t), z_2(t), w(t), v_1(t), v_2(t)\big), \\ \dot{z}_3(t)=k\big(z_2(t), v_2(t)\big), \\ X_0\big(z_0(h)\big)=0, \\ z_1(0)=z_0(h),\; z_2(0)=z_1(h), \\ z_3(h)-z_3(0)\leqslant b, \end{cases}$$

where $f_1=f$ and $f_2(z_1, z_2, w, v_1, v_2)=f(z_2, f(z_1, w, v_1), v_2)$.

Clearly, Theorem 2.1 is directly applicable to this problem, provided that appropriate regularity assumptions are made. Hence if $w^o(\cdot)$, $x^o(0)$, $u^o(\cdot)$ are optimal then there exist nontrivial $(\bar{\alpha}, \alpha, \psi)$, $\bar{\alpha}\leqslant 0$, $\alpha=(\alpha_0, \alpha_1, \alpha_2, \alpha_3)\in R^{3n+1}$, $\alpha_3\leqslant 0$, $\psi=(\psi_0, \psi_1, \psi_2, \psi_3)\colon [0, h]\to R^{3n+1}$ such that for corresponding optimal values $x^o(\cdot)$, $z_1^o(0)$, $v^o(\cdot)$, $z^o(\cdot)$ the following conditions hold:

(i) $|\bar{\alpha}|+|\alpha|>0$,

(ii) $\dot{\psi}_0(t)=0$, $\dot{\psi}_3(t)=0$,

$$\dot{\psi}_1^*(t)=\psi_1^*(t)\, f_{1z_1}(^o)-\psi_2^*(t)\, f_{2z_1}(^o),$$

$$\dot{\psi}_2^*(t)=\psi_2^*(t)\, f_{2z_2}(^o)-\psi_3\, k_{z_2}(^o),$$

for a.a. $t \in [0, h]$, and transversality conditions

$$\psi(0) = \begin{bmatrix} 0 \\ -\alpha_1 \\ -\alpha_2 \\ \alpha_3 \end{bmatrix}, \quad \psi(h) = \begin{bmatrix} -\alpha_1 + X^*_{0z_0}\alpha_0 \\ -\alpha_2 \\ \bar{\alpha}\bar{X}^*_{z_2}(h) \\ \alpha_3 \end{bmatrix},$$

(iii) maximum condition

$$H\big(t, w^o(t), v^o(t)\big) = \max_{\substack{w \in \mathscr{F} \\ v \in U \times U}} H(t, w, v) \text{ a.e. on } [0, h],$$

where

$$H(t, w, v) = \psi_o^* w + \psi_1^*(t) f_1\big(z_1^o(t), w, v_1\big) + \psi_2^*(t) f_2\big(z_1^o(t), z_2^o(t), w, v_1, v_2\big) + \\ + \psi_3 k\big(z_2^o(t), v_2\big).$$

Note that here ψ_0 does not correspond to performance index; this role plays $\bar{\alpha}$. Furthermore, from these conditions, it follows that

$$\psi_0(t) = \psi_0 = 0,$$

$$\psi_3(t) = \alpha_3 = \text{const.},$$

$$\psi_1(0) = -X^*_{0z_0(h)}\alpha_0,$$

$$\psi_1(h) = \psi_2(0),$$

$$\psi_2(h) = \bar{\alpha}\bar{X}_{z_2(h)}.$$

Using the equations above, the notation $\psi_3 = p_a$ and (4.16) we finally obtain the following form of maximum rpinciple.

Maximum principle for the problem of Example 7.2.

If $w^o(t) = \dot{\varphi}^o(t-h)$, $x^o(0)$, $u^o(t)$ is the solution to optimal control problem of Example 7.2 then there exist a nontrivial quadruple $\big(p(\cdot), p_a, \alpha_0, \bar{\alpha}\big)$, where $p_a \leqslant 0$, $\bar{\alpha} \leqslant 0$, $\alpha_0 \in R^n$ and $p\colon [0, T] = [0, 2h] \to R^n$, such that

(iv) the function $p(\cdot)$ is absolutely continuous on $[0, T] = [0, 2h]$ and satisfies a.e. on this interval the following differential-difference equation

$$\dot{p}^*(t) = -p^*(t) f_x\big(x^o(t), \dot{x}^o(t-h), u^o(t)\big) - p(t+h) f_{\dot{x}}\big(x^o(t+h), \dot{x}^o(t), \\ u^o(t+h)\big) - p_a(t) k_x\big(x^o(t), u^o(t)\big)$$

with boundary conditions

$$p(t) = 0 \text{ on } (T, T+h),$$

$$p(T) = \bar{\alpha}\bar{X}^*_x\big(x^o(T)\big),$$

$$p(0) = -X^*_{0x}\big(x^o(0)\big)\alpha_0.$$

The function $p_a(t)$ is defined by (5.1).

(v) The maximum condition holds

$$p(t)\,f\big(x^o(t), w^o(t), u^o(t)\big)+p(t+h)\,f\big(x^o(t+h), f(x^o(t), w^o(t), u^o(t)), u^o(t+h)\big)+p_a\,k\big(x^o(t+h), u^o(t+h)\big)$$
$$=\max_{\substack{w\in\mathcal{F}\\ u_1,u_2\in U}} p(t)\,f\big(x^o(t), w, u_1\big)+p(t+h)\,f\big(x^o(t+h), f\big(x^o(t), w, u_1\big), u_2\big)+p_a\,k\big(x^o(t+h), u_2\big)$$

for almost all $t\in[0, h]$.

8. Concluding remarks

A general class of optimal control problems for nonlinear systems with possible delays in the state and control variables has been considered. Standard hypotheses on regularity of the functions defining the problem has been assumed. The controls has been taken from the space $L^\infty(0, T; U)$, U — given nonempty subset of R^m. The final time T and the lags of the system has been assumed commensurable. It does not seem to be, however, a restrictive assumption from the point of view of applications. The aim of the paper was to develope a general procedure for obtaining necessary optimality conditions for time-lag systems when having appropriate conditions for non-delayed systems. It has been shown that under commensurability assumption it is possible to construct in all practical cases an equivalent optimal control problem for some system without delays.

This reformulation has been described in Section 3 for the case of delays in state and control variables but it does apply as well to neutral systems, when the delayed derivative of state variable is present in system equation. One basic problem has been chosen to show how the method works; the problem of controlling to a complete final state $x_T(\cdot)$ in a given ball in function space. The spaces C, L^2 and $W^2_{(1)}$ has been considered in details in Sections 4 and 5. The necessary optimality conditions has been derived in the form of pointwise maximum principle with absolutely continuous adjoint variable. It occurs that the cases of L^2 and $W^2_{(1)}$ are simplest than the case of C-space in the sense that adjoint variables corresponding to final state constraints are constants only. In Section 6 various possible generalizations has been described, including the cases of neutral systems, constraints on initial conditions, phase variable equality and inequality constraints, fixed final complete state and others. Two examples has been given in Section 7, one concerning linear system, quadratic functional and the final state $x_T(\cdot)$ in a ball in $W^2_{(1)}$, second with system equation of neutral type. The major limitation of the method presented is that it does not apply for systems with distributed lags e.g. integro-differential equations of the type

$$\dot{x}(t)=\int_h^0 k\big(t, x(t+\Theta), \Theta\big)\,d\Theta+f\big(x(t), u(t), t\big)$$

or for systems with time-varying lags.

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Received, January 1976

Sterowanie układami z opóźnieniem przy ograniczeniach w przestrzeni funkcyjnej. Część 1. Warunki konieczne optymalności

W pracy rozpatrzono klasę problemów optymalizacji dynamicznej dla układów z opóźnieniami zmiennej stanu i sterowania o postaci $\dot{x}(t)=f(x(t), x(t-h_1), ..., x(t-h_s), u(t), u(t-h_1), ..., u(t-h_s), t)$, $t\in[0, T]$ przy całkowym wskaźniku jakości i ograniczeniach uwzględniających ograniczenia w przestrzeni funkcyjnej zupełnych stanów końcowych. Założono współmierność czasu końcowego T i opóźnień h_i, $i=1, ..., s$. Warunek ten jest w praktyce niemal zawsze spełniony, gdyż zwykle przyjmuje się wartości liczbowe dla T, h_i w postaci liczb o skończonej reprezentacji w układzie dziesiętnym. Wykazano, że po wprowadzeniu równoważnego układu bez opóźnień z dodatkowymi dwupunktowymi warunkami brzegowymi można sformułować nowy równoważny

problem nie zawierający jawnie opóźnień. Po zastosowaniu istniejących w literaturze wyników dotyczących układów bez opóźnień otrzymano warunki konieczne optymalności w postaci zasady maksimum Pontriagina, które można przedstawić w postaci właściwej dla układów z opóźnieniami przypominającej warunki otrzymane przez Kharatishvili. Dokładne i ścisłe rozważania przeprowadzono dla specjalnego problemu, jak się wydaje, dość ważnego praktycznie: sterowanie do końcowego stanu zupełnego $x_T(\cdot)$ przy warunku, że stan ten należy do zadanej kuli w przestrzeni funkcyjnej. Rozważono trzy przypadki przestrzeni funkcyjnej stanów końcowych: przestrzeń $C[T-h_s, T; \mathbf{R}^n]$ funkcji ciągłych o wartościach w R^n, przestrzeń $L^2[T-h_s, T; R^n]$ oraz przestrzeń Sobolewa $W^2_{(1)}[T-h_s, T; R^n]$ funkcji absolutnie ciągłych o pochodnej w $L^2[T-h_s, T; R^n]$. We wszystkich przypadkach otrzymano warunki konieczne optymalności w postaci warunku maksimum hamiltonianu przy czym zmienne sprzężone są absolutnie ciągłe w przypadku normy w L^2 lub $W^2_{(1)}$ oraz przedziałami absolutnie ciągłe dla przypadku normy typu suprémum w przestrzeni C. Otrzymane rezultaty są silniejsze niż dotychczas znane w literaturze. W rozdziale 6 pokazano, że w analogiczny sposób można uzyskać warunki optymalności dla większości spotykanych w teorii optymalizacji dynamicznej problemów z opóźnieniami, wliczając w to takie problemy jak:

1. Równościowe i nierównościowe ograniczenia skończenie wymiarowe typu $X^j(x(t_0), \ldots, x(t_l))=0$ dla $j=1, \ldots, d_1$ oraz ≤ 0 dla $j=d_1+1, \ldots, d$.
2. Ograniczenia dla warunków początkowych typu $x(t) \in F(t)$, $\forall t \in [-h_s, 0]$.
3. Ogólne ograniczenia dla stanu końcowego w przypadkach
   (a) $Q_i(x(t), t) \leq 0$ $\forall t \in [T-h_s, T]$,
   (b) ograniczenia funkcjonalne
   $q_i(x_T(\cdot))=0$ dla $i=1, \ldots, c_1$ oraz ≤ 0 dla $i=c_1+1, \ldots, c$,
   (c) stan końcowy ustalony $x(t)=\xi(t)$ dla $t \in [T-h_s, T]$,
   (d) ograniczenia operatorowe równościowe
   $P(x(t), t)=0$ dla $t \in [T-h_s, T]$.
4. Wskaźnik jakości całkowy z opóźnionymi zmiennymi stanu i sterowania pod całką.
5. Układy z opóźnieniami opisywane równaniami typu neutralnego. Sterowanie do kuli w przestrzeni $W^2_{(1)}$ zilustrowano przykładem numerycznym (Przykład 7.1), natomiast Przykład 7.2 pokazuje dokładniej niektóre z możliwych uogólnień 1—5.

Управление системами с запаздыванием и с ограничениями в функциональным пространстве. Част I. Необходимые условия оптимальности

Рассматривается система вида $\dot{x}(t)=f(x(t), x(t-h_1), \ldots, x(t-h_s), u(t), u(t-h_1), \ldots, u(t-h_s), t)$, $t \in [0, T]$, для которой формулируется проблема оптимального управления с интегральным функционалом качества и конечным состоянием $x_T(\cdot)$ приданлежащим некоторому шару в функциональном пространстве. Применяются три типа пространств: пространство непрерывных функций $C(T-h_s, T; R^n)$ определенных на $[T-h_s, T]$ со значениями в R^n, пространство функций суммируемых с квадратом $L^2(T-h_s, T; R^n)$ и пространство Соболева $W^2_{(1)}(T-h_s, T; R^n)$ абсолютно непрерывных функций с производной в L^2. Предлагается что числа T, h_i, $i=1, \ldots, s$ являются соизмеримыми. Это предложение исполняется на практике почти всегда. После применения эквивалентной системы без запаздывания с добавочными двуграничными условиями оказывается что поставленные проблемы решаются с помощью существующих в литературе достаточно общих и мощных результатов. Полученные в работе необходимые условия оптимальности в форме обычного принципа максимума с абсолютно непрерывными сопряженными переменными являются, в рассматриванном классе проблем, более мощными чем известные в литературе. В разделе 6 работы показано несколько обобщений. Обобщаются типы ограничений, функционал качества, рассматриваются системы нейтрального типа. Два примера, один численный, второй теоретический, иллюстрируют полученные результаты.