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# Optimal control for quadratic problem with neutral system equation

by

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In the paper the step method is applied to solve an optimal control problem with a linear neutral equation of the system and a quadratic cost. The delays are constant and commensurable. A formula for the optimal feedback operator is derived. For the open loop system a numerical algorithm is presented. A computational algorithm is also constructed for the optimal feedback operator in case of infinite control time.

## 1. Introduction

The problem of optimal control for systems described by linear or affine functional differential equations with quadratic or linear-quadratic performance indices was considered in a number of papers. It was established that the value of optimal control at any moment of time is a linear (or affine, in case of affine systems) function of the state of the system at the same moment. This function will be called the optimal feedback operator. Probably the most complete results concerning affine hereditary systems with linear-quadratic performance indices may be found in [6] and [7]. Systems with a lag occurring in the trajectory and in the control function were considered in [9]. Datko [4] studied a neutral system and obtained existence and uniqueness results for the optimal feedback operator; however, no detailed characterization was presented. For hereditary systems a set of Riccati-type differential equations was derived (e.g. [6], [7]), and numerical methods of solution were studied [1], [6], [8], [16]. Another approach was used in [13], [9]. It consisted in the construction of Fredholm integral equations of the second kind for the feedback operator. These techniques were used to overcome the difficulties connected with the fact that both advanced and delayed terms appear in the canonical system of equations for such a control problem.

In [10] the step method was applied to a problem with one constant delay in the trajectory and relatively simple formulas were obtained for the optimal feedback operator, as well as numerical algorithms. The purpose of this paper is to generalize those results to neutral systems with constant commensurable delays in the trajectory and its derivative and in the control. First, the original problem is transformed into a problem without delays with split boundary conditions. Next the system of canonical equations is derived and solved in an explicit form. Basing on these results the optimal feedback operator is constructed and an algorithm for computation of the optimal control presented in case of finite control time. The questions of existence and uniqueness are discussed. For the infinite control time problem a numerical iterative algorithm is presented. The considerations are illustrated with two examples.

The standard vector-matrix notation is used. $A'$ denotes the transpose of $A$. A function $f\colon [a, b]\to R^s$ is called piece-wise continuous (p.w.c.) if the number of discontinuities is finite and for every $t\in[a, b]$, $f(t)=f(t+0)$ and/or $f(t)=f(t-0)$. We define $(a,a]=\emptyset$.

## 2. Formulation of the problem

Consider an optimal control problem for a linear system with commensurable delays in the trajectory, its derivative, and in the control

$$\sum_{i=0}^{r} [A^i(t)\,\dot{x}(t-i\tau)+B^i(t)\,x(t-i\tau)+C^i(t)\,u(t-i\tau)]=0, \qquad t\in[0, T], \tag{1}$$

$$x(t)=\varphi(t),\ \ t\in[-r\tau, 0],$$

$$\dot{x}(t)=\begin{cases}\dfrac{dx(t)}{dt}, & t>0,\\ \mu(t), & t\in[-r\tau, 0),\end{cases}$$

$$u(t)=\eta(t), \qquad t\in[-r\tau, 0),$$

$$x(t)\in R^n, \qquad u(t)\in R^m, \quad \tau>0, \qquad r\in N.$$

The functions $\varphi$, $\eta$, $\mu$, $A^i$, $B^i$, $C^i$ are p.w.c., $A^0(t)$ has a bounded inverse everywhere in [0, T]. The admissible controls $u$ are square summable in [0, T].

A quadratic cost functional is minimized subject to Eq. (1)

$$S(u)=\int_0^T [x'(t)\,W(t)\,x(t)+u'(t)\,U(t)\,u(t)]\,dt+x'(T)\,Qx(T). \tag{2}$$

The matrix-valued functions $W$ and $U$ are p.w.c., $W(t)$, $U(t)$, $Q$ are symmetric, $W(t), Q\geqslant 0$, $U(t)>0$ and has a bounded inverse everywhere in $[0, T]$. Let

$$x_t=\{x(t+s),\ s\in[t-r\tau, t]\},$$

$$\dot{x}_t=\{\dot{x}(t+s),\ s\in(t-r\tau, t)\}$$

$$u_t=\{u(t+s),\ s\in[t-r\tau, t)\}.$$

The state of the system (1) at time $t$, denoted by $X(t)$, is a triple $X(t)=(\dot{x}_t;\, x_t;\, u_t)$.

The optimal control problem (1), (2) will be also considered under an additional

*Assumption 1*:

(i) $\varphi$ is absolutely continuous and $\mu = \frac{d\varphi}{dt}$ is p.w.c. in $[-r\tau, 0]$;

(ii) $A^i$, $i=0, \ldots, r$, are absolutely continuous in $[0, T]$ with p.w.c. first derivatives.

If Assumption 1 (i) is valid, the state of the system (1) is defined as a pair $X(t) = (x_t; u_t)$.

LEMMA 1. For every admissible initial state $X(0)$ and every admissible control $u$ there exists a unique continuous solution $x$ of (1).

Proof. In every interval $(k\tau, (k+1)\tau]$, $k=0, 1, \ldots$ Eq. (1) may be solved as an affine ordinary differential equation with a square summable right-hand side.

The first problem considered below is that of optimal control synthesis. Our purpose is to find the mapping $L_t$ that connects the value of optimal control $u(t)$ with the state $X(t)$

$$u(t) = L_t X(t). \tag{3}$$

The same approach yields an effective method of determining the optimal control in an open loop system, i.e. as an explicit function of time. The last problem discussed in the paper is the optimal control synthesis for stationary systems in case of infinite control time, $T \to \infty$. Then the optimal feedback operator (3) does not depend on time and is denoted by $L^{\infty}$.

## 3. Step method

In order to express the optimal control $u(t_0)$ in terms of the state $X(t_0)$ we consider the optimization problem (1), (2) in the interval $[t_0, T]$ for an arbitrary $t_0 \in [0, T)$. The cost functional to be minimized is

$$S_{t_0}(u) = \int_{t_0}^{T} [x'(t)\, W(t)\, x(t) + u'(t)\, U(t)\, u(t)]\, dt + x'(T)\, Q x(T). \tag{4}$$

New variables are introduced which satisfy a system of differential equations equivalent to (1) in $[t_0, T]$, but with no deviations of argument. Let $k$ be an integer and $\vartheta$ a real such that

$$t_0 \in [T - k\tau, T - (k-1)\tau) \tag{5}$$

$\vartheta = t_0 + k\tau - T$, $0 \leqslant \vartheta < \tau$.

Denote

$$\begin{aligned} x_i(s) &= x(T - i\tau + s), \quad i = 1, \ldots, k, \\ u_i(s) &= u(T - i\tau + s), \\ y_i(s) &= x(t_0 - i\tau - \vartheta + s), \quad i = 1, \ldots, r, \\ v_i(s) &= u(t_0 - i\tau - \vartheta + s). \end{aligned} \tag{6}$$

Let $q$ be an integer, $1 \leqslant q \leqslant k$. Denote

$$\tilde{x}_q = \begin{pmatrix} x_1 \\ \vdots \\ x_q \end{pmatrix}, \quad \tilde{u}_q = \begin{pmatrix} u_1 \\ \vdots \\ u_q \end{pmatrix}, \quad y = \begin{pmatrix} y_1 \\ \vdots \\ y_r \end{pmatrix}, \quad v = \begin{pmatrix} v_1 \\ \vdots \\ v_r \end{pmatrix}, \quad w = \begin{pmatrix} \dot{y} \\ y \\ v \end{pmatrix} \tag{7}$$

$$A_i^j(s) = A^j(T - i\tau + s), \; j = 0, \ldots, r, \; i = 1, \ldots, q, \; A_i^j = 0, \; j > r.$$

$B_i^j$, $C_i^j$, $U_i$, $W_i$ are defined in the same manner.

$$\tilde{A}_q = \begin{pmatrix} A_1^0 & A_1^1 & \ldots & A_1^{q-1} \\ 0 & A_2^0 & \ldots & A_2^{q-2} \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \ldots & A_q^0 \end{pmatrix}$$

Replacing $A$ by $B$ and $C$ we obtain the matrices $\tilde{B}_q$ and $\tilde{C}_q$ respectively.
Let

$$\tilde{W}_q = \operatorname{diag}(W_1, \ldots, W_q),$$

$$\tilde{U}_q = \operatorname{diag}(U_1, \ldots, U_q),$$

$$d_q = (0\; I) \quad ((q-1)\, n \times qn \text{ — matrix}),$$

$$e_q = (I\; 0) \quad ((q-1)\, n \times qn \text{ — matrix}).$$

LEMMA 2. The matrix $\tilde{A}_q(s)$ is nonsingular for every $q$ and every $s \in [0, \tau]$.

Proof. This follows from the nonsingularity of $A_i^0(s)$, $i = 1, \ldots, k$.
Denote

$$\hat{B}_q = -\tilde{A}_q^{-1} \tilde{B}_q, \quad \hat{C}_q = -\tilde{A}_q^{-1} \tilde{C}_q, \quad a_q^i = -\tilde{A}_q^{-1} \begin{pmatrix} A_1^{i+q} \\ \vdots \\ A_q^{i+1} \end{pmatrix}. \tag{8}$$

Matrices $b_q^i$ and $c_q^i$ are defined similarly, $A$ replaced by $B$ and $C$ respectively.
Let

$$\delta_q = (a_q^0 \ldots a_q^{r-1}\; b_q^0 \ldots b_q^{r-1}\; c_q^0 \ldots c_q^{r-1}). \tag{9}$$

Using the step method the original problem is transformed in a problem with split boundary conditions but without delays. Unless otherwise stated, all further results are valid for $k \geqslant 2$. In case $k = 1$ the optimization problem is trivial since the system equation (1) may be treated as an affine equation without deviations of argument.

THEOREM 1. Minimization of the functional (4) subject to Eq. (1) in $[t_0, T]$ with the initial condition $X(t_0)$ is equivalent to minimization of the functional

$$J(\tilde{u}_{k-1}, \tilde{u}_k) = \int_0^{\vartheta} [\tilde{x}'_{k-1}(s)\, \tilde{W}_{k-1}(s)\, \tilde{x}_{k-1}(s) + \tilde{u}'_{k-1}(s)\, \tilde{U}_{k-1}(s)\, \tilde{u}_{k-1}(s)]ds +$$

$$+ \int_{\vartheta}^{\tau} [\tilde{x}'_k(s)\, \tilde{W}_k(s)\, \tilde{x}_k(s) + \tilde{u}'_k(s)\, \tilde{U}_k(s)\, \tilde{u}_k(s)]\, ds + x'_1(\tau)\, Q x_1(\tau), \tag{10}$$

subject to

$$\dot{\tilde{x}}_{k-1}(s)=\hat{B}_{k-1}(s)\,\tilde{x}_{k-1}(s)+\hat{C}_{k-1}(s)\,\tilde{u}_{k-1}(s)+\delta_{k-1}(s)\,w(s+\tau),\ s\in[0,\vartheta] \tag{11}$$

$$\dot{\tilde{x}}_k(s)=\hat{B}_k(s)\,\tilde{x}_k(s)+\hat{C}_k(s)\,\tilde{u}_k(s)+\delta_k(s)\,w(s),\ \ s\in[\vartheta,\tau],$$

$$\tilde{x}_{k-1}(\vartheta)=e_k\,\tilde{x}_k(\vartheta),\ \tilde{x}_{k-1}(0)=d_k\,\tilde{x}_k(\tau),\ x_k(\vartheta)=x(t_0).$$

Proof. Substituting (6), (7), (8) and (9) to (1) and (4) we obtain (10) and (11). Hence every solution of (1) is also, in virtue of (6) and (7), a solution of (11) in $[t_0, T]$. The corresponding values of (10) and (4) are equal. Conversely, if $w$ is regular enough, every solution of (11) yields a solution of (1) in $[t_0, T]$ with the same values of the cost functionals (10) and (4).

## 4. Adjoint equations

A usual variational technique will be used to obtain adjoint equations for the system (10), (11). Let $\tilde{x}_{k-1}, \tilde{x}_k$ be a solution of (11) corresponding to an admissible control $\tilde{u}_{k-1}, \tilde{u}_k$. Let $\delta\tilde{u}_{k-1}, \delta\tilde{u}_k$ be an admissible variation of the control and $\delta\tilde{x}_{k-1}, \delta\tilde{x}_k$ — the corresponding variation of the trajectory.

Lemma 3. The variation of the trajectory satisfies the system of equations

$$\delta\dot{\tilde{x}}_{k-1}(s)=\hat{B}_{k-1}(s)\,\delta\tilde{x}_{k-1}(s)+\hat{C}_{k-1}(s)\,\delta\tilde{u}_{k-1}(s),\ \ s\in[0,\vartheta], \tag{12}$$

$$\delta\dot{\tilde{x}}_k(s)=\hat{B}_k(s)\,\delta\tilde{x}_k(s)+\hat{C}_k(s)\,\delta\tilde{u}_k(s),\ s\in[\vartheta,\tau],$$

$$\delta\tilde{x}_{k-1}(\vartheta)=e_k\,\delta\tilde{x}_k(\vartheta),\ \delta\tilde{x}_{k-1}(0)=d_k\,\delta x_k(\tau),\ \delta\tilde{x}_k(\vartheta)=0.$$

Proof. By subtraction of Eqs. (11), written for the control $\tilde{u}_{k-1}, \tilde{u}_k$, from the same equations written for $\tilde{u}_{k-1}+\delta\tilde{u}_{k-1}, u_k+\delta\tilde{u}_k$.
It is easy to verify that the variation of $J$ (4) has the form

$$\delta J(\tilde{u}_{k-1}, \tilde{u}_k, \delta\tilde{u}_{k-1}, \delta\tilde{u}_k)=2\Big[\delta J_1+\int_0^{\vartheta}\tilde{u}'_{k-1}(s)\,\tilde{U}_{k-1}(s)\,\delta\tilde{u}_{k-1}(s)\,ds+ \\ +\int_{\vartheta}^{\tau}\tilde{u}'_k(s)\,\tilde{U}_k(s)\,\delta\tilde{u}_k(s)\,ds\Big] \tag{13}$$

where

$$\delta J_1=\int_0^{\vartheta}\tilde{x}'_{k-1}(s)\,\tilde{W}_{k-1}(s)\,\delta\tilde{x}_{k-1}(s)\,ds+\int_{\vartheta}^{\tau}\tilde{x}'_k(s)\,\tilde{W}_k(s)\,\delta\tilde{x}_k(s)\,ds+ \\ +x'_1(\tau)\,Q\delta x_1(\tau), \tag{14}$$

$\delta x_1$ denotes the variation of $x_1$.

Lemma 4. $\delta J_1$ can be expressed in the form

$$\delta J_1=\int_0^{\vartheta}\tilde{p}'_{k-1}(s)\,\hat{C}_{k-1}(s)\,\delta\tilde{u}_{k-1}(s)\,ds+\int_{\vartheta}^{\tau}\tilde{p}'_k(s)\,\hat{C}_k(s)\,\delta\tilde{u}_k(s)\,ds \tag{15}$$

where $\tilde{p}_q$, $q=k-1, k$, are adjoint variables

$$\tilde{p}_q=\begin{pmatrix} p_1 \\ \vdots \\ p_q \end{pmatrix}, \qquad p_i(s)\in R^n, \qquad i=1, \dots, q$$

satisfying the system of adjoint equations

$$\begin{aligned} \dot{\tilde{p}}_{k-1}(s) &= -\hat{B}'_{k-1}(s)\,\tilde{p}_{k-1}(s)-\tilde{W}_{k-1}(s)\,\tilde{x}_{k-1}(s), \quad s\in[0, \vartheta], \\ \dot{\tilde{p}}_k(s) &= -\hat{B}'_k(s)\,\tilde{p}_k(s)-\tilde{W}_k(s)\,\tilde{x}_k(s), \quad s\in[\vartheta, \tau], \\ \tilde{p}_{k-1}(\vartheta) &= e_k\,\tilde{p}_k(\vartheta), \\ \tilde{p}_{k-1}(0) &= d_k\,\tilde{p}_k(\tau) \\ p_1(\tau) &= Qx_1(\tau). \end{aligned} \tag{16}$$

Proof. Let $\Pi_q$ and $\psi_q$, $q=k-1, k$, be the fundamental solutions of (12) and (16) respectively. Of course $\Pi_q(s, t)=\psi'_q(t, s)$. Denote $\gamma_q=\tilde{W}_q\,\tilde{x}_q$, $\beta_q=\tilde{C}_q\,\delta\tilde{u}_q$. Then

$$\delta\tilde{x}_{k-1}(s)=\Pi_{k-1}(s, 0)\,\delta\tilde{x}_{k-1}(0)+\int_0^s \Pi_{k-1}(s, \sigma)\,\beta_{k-1}(\sigma)\,d\sigma,$$

$$\delta\tilde{x}_k(s)=\Pi_k(s, \vartheta)\,\delta\tilde{x}_k(\vartheta)+\int_\vartheta^s \Pi_k(s, \sigma)\,\beta_k(\sigma)\,d\sigma,$$

$$\tilde{p}_{k-1}(s)=\psi_{k-1}(s, \vartheta)\,\tilde{p}_{k-1}(\vartheta)-\int_\vartheta^s \psi_{k-1}(s, \sigma)\,\gamma_{k-1}(\sigma)\,d\sigma,$$

$$\tilde{p}_k(s)=\psi_k(s, \tau)\,\tilde{p}_k(\tau)-\int_\tau^s \psi_k(s, \sigma)\,\gamma_k(\sigma)\,d\sigma.$$

After substitution of these expressions to the right-hand sides of (14) and (15) and after some transformations one obtains the identity (15), what completes the proof.

## 5. Optimal control. Existence, uniqueness and canonical equations

Denote

$$\alpha_q=\begin{pmatrix} \hat{B}_q & -\hat{C}_q\,\tilde{U}_q^{-1}\,\hat{C}'_q \\ -\tilde{W}_q & -\hat{B}'_q \end{pmatrix}, \qquad \bar{\delta}_q=\begin{pmatrix} \delta_q \\ 0 \end{pmatrix}, \qquad z_q=\begin{pmatrix} \tilde{x}_q \\ \tilde{p}_q \end{pmatrix}. \tag{17}$$

THEOREM 2. There exists a unique optimal control $\tilde{u}_{k-1}, \tilde{u}_k$ in (10, (11) for every $\vartheta\in[0, \tau)$

$$\begin{aligned} \tilde{u}_{k-1}(s) &= -\tilde{U}_{k-1}^{-1}(s)\,\hat{C}'_{k-1}(s)\,\tilde{p}_{k-1}(s), \quad s\in[0, \vartheta], \\ \tilde{u}_k(s) &= -\tilde{U}_k^{-1}(s)\,\hat{C}'_k(s)\,\tilde{p}_k(s), \quad s\in[\vartheta, \tau]. \end{aligned} \tag{18}$$

The adjoint variables $\tilde{p}_{k-1}, \tilde{p}_k$ and the optimal trajectory $\tilde{x}_{k-1}, \tilde{x}_k$ satisfy the system of canonical equations

$$\begin{aligned}
\dot{z}_{k-1}(s) &= \alpha_{k-1}(s)\, z_{k-1}(s) + \bar{\delta}_{k-1}(s)\, w(s+\tau), \quad s \in [0, \vartheta], \qquad (19)\\
\dot{z}_k(s) &= \alpha_k(s)\, z_k(s) + \bar{\delta}_k(s)\, w(s), \quad s \in [\vartheta, \tau],\\
z_{k-1}(\vartheta) &= \operatorname{diag}(e_k, e_k)\, z_k(\vartheta),\\
z_{k-1}(0) &= \operatorname{diag}(d_k, d_k)\, z_k(\tau),\\
x_k(\vartheta) &= x(t_0),\\
p_1(\tau) &= Qx_1(\tau).
\end{aligned}$$

Proof. The solution $x(t)$, $t \geqslant t_0$, of (1) is a linear function of the pair $(X(t_0);\ u(s),\ s \in [t_0, t))$. Hence the solution $\tilde{x}_{k-1}, \tilde{x}_k$ of (11) depends linearly on $(w;\ \tilde{u}_{k-1}(s),\ s \in [0, \vartheta);\ \tilde{u}_k(s),\ s \in [\vartheta, \tau])$. Then the functional (10) is strictly convex. By usual arguments [12] we obtain that there exists a unique optimal control. The functional $J$ (10) is Fréchet differentiable, therefore a necessary and sufficient condition of optimality for a control is that the variation $\delta J$ (13) must be equal to zero for every pair $(\delta\tilde{u}_{k-1};\ \delta\tilde{u}_k)$. Hence we get (18). The canonical equations (19) result from (11), (16) and (18).

Corollary 1. For every $t_0 \in [0, T)$ there exists a unique control $u$ that minimizes (4) subject to (1) in $[t_0, T]$. The relations (6), (7) determine a one-to-one correspondence between $u$ and the control $\tilde{u}_{k-1}, \tilde{u}_k$ that minimizes (10).

Proof. This is a consequence of the fact that the problems (1), (4) and (10), (11) are equivalent.

Corollary 2. The canonical equations (17) have a unique solution $z_{k-1}(s)$, $s \in [0, \vartheta)$, $z_k(s)$, $s \in [\vartheta, \tau]$.

Proof. As the optimal control $\tilde{u}_{k-1}, \tilde{u}_k$ is unique, we obtain from (19) a unique optimal trajectory $\tilde{x}_{k-1}, \tilde{x}_k$. Then the solution $\tilde{p}_{k-1}, \tilde{p}_k$ of (16) is also unique, hence there exists a unique $z_{k-1}, z_k$ satisfying (19).

## 6. Basic algebraic equation

$\Phi_q$, $q = k-1, k$, will denote the fundamental solutions of Eqs. (19)

$$\frac{\partial}{\partial s}\Phi_q(s, t) = \alpha_q(s)\, \Phi_q(s, t), \quad \Phi_q(t, t) = I.$$

Then we obtain from (19)

$$z_{k-1}(\vartheta) = \Phi_{k-1}(\vartheta, 0)\, z_{k-1}(0) + \int_0^{\vartheta} \Phi_{k-1}(\vartheta, s)\, \bar{\delta}_{k-1}(s)\, w(s+\tau)\, ds, \qquad (20)$$

$$z_k(\tau) = \Phi_k(\tau, \vartheta)\, z_k(\vartheta) + \int_{\vartheta}^{\tau} \Phi_k(\tau, s)\, \bar{\delta}_k(s)\, w(s)\, ds.$$

These, together with the boundary conditions in (19), yield a linear algebraic system of equations. Let

$$\Phi_k = \begin{array}{c} \left[\begin{array}{ccc} \Phi_k^1 & \Phi_k^2 & \Phi_k^3 \\ \Phi_k^4 & \Phi_k^5 & \Phi_k^6 \\ \Phi_k^7 & \Phi_k^8 & \Phi_k^9 \\ \Phi_k^{10} & \Phi_k^{11} & \Phi_k^{12} \end{array}\right] \begin{array}{l} n \\ (k-1)\,n \\ n \\ (k-1)\,n \end{array} \\ \begin{array}{ccc} (k-1)\,n & n & kn \end{array} \end{array} \tag{21}$$

$$\Phi_k^A = (\Phi_k^7 - Q\Phi_k^1,\ \Phi_k^8 - Q\Phi_k^2)$$

$$\Phi_k^B = \begin{pmatrix} \Phi_k^4 & \Phi_k^6 \\ \Phi_k^{10} & \Phi_k^{12} \end{pmatrix}, \quad \Phi_k^C = \begin{pmatrix} \Phi_k^5 \\ \Phi_k^{11} \end{pmatrix}, \quad \Phi_k^D = \begin{pmatrix} \Phi_k^4 & \Phi_k^5 \\ \Phi_k^{10} & \Phi_k^{11} \end{pmatrix},$$

$$\Phi_k^E = \begin{bmatrix} \Phi_k^1 & \Phi_k^2 \\ \Phi_k^4 & \Phi_k^5 \\ \Phi_k^7 & \Phi_k^8 \\ \Phi_k^{10} & \Phi_k^{11} \end{bmatrix}.$$

After simple elimination of variables, matrix inversion excluded, we obtain the basic algebraic system of equations which will play a fundamental role in further investigations

$$\Delta_k \begin{pmatrix} \tilde{x}_{k-1}(\vartheta) \\ \tilde{p}_k(\vartheta) \end{pmatrix} = \begin{pmatrix} P_1 \\ P_2 \end{pmatrix} \tag{22}$$

$$P_1 = [\Phi_k^8(\tau, \vartheta) - Q\Phi_k^2(\tau, \vartheta)]\, x(t_0) + \int_{\vartheta}^{\tau} \Phi_k^A(\tau, \sigma)\, \delta_k(\sigma)\, w(\sigma)\, d\sigma, \tag{23}$$

$$P_2 = \Phi_{k-1}(\vartheta, 0)\, \Phi_k^C(\tau, \vartheta)\, x(t_0) + \int_0^{\vartheta} \Phi_{k-1}^E(\vartheta, \sigma)\, \delta_{k-1}(\sigma)\, w(\sigma+\tau)\, d\sigma +$$
$$+ \Phi_{k-1}(\vartheta, 0) \int_{\vartheta}^{\tau} \Phi_k^D(\tau, \sigma)\, \delta_k(\sigma)\, w(\sigma)\, d\sigma,$$

$$\Delta_k = \begin{pmatrix} Q\Phi_k^1(\tau, \vartheta) - \Phi_k^7(\tau, \vartheta) & Q\Phi_k^3(\tau, \vartheta) - \Phi_k^9(\tau, \vartheta) \\ & e_{2k-1} - \Phi_{k-1}(\vartheta, 0)\, \Phi_k^B(\tau, \vartheta) \end{pmatrix}.$$

In the case $k=1$ we get $\Delta_1 = Q\Phi_1^3(\tau, \vartheta) - \Phi_1^9(\tau, \vartheta)$ and the basic equation takes the form

$$\Delta_1 \tilde{p}_1(\vartheta) = P_1. \tag{24}$$

LEMMA 5. The coefficient matrices $\Delta_k$ in (22) and $\Delta_1$ in (24) are nonsingular for every $\vartheta$.

Proof. Let us fix our attention on (22). If for a certain $\vartheta$ the matrix $\Delta_k$ were singular, the solution to (22) either would not exist or would not be unique — contrary to Theorem 2.

### 7. Optimal feedback operatory

Denote

$$\Delta_k^{-1} = \begin{array}{cc} \begin{pmatrix} \Gamma_k^1 & \Gamma_k^2 \\ \Gamma_k^3 & \Gamma_k^4 \end{pmatrix} & \begin{array}{l} (k-1)\,n \\ kn \end{array} \\ \begin{array}{cc} n & 2(k-1)\,n \end{array} & \end{array} . \tag{25}$$

Let $H(t)$ be a $(kn \times n)$-matrix consisting of the last $n$ columns of $\hat{C}_k(\vartheta)$. Let for $t<T-\tau$:

$$G(t) = \Gamma_k^3 [\Phi_k^8(\tau, \vartheta) - Q\Phi_k^2(\tau, \vartheta)] + \Gamma_k^4\, \Phi_{k-1}(\vartheta, 0)\, \Phi_k^C(\tau, \vartheta). \tag{26}$$

$$G_i^a(t, s) = [\Gamma_k^3\, \Phi_k^A(\tau, s+\vartheta+\tau) + \Gamma_k^4\, \Phi_{k-1}(\vartheta, 0)\, \Phi_k^D(\tau, s+\vartheta+\tau)]\, a_k^i(s+\vartheta+\tau),$$
$$s \in [-\tau, -\vartheta],$$

$$G_i^a(t, s) = \Gamma_k^4\, \Phi_{k-1}^E(\vartheta, s+\vartheta)\, a_{k-1}^i(s+\vartheta), \quad s \in (-\vartheta, 0].$$

If $t \geqslant T-\tau$

$$G(t) = \Gamma_1^3 [\Phi_1^8(\tau, \vartheta) - Q\Phi_1^2(\tau, \vartheta)]. \tag{27}$$

$$G_i^a(t, s) = \Gamma_1^3\, \Phi_1^A(\tau, s+\vartheta+\tau)\, a_1^i(s+\vartheta+\tau), \quad s \in [-\tau, -\vartheta),$$

$$G_i^a(t, s) = 0, \quad s \in (-\vartheta, 0].$$

Replacing the letter $a$ in (26) and (27) by $b$ and $c$ we obtain $G_i^b$ and $G_i^c$ respectively.

THEOREM 3. For every $t \in [0, T]$ the optimal control $u(t)$ can be synthesized in the form

$$u(t) = -U^{-1}(t)\, H'(t) \Big\{ G(t)\, x(t) +$$
$$+ \sum_{i=0}^{r-1} \int_{-\tau}^{0} [G_i^a(t, s)\, \dot{x}(t-i\tau+s) + G_i^b(t, s)\, x(t-i\tau+s) + G_i^c(t, s)\, u(t-i\tau+s)]\, ds \Big\}. \tag{28}$$

The trajectory $x$ and control $u$ in the right-hand side are optimal.

Proof. (28) results immediately from (18) and from substitution of (6), (8), (26) and (27) into the relations $\tilde{p}_k(\vartheta) = \Gamma_k^3 P_1 + \Gamma_k^4 P_2$ and $\tilde{p}_1(\vartheta) = \Gamma_1^3 P_1$ obtained from (22) and (24).

REMARK. If the system (1) has no delays in derivatives, a reduction of dimensionality is possible. Let $\kappa = \min(r, k)$. Then $H(t)$ is a $(\kappa n \times n)$-matrix consisting of the last $n$ columns and last $\kappa n$ rows of $\hat{C}_k$; $\Gamma_k^3$ and $\Gamma_k^4$ are replaced by $\bar{\Gamma}_k^3$ and $\bar{\Gamma}_k^4$, consisting of the last $\kappa n$ rows of the former ones respectively.

Interesting results are obtained under more severe assumptions which make possible to remove the derivatives from (28).

THEOREM 4. Let Assumption 1 be valid. Then the optimal control $u(t)$, $t \in [0, T]$, can be expressed in the form

$$u(t) = -U^{-1}(t)\, H'(t) \Big\{ \sum_{i=0}^{r} g_i^1(t)\, x(t - i\tau) + \sum_{i=0}^{r-1} \Big[ g_i^2(t)\, x(t - i\tau - \vartheta) + \\ + \int_{-\tau}^{0} \big( g_i^b(t, s)\, x(t - i\tau + s) + g_i^c(t, s)\, u(t - i\tau + s) \big)\, ds \Big] \Big\}, \tag{29}$$

$$g_0^1(t) = G(t) + G_0^a(t, 0), \tag{30}$$

$$g_i^1(t) = G_i^a(t, 0) - G_{i-1}^a(t, -\tau),\ i = 1, \dots, r-1,$$

$$g_r^1(t) = -G_{r-1}^a(t, -\tau),$$

$$g_i^2(t) = -\Gamma_k^3\, Q\, \check{a}_k^i(\tau) \text{ if } \vartheta \neq 0,\ i = 0, \dots, r-1,$$

$$g_i^2(t) = 0 \text{ if } \vartheta = 0,$$

$\check{a}_k^i$ consists of the $u$ upper rows of $a_k^i$,

$$g_i^b(t, s) = G_i^b(t, s) - \frac{\partial}{\partial s} G_i^a(t, s),\quad s \neq -\vartheta,\ i = 0, \dots, r-1,$$

$$g_i^c(t, s) = G_i^c(t, s).$$

Proof. The formula (29) is obtained by integration by parts of the appropriate right-hand side term in (28).

## 8. Optimal control in the open-loop system

Assuming $t_0 = 0$ we shall determine the optimal control $u$ in the interval $[0, T]$ as an explicit function of time. Formulas analogous to (28) or (29) can be derived, however, as they do not seem very useful, a numerical algorithm will be presented instead. This algorithm yields the optimal control and trajectory provided $k > 1$.

1. Determine an integer $k$, $(k-1)\tau < T \leqslant k\tau$, and $\vartheta = k\tau - T$.
2. Calculate $\Phi_k(\tau, \vartheta)$ and $\Phi_{k-1}(\vartheta, 0)$:

$$\frac{\partial}{\partial s} \Phi_{k-1}(s, 0) = \alpha_{k-1}(s)\, \Phi_{k-1}(s, 0),\quad \Phi_{k-1}(0, 0) = I,$$

$$\frac{\partial}{\partial s} \Phi_k(s, \vartheta) = \alpha_k(s)\, \Phi_k(s, \vartheta),\quad \Phi_k(\vartheta, \vartheta) = I.$$

3. Calculate the matrix $\Delta_k$ (23).
4. Compute

$$f_{k-1}(\vartheta) = \int_0^{\vartheta} \Phi_{k-1}(\vartheta, s)\, \bar{\delta}_{k-1}(s)\, w(s + \tau)\, ds,$$

$$f_k(\tau) = \int_{\vartheta}^{\tau} \Phi_k(\tau, s)\, \bar{\delta}_k(s)\, w(s)\, ds,$$

from

$$\dot{f}_{k-1}(s)=\alpha_{k-1}(s) f_{k-1}(s)+\bar{\delta}_{k-1}(s)\, w(s+\tau), f_{k-1}(0)=0,$$

$$\dot{f}_k(s)=\alpha_k(s) f_k(s)+\bar{\delta}_k(s)\, w(s), f_k(\vartheta)=0.$$

5. Compute $P_1$ and $P_2$ (23).
6. Solve the basic system (22) to obtain $\tilde{x}_{k-1}(\vartheta)$ and $\tilde{p}_k(\vartheta)$.
7. Calculate $z_k(\vartheta)=(\tilde{x}'_{k-1}(\vartheta),\ x'(0), \tilde{p}'_k(\vartheta))'$ and $z_k(s)$, $s \in [\vartheta, \tau]$, from (19).
8. Calculate $z_{k-1}(\vartheta)=(\tilde{x}'_{k-1}(\vartheta), \tilde{p}'_{k-1}(\vartheta))'$ and $z_{k-1}(s)$, $s \in [0, \vartheta]$, by backward integration of the first of Eqs. (19).
9. Determine the optimal trajectory and adjoint variable from (17), (6) and (7) and then — the optimal control (18), (6), (7).

## 9. Optimal control synthesis in case of infinite control time

We shall confine ourselves to constant systems, with constant coefficient matrices in the system equation and in the cost functional

$$\sum_{i=0}^{r} [A^i \dot{x}(t-i\tau)+B^i x(t-i\tau)+C^i u(t-i\tau)]=0,\ t \in [t_0, \infty), \tag{31}$$

$$S(u)=\lim S^R(u),\ R \to \infty,$$

$$S^R(u)=\int_{t_0}^{t_0+R} [x'(t)\, Wx(t)+u'(t)\, Uu(t)]\, dt+x'(t_0+R)\, Qx(t_0+R). \tag{32}$$

Assumptions on the coefficient matrices and initial conditions are the same as in (1), (2). Admissible controls are square summable in $[t_0, \infty)$.

Denote by $L^R$ the initial value (at time $t_0$) of the optimal feedback operator that minimizes $S^R$ subject to (31). As the system is constant, $L^R$ does not depend on $t_0$. By arguments similar to [7] it will be shown in a future paper that if the system (31) is asymptotically stable and/or controllable, $L^R \to L^\infty$ when $R \to \infty$. $L^\infty$ is a constant feedback operator that minimizes $S$ subject to (31). For convenience in computations $R$ should be an integer multiple of the delay, $R=k\tau$. It is advisable to calculate the operator $L^{k\tau}$ for several $k$'s. As $k$ increases $L^{k\tau}$ approaches a constant value. The calculations should be finished if $L^{k\tau}$ may be considered constant within a given accuracy. For every finite $k$ the operator $L^{k\tau}$ is determined according to the following algorithm.

1. Determine $\Phi_k(\tau, s)$, $s \in [0, \tau]$, by backward integration of $\frac{\partial}{\partial s}\ \Phi_k(\tau, s)= -\alpha_k(\Phi_k(\tau, s)$, $\Phi_k(\tau, \tau)=I$.
2. Calculate the matrix $\Delta_k$ (23).
3. Calculate $\Gamma_k^3$ amd $\Gamma_k^4$ (25).
4. Determine $L^{k\tau}$ according to (29).

## 10. Examples

*Example 1.* Determine the optimal control for the system

$$\dot{x}(t)=\dot{x}(t-2)+u(t),\quad t\in[0,3],$$

$$S(u)=x^2(3)+\int_0^3 u^2(t)\,dt,$$

as a function of the initial conditions $x(s)$, $s\in[-2,0]$.

The algorithm of Section 8 will be followed. We get $k=2$, $\vartheta=1$. The canonical equations have the form

$$\dot{z}_1(s)=\begin{pmatrix}0 & -1\\ 0 & 0\end{pmatrix} z_1(s)+\begin{pmatrix}1 & 0 & 0\\ 0 & 0 & 0\end{pmatrix} w(s+2),\quad s\in[0,1),$$

$$\dot{z}_2(s)=\begin{pmatrix}0 & 0 & -2 & -1\\ 0 & 0 & -1 & -1\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{pmatrix} z_2(s)+\begin{pmatrix}-1 & 0 & 0 & 0\\ 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{pmatrix} w(s),\quad s\in[1,2].$$

The basic algebraic equation is

$$\begin{pmatrix}1 & -3 & -1\\ 1 & 1 & 2\\ 0 & 1 & -1\end{pmatrix}\begin{pmatrix}\tilde{x}_1(1)\\ \\ \tilde{p}_2(1)\end{pmatrix}=\begin{pmatrix}\int_1^2 \dot{y}(s)\,ds\\ x(0)+\int_0^1 \dot{y}(s+2)\,ds+\int_1^2 \dot{y}(s)\,ds\\ 0\end{pmatrix}.$$

Finally we obtain the optimal control $u(t)=2V$, $t\in[0,1)$, $u(t)=V$, $t\in[1,3]$, $V=-\frac{1}{7}[x(0)+\int_{-1}^{0} x(s)\,ds]$.

*Example 2.* Consider a system

$$\dot{x}(t)=x(t-1)+u(t),$$

$$x(t)=1,\ t\leqslant 0,$$

$$S^k(u)=x^2(k)+\int_0^k u^2(t)\,dt,\ k>1,\ k\in N.$$

According to (29) the operator $L^k$ is determined by $g_0^1(0)$ and $g_0^b(0,s)$, $s\in[-1,0]$. The results in Table 1 show how $L^k$ approaches the constant value $L^\infty$. In each

**Table 1**

| $k$ \ $s$ | —.1 | —.8 | —.5 | —.2 | .0 |
|---|---|---|---|---|---|
| 2 | —.923 | —.830 | —.692 | —.554 | —.461 |
| 4 | —1.106 | —.987 | —.833 | —.703 | —.628 |
| 6 | —1.132 | —1.010 | —.852 | —.719 | —.642 |
| 8, 10 | —1.134 | —1.013 | —.854 | —.721 | —.643 |

case $g_0^1(0)=g_0^b(0,-1)$, so Table 1 contains only $g_0^b(0, s)$ for various $s$ and $k$. Table 2 contains optimal ($S^k_{opt}$) and suboptimal ($S^k_{sub}$) values of the functional $S^k$. The sub-optimal ones are obtained by the application of the feedback operator $L^k$.

**Table 2**

| $k$ | $S^k_{opt}$ | $S^k_{sub}$ |
|---|---|---|
| 2 | 2.826 | 3.020 |
| 6 | 3.518 | 3.525 |
| 10 | 3.527 | 3.528 |

## 11. Conclusions

The quadratic optimal control problem for a system described by linear neutral equations with constant commensurable delays in the trajectory, its derivative and control has been solved. An explicit formula for the optimal feedback control and a computational algorithm for the optimal control in the open-loop system have been presented. Due to use of the step method, the results have been obtained in a relatively simple form from the computational point of view. In case of in-finite control time a numerical algorithm for determining the optimal feedback operator has been constructed. The numerical results obtained hitherto suggest that the algorithms based on the step method are quicker and more accurate than those based on Riccati or Fredholm equations, at least for large values of $\tau/T$.

## References

1. Aggarwal J. K.: Computation of optimal control for time delay systems. *IEEE Trans. Automatic Control* **AC-15** (1970) 683—685.
2. Alekal Y., Brunovský P., Chyung D. H., Lee E. B.: The quadratic problem for systems with time delays. *IEEE Trans. Automatic Control* **AC-16** (1971) 673—687.
3. Banks H. T., Jacobs M. Q.: An attainable sets approach to optimal control of functional differential equations with function space terminal conditions. *J. Differential Equations* **13** (1973) 127—149.
4. Datko R.: Neutral autonomous functional equations with quadratic cost. *SIAM J. Control* **12** (1974) 70—82.
5. Delfour M. C., Mitter S. K.: Controllability, observability and optimal feedback control of affine hereditary systems. *SIAM J. Control* **10** (1972) 298—328.
6. Delfour M. C.: Numerical solution of the optimal control problem for linear hereditary differential systems with a linear-quadratic cost function and approximation of the Riccati differential equation. Report CRM-408, Centre de Recherches Mathématiques, Université de Montreal, June 1974.
7. Delfour M. C., McCalla C., Mitter S. K.: Stability and the infinite-time quadratic cost problem for linear hereditary differential systems. *SIAM J. Control* **13** (1975) 48—88.
8. Eller D. H., Aggarwal J. K., Banks H. T.: Optimal control of linear time-delay systems. *IEEE Trans. Automatic Control* **AC-14** (1969) 678—687.

9. KOIVO H. N., LEE E. B.: Controller synthesis for linear systems with retarded state and control variables and quadratic cost. *Automatica* 8 (1972) 203—208.
10. KORYTOWSKI A.: Sterowanie optymalne i suboptymalne w problemie kwadratowym z opóźnieniem. Proc. VI National Conf. on Automatic Control, Vol. 1. Poznań 1974.
11. KUSHNER H. J., BARNEA D. I.: On the control of a linear functional-differential equation with quadratic cost. *SIAM J. Control* 8 (1970) 257—272.
12. LIONS J. L.: Contrôle optimal de systèmes gouvernés par des équations aux dérivées partielles. Paris 1968.
13. MANITIUS A.: Optymalne sterowanie procesami z opóźnieniami zmiennych stanu. Ph. D. Thesis, Technical University of Warsaw 1968.
14. MANITIUS A.: Optimum control of linear time-lag processes with quadratic performance indexes. Proc. IV Cong. IFAC, Warszawa 1969.
15. ROSS D. W., FLÜGGE-LOTZ J.: An optimal control problem for systems with differential-difference equation dynamics. *SIAM J. Control* 4 (1969) 609—623.
16. ROSS D. W.: Controller design for time-lag systems via a quadratic criterion. *IEEE Trans. Automatic Control* **AC-16** (1971) 664—672.
17. SOLIMAN M. A., RAY W. H.: Optimal feedback control for linearquadratic systems having time delays. *Int. J. Control* 15 (1972) 609—627.
18. WIERZBICKI A. P., HATKO A.: Computational methods in Hilbert space for optimal control problems with delays. Proc. V Conf. on Optimization Techniques, R. Conti and P. Ruberti, eds., 304—318. New York 1973.

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## Sterowanie optymalne zadania kwadratowego dla układu opisywanego równaniami neutralnymi

Zastosowano metodę krokową dla rozwiązania zadania sterowania optymalnego układu liniowego opisywanego liniowymi równaniami neutralnymi z kwadratowym wskaźnikiem jakości. Opóźnienia są stałe i współmierne. Wyprowadzono wyrażenie na optymalny operator sprzężenia zwrotnego. Dla układu otwartego zaproponowano algorytm numeryczny. W przypadku nieskończonego czasu sterowania podano również algorytm obliczeniowy dla optymalnego operatora sprzężenia zwrotnego.

## Оптимальное управление для квадратической задачи в случае системы описываемой нейтральными уравнениями

В статье используется шаговый метод для решения задачи оптимального управления линейной системы, описываемой линейным нейтральными уравнениями с квадратическим показателем качества. Запаздывания являются постоянными и соизмеримыми. Выводится формула для оптымального оператора обратной связи. Для разомкнутой системы предлагается численный алгоритм. Для случая бесконечного времени управлензя дается также алгоритм вычисления оптимального оператора обратной саязи.

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