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Convergence Conditions for the Interaction Balance Algorithm Based on an Approximate Mathematical Model

by

ANDRZEJ RUSZCZYŃSKI

Technical University of Warsaw
Institute of Automatic Control

In the paper a new algorithm of real process coordination is presented. The algorithm is based on mathematical models of subsystems and takes advantage of the feedback from the real system. Convergence of the algorithm has been proved under Kantorovich-type conditions and certain assumptions limiting the difference between the mathematical model and the real system. All considerations are made in Hilbert spaces. Encouraging computational results have been obtained.

1. Introduction

A problem of finding the optimal control for a complex system consisting of several subsystems interconnected in a certain way can be solved by well known multilevel methods of optimization. In practice however mathematical models at our disposal are only an approximate image of reality. Consequently the model-optimal control will not be optimal for the real system (in general).

The Interaction Balance Method with Feedback (IBMF), shortly presented in sec. 3 of the paper, is one of the attempts to adapt multilevel methods to real process coordination. The method (firstly suggested in [1] and investigated in [4]) is a base for our further considerations. It will be shown that the coordination condition for this method is an operator equation which is extremely difficult to solve. The main difficulties are: incomplete knowledge of the real system equations and nondifferentiability of certain mappings involved. Classical methods of solving operator equations, as for instance Newton-like methods, cannot be applied in this case. Nevertheless, it is possible to construct an algorithm which can solve our equation (sec. 4). It is based on the ideas suggested in [6]. Moreover, convergence of the algorithm may be proved while imposing certain conditions on the difference between the mathematical model and the real system. The convergence analysis is carried on in sec. 6 after the examination of properties of mappings used for purpose of this analysis. Conditions formulated in theorem 2 resemble the Kantorovich condi-

tions for the Newton method. The convergence theory — initially finite-dimensional space oriented — turned out to be valid in Hilbert spaces. In sec. 7 results of some numerical experiments are presented.

2. Formal description of the system

We shall consider a system consisting of several subsystems (objects) being mutually interrelated in some way. We can distinguish 3 types of variables in each subsystem, namely: u^i — local inputs (originating from other subsystems), c^i — local controls, y^i — local outputs (upper index i indicates the subsystem number). Let equations of the subsystems have the following form:

$$y^i = f_*^i(c^i, u^i), \quad i=1, 2, ..., N, \tag{1}$$

where $y^i \in Y^i$, $c^i \in C^i$, $u^i \in U^i$, $f_*^i: C^i \times U^i \to Y^i$ and Y^i, C^i, U^i are Hilbert spaces.

Let us assume that the equations (1) are not exactly known to us and we have at our disposal only approximate models of the subsystems in the form as follows:

$$y^i = f^i(c^i, u^i), \quad i=1, 2, ..., N, \tag{2}$$

whereas $f^i: C^i \times U^i \to Y^i$ and $f^i \neq f_*^i$ in general.

Furthermore we assume that the structure of both real system and model is described by the following equations:

$$u^i = \sum_{j=1}^{N} M_j^i y^j, \quad i=1, 2, ..., N, \tag{3}$$

where $M_j^i: Y^j \to U^i$ is a bounded linear operator.

Let us take for brevity: $c=(c^1, c^2, ..., c^N)$, $u=(u^1, u^2, ..., u^N)$, $y=(y^1, y^2, ..., y^N)$ whereas $c \in C = \mathop{X}_{i=1}^{N} C^i$, $u \in U = \mathop{X}_{i=1}^{N} u^i$, $y \in Y = \mathop{X}_{i=1}^{N} Y^i$. Thus we can write the equations (1) through (3) in the compact form:

real objects:

$$y = f_*(c, u) \tag{1a}$$

mathematical models of objects:

$$y = f(c, u) \tag{2a}$$

system structure:

$$u = My. \tag{3a}$$

The set of equations (1a), (3a) gives description of the real system and the set of equations (2a), (3a) gives description of the mathematical model at our disposal. We assume that the system has a defined performance index in the form of

$$Q(c, u) = \sum_{i=1}^{N} Q^i(c^i, u^i) \tag{4}$$

where $Q: C \times U \to R^1$, $Q^i: C^i \times U^i \to R^1$.

The problem of providing the optimal control consists in finding such $\hat{c}$ which satisfies the following conditions:

1° there is $u=u_*(\hat{c})$ which satisfies (1a), (3a)

2° $(\hat{c}, u_*(\hat{c})) \in CU \subset C \times U$

where CU is the feasible set for the real system

3° for each c satisfying 1° and 2° there is

$$Q(\hat{c}, u_*(\hat{c})) \leqslant Q(c, u_*(c)).$$

Concerning the constraints which appear under 2° let us assume that they can be written in the form of local constraints:

$$((c, u) \in CU) \Leftrightarrow \forall i((c^i, u^i) \in (CU^i)). \tag{5}$$

The problem formulated above is extremely difficult to solve due to incomplete knowledge of the equations (1a). Therefore we will search for suboptimal solutions from which, however, we will require to satisfy the constraints (condition 2°). At the beginning let us assume uniqueness of the solutions of model and system equations:

A1)
$$\forall (c \in P_c(CU))\ \exists!\ (u_m(c) \in U)\ \exists!\ (u_*(c) \in U)$$
$$u_m(c) = Mf(c, u_m(c))$$
$$u_*(c) = Mf_*(c, u_*(c))$$

where P_c: $C \times U \to C$ is the projection onto C. Quantifier $\exists!$ means: "there is exactly one".

3. Interaction Balance Method with Feedback (IBMF)

The IBMF proposed in [1] employs forms of objects (4) and constraints (5) which are convenient for decomposing the problem and then carrying on the coordination. They afford possibility to formulate independent local problems of the lower level as follows:

I) for a given $p=(p^1, p^2, ..., p^N) \in U$ find

$$\min_{c^i, u^i \in CU^i} [L^i(c^i, u^i, p) = Q^i(c^i, u^i) + \langle p^i, u^i \rangle - \sum_{j=1}^{N} \langle p^j, M_i^j f^i(c^i, u^i) \rangle] \tag{6}$$

where p (prices) are coordinating variables. Meaning of these variables will be soon fully discussed. Let the solution of the set of problems I) (for $i=1, 2, ..., N$) be a pair $(\bar{c}(p), \bar{u}(p))$. The controls $\bar{c}(p)$ obtained in such a way are applied to the real system which results in establishing of interactions (flows) $u_*(\bar{c}(p))$. The task of upper level (coordinator) is

II) find $\tilde{p} \in U$ being such that

$$\bar{u}(\tilde{p}) = u_*(\bar{c}(\tilde{p})). \tag{7}$$

The principle of IBMF is illustrated with Fig. 1. It is obvious that after completing the coordination (satisfying the condition (7)) the controls $\bar{c}(\tilde{p})$ will not be the optimal ones since the local problems I) are based on approximate models (2). All the same the condition under 2° will be satisfied — it means that the IBMF control will be feasible for the real system. Furthermore the IBMF control has substantially better properties as far as the optimality is concerned than the open loop control.

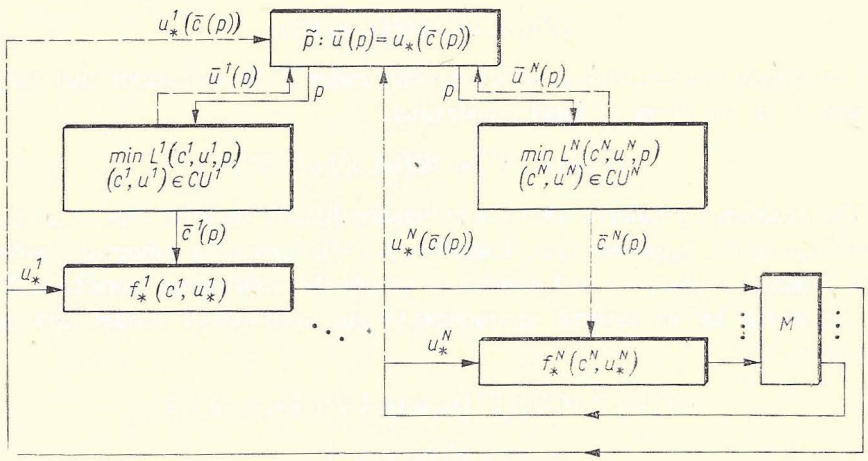

Fig. 1. The principle of IBMF

We will not stay longer at this subject-matter; a mathematical analysis of it and some numerical results are contained in [4].

The question which will be answered in the present work is how to find the coordinating values $\tilde{p}$.

In order to simplify the notation let us denote $x=(c, u)$, $x \in X = C \times U$ and in the similar fashion $\bar{x}(p) = (\bar{c}(p), \bar{u}(p))$. We shall write $Q(x)$ meaning $Q(c, u)$. We introduce the operators

$$G: X \to U, \quad G(c, u) = u - Mf(c, u),$$

$$D: X \to U, \quad D(c, u) = u - u_m(c),$$

$$D_*: X \to U, \quad D_*(c, u) = u - u_*(c).$$

While using such denotations the local problems of IBMF may be written in the form

Ia) $$\min_{x \in CU} [Q(x) + \langle p, G(x) \rangle] \tag{8}$$

and the coordinator problem in the form

IIa) find $\tilde{p}$ so that

$$D_*(\bar{x}(\tilde{p})) = 0 \tag{9}$$

where $\bar{x}(p)$ is the solution of (8).

4. An Algorithm of Coordination

The equation (9) is an operator equation provided that the lower level solutions $\bar{x}(p)$ are unique for p from a certain neighbourhood of p^o. This equation however is very difficult to solve since:

a) we know a little of the operator D_* — only its approximation is known

b) the function $\bar{x}(\cdot)$ is unsufficiently regular. Even under assumption that functions f and Q are repeatedly differentiable the $\bar{x}(\cdot)$ may not be differentiable at all if there are any inequality constraints in the definition of the set CU.

Therefore we take the following assumptions:

A2) Functions f and Q are continuously twice Frechet differentiable with respect to both variables together.

A3) For each $c \in P_c(CU)$ the operator $F: U \to U$, $F = I - Mf_u(c, u_m(c))$ has a bounded inverse. The symbol f_u indicates the Frechet derivative with respect to u.

A4) Feasible sets are expressible in the form

$$CU^i = \{(c^i, u^i): h_j(c^i, u^i) \leqslant 0, \quad j \in J^i\}, \quad i = 1, \ldots, N,$$

where $h_j: c^i \times U^i \to R^1$ are continuously twice Frechet differentiable convex functionals and J^i is a finite set of indexes.

In order to define the algorithm of coordination we shall generalize some ideas presented in [4], [6]. They lead to the following procedure:

1) Basing on the mathematical model solve the equation

$$D(\bar{x}(p)) = 0$$

or, which is equivalent, the equation

$$G(\bar{x}(p)) = 0$$

It can be done by the classical price method (see [1]).

2) Use the prices p^o obtained in this way for finding $x^o = \bar{x}(p^o)$ and set $D_0 = D_x(x^o)$, $G_0 = G_x(x^o)$.

3) Select successive prices according to the following principle

$$p^{k+1} = p^k + \varepsilon E_0 D_*(\bar{x}(p^k)) \tag{10}$$

where

$$E_0 = (D_0 A G_0^*)^{-1} \tag{11}$$

and A is a self-adjoint operator, strongly positively defined on $\mathscr{R}(G_0^*)$. ($\mathscr{N}(\cdot)$, $\mathscr{R}(\cdot)$ denote the null space and the range of a linear operator respectively and G_0^* denotes the adjoint operator of G_0, $G_0^*: U \to X$). The number ε is sufficiently small and is constant during the coordination.

It may be proved that A3 implies $\mathscr{R}(G_0) = U$ and, consequently, $\overline{\mathscr{R}(G_0^*)} = \mathscr{R}(G_0^*)$ (the closed range theorem, [5]). Next, it is possible to demonstrate that the operator E_0 defined by (11) is well defined and bounded (Banach's inverse operator theorem, [5]). We shall omit particulars of the proofs.

The central problem which arises now is the question of convergence of the above algorithm. It will be discussed in sec. 6 and now we will recall very shortly properties of the function $\bar{x}(\cdot)$ which will be used later on. We will make also assumptions being necessary for further considerations.

5. Relevant assumptions

Our further considerations will be limited to prices from a certain neighbourhood $\Omega(p^o)$ of the initial prices. The lower level variables are assumed to attain values from a certain neighbourhood $\omega(x^o)$ of the initial solution. We assume that

A5) $$\exists(\lambda>0)\ \forall\big(x\in CU\setminus\omega(x^o)\big)$$
$$Q(x)+\langle p^o, G(x)\rangle\geqslant Q(x^o)+\langle p^o, G(x^o)\rangle+\lambda$$

and

A6) $$\exists(\vartheta>0)\ \forall\big(x\in\omega(x^o)\big)\ \forall\big(p\in\Omega(p^o)\big)$$
$$Q_{xx}(x)+\langle p, G(x)\rangle_{xx}\geqslant\vartheta I$$

i.e. lower level problems are uniformly convex in this sets.

The assumptions A5 and A6 imply that for p being sufficiently close to p^o there is a continuous function $\bar{x}(\cdot)$ the values of which are minima of the function (8) as well as $\bar{x}(p)\in\omega(x^o)$ for these p. Therefore we may assume that the sets $\Omega(p^o)$, $\omega(x^o)$ are chosen so as to satisfy the relation $\bar{x}\big(\Omega(p^o)\big)\subset\omega(x^o)$.

Let $p_1, p_2\in\Omega(p^o)$ and $p_t=tp_2+(1-t)p_1$ at $0\leqslant t\leqslant 1$. We shall consider the behaviour of $\bar{x}(p_t)$ as the function of a real variable t. Let us define for $p\in\Omega(p^o)$ the set

$$I^o(p)=\{i\colon h_i\big(\bar{x}(p)\big)=0\}.$$

Let $I^o(p)=\{i_1, i_2, \ldots, i_l\}$. We define now the operator $H\colon\Omega(p^o)\times X\to R^l$ as follows

$$H(p, x)=\big(h_{i_1}(x), h_{i_2}(x), \ldots, h_{i_l}(x)\big).$$

The next assumption is typical for such kind of problems:

A7) For $p\in\Omega(p^o)$ the derivative H_x of the operator $H(p, x)$ with respect to x taken at the point $(p, \bar{x}(p))$ is an epimorphism.

In other words we assume that at $\bar{x}(p)$ the gradients of active constraints are linearly independent. This asumption together with the previous ones give us the warranty for existence and uniqueness of Lagrange multipliers for lower level problems (see [3]). Let $\mu\in R^l$ be these multipliers. Let us introduce, for fixed p, the operator

$$W(p)\colon X\to X,\ W(p)=Q_{xx}(\bar{x}(p))+\langle p, G(x)\rangle_{xx}|_{(p,\bar{x}(p))}+\langle\mu, H(p, x)\rangle_{xx}|_{(p,\bar{x}(p))}.$$

Due to convexity of constraints and inequality $\mu\geqslant 0$, the operator $\langle\mu, H(p, x)\rangle_{xx}$ is positively semi-defined. Therefore, according to A6, the operator $W(p)$ with $p\in\Omega(p^o)$ is positively defined.

Let us assume that the set $I^o(p_t)$ remains constant within an open interval $(t_0-\tau, t_0+\tau)$. Then by means of direct transformations and the implicit function theorem [7] we can prove that the function $\bar{x}(p_t)$ is differentiable with respect to t at the point t_0 and that

$$\frac{d\bar{x}(p_t)}{dt}(t_0) = -B(p_{t_0})\, G_x^*(\bar{x}(p_{t_0}))\,(p_2-p_1) \tag{12}$$

where

$$B(p) = \begin{cases} W^{-1} & \text{for} \quad I^o(p)=\emptyset \\ W^{-1} - W^{-1} H_x^* (H_x w^{-1} H_x^*)^{-1} H_x w^{-1} & \text{for} \quad I^o(p)\neq\emptyset. \end{cases}$$

For brevity we have denoted $w = w(p)$, $H_x = H_x(p, \bar{x}(p))$. Furthermore, as it is shown in [4], the operator $B(p)$ is nonnegative defined, $\|B(p)\| \leqslant \|w^{-1}(p)\|$ and if $I^o(p)\neq\emptyset$ then

$$N(B(p)) = \mathscr{R}(H_x^*) \tag{13}$$

Next we assume that

A8) For any $p_1, p_2 \in \Omega(p^o)$ the section

$$[p_1, p_2] = \{p_t = tp_2 + (1-t)\,p_1,\ 0 \leqslant t \leqslant 1\}$$

can be divided into a countable number of sub-intervals so as within each of them the set $I^o(p_t)$ is constant.

This assumption means simply that we are dealing with the model having such property that the solution $\bar{x}(p_t)$ jumps countable number of times from one wall of the feasible set to another. It should be stressed that the above assumption is purely technical. It is necessary for some mathematical considerations relating to the formula

$$\bar{x}(p_{t_2}) - \bar{x}(p_{t_1}) = \int_{t_1}^{t_2} \frac{d\bar{x}(p_t)}{dt}\, dt$$

(see [7]). From practical point of view the assumption A8 makes no essential restrictions on the problem. It is very difficult to construct the model of the system, the constraints and the cost function which do not satisfy A8. It seems impossible that such a model would be constructed in any practical problem. Finally it should be stressed again that A8 refers to the mathematical model only.

We shall assume also that in $\Omega(p^o)$ active constraints and the equations of the model show the property of uniform linear independence:

A9)
$$\exists(\delta>0)\,(p\in\Omega(p^o))\ \forall\big(v\in\mathscr{R}(G_x^*(\bar{x}(p^o)))\big)$$
$$d\big(v, \mathscr{R}(H_x^*(p, \bar{x}(p)))\big) \geqslant \delta\,\|v\|$$

where $d\colon X\times 2^x \to R^1$ is the distance between the point and the set involved.

Roughly speaking, this assumption has as its purpose to give a sufficient freedom in manipulation with values $\bar{x}(p)$ through changes in p as it is seen from (12) and (13).

The assumption A9 needs some comments. We assume that $\mathscr{R}(G_x^*) \cap \mathscr{R}(H_x^*) = \{0\}$. It means that the equation

$$G_x^* \, du + H_x^* \, dh = 0$$

has the unique solution $du=0$, $dh=0$ in $U \times R^l$. The ranges of G_x^* and H_x^* being closed, the above is equivalent to the solvability of the set of equations

$$G_x \, dx = du$$

$$H_x \, dx = dh$$

for any $du \in U$, $dh \in R^l$ (see [5]). The last property in the finite dimensional case means that the gradients of system equations (rows of the matrix G_x) and the gradients of active constraints (rows of the matrix H_x) are linearly independent. It is commonly assumed in various works in the field [3]. (System equations are equality constraints for the problem). We assume little more: the angle between the two subspaces spanned by the two groups of gradients is always greater than a positive angle $\varphi = \arcsin \delta$.

6. Convergence analysis

Like many other methods for solving nonlinear operator equations the algorithm (10) will converge under certain assumptions limiting the distance between the initial point and the solution. However, all assumptions which limit this distance directly seem to be of little value, because the solution is not known. Therefore we shall avoid assumptions of this kind. We shall try to prove convergence after the lines of the famous Kantorovich proof made for the Newton method. (see [2]).

Let us denote

$$\begin{aligned} V_*(p) &= p + \varepsilon E_0 \, D_*\big(\bar{x}(p)\big) \\ V(p) &= p + \varepsilon E_0 \, D\big(\bar{x}(p)\big) \end{aligned} \tag{14}$$

It will be demonstrated that the operator V_* has contraction mapping properties. We define in U a new scalar product as follows

$$\langle u, v \rangle_0 = \langle G_0^* u, A G_0^* v \rangle$$

It can be easily verified (A is positive on $\mathscr{R}(G_0^*)$) that it satisfies all scalar product axioma and that the norm $\|\cdot\|_0$ induced by it is equivalent to the original one in U.

THEOREM 1. Let the assumptions A1 through A10 be satisfied and let

(i) $\exists (m_A > 0, M_A > 0) \, \forall \big(x \in \mathscr{R}(G_0^*)\big)$

$$m_A \|x\|^2 \leqslant \langle x, Ax \rangle \leqslant M_A \|x\|^2$$

(ii) $\exists (M_w > 0) \, \forall \big(p \in \Omega(p^o)\big) \, \forall (z \in X)$

$$\langle z, W(p) z \rangle \leqslant M_w \|z\|^2$$

(iii) $\exists (L>0)\ \forall (p_1, p_2 \in \Omega(p^o))$

$$\|D_x(\bar{x}(p_1))\, B(p_2)\, G_x^*(\bar{x}(p_1)) - D_x(\bar{x}(p_2))\, B(p_2)\, G_x^*(\bar{x}(p_2))\|_0 \leqslant L\, \|p_1 - p_2\|_0$$

Then the operator V defined by (14) satisfies the following inequalities

$$\|V(p_2) - V(p_1)\|_0 \leqslant (\alpha(\varepsilon) + \varepsilon\, \|E_0\|_0\, L \max(\|p_2 - p^o\|_0, \|p_1 - p^o\|_0))\, \|p_2 - p_1\|_0 \tag{15}$$

$$\|V(p) - p^o\| \leqslant \tfrac{1}{2}\, \varepsilon\, \|E_0\|_0\, L\, \|p - p^o\|_0^2 + \alpha(\varepsilon)\, \|p - p^o\|_0 \tag{16}$$

where $p, p_1, p_2 \in \Omega(p^o)$ and

$$\alpha(\varepsilon) = \left(1 - \varepsilon \frac{2\delta^2}{M_A\, M_w} + \varepsilon^2 \frac{1}{w^2\, m_A^2}\right)^{1/2}.$$

Proof. We define the function $V_0 \colon U \times U \to U$ as follows

$$V_0(p_1, p_2) = \left(I - \varepsilon E_0\, D_0 \int_0^1 B(p_t)\, dt\, G_0^*\right)(p_2 - p_1) \tag{17}$$

where $p_t = tp_2 + (1-t)\, p_1$. Existence of the integral $\int_0^1 B(p_t)\, dt$ results from A8. It has been proved in [6] that

$$\|V_0(p_1, p_2)\|_0 \leqslant \alpha(\varepsilon)\, \|p_1 - p_2\|_0$$

if $p_1, p_2 \in \Omega(p^o)$. Now it is easy to prove (15) and (16). We have

$$V(p_2) - V(p_1) = V_0(p_1, p_2) +$$

$$+ \varepsilon E_0 \int_0^1 [D_0\, B(p_t)\, G_0^* - D_x(\bar{x}(p_t))\, B(p_t)\, G_x^*(\bar{x}(p_t))](p_2 - p_1)\, dt \tag{18}$$

From (18), (iii) and (17) we obtain

$$\|V(p_2) - V(p_1)\|_0 \leqslant \alpha(\varepsilon)\, \|p_2 - p_1\|_0 + \varepsilon\, \|E_0\|_0 \int_0^1 L\, \|p_t - p^o\|_0\, dt\, \|p_2 - p_1\|_0. \tag{19}$$

The formulae (15) and (16) immediately result from (19) because

$$\int_0^1 \|p_t - p^o\|_0\, dt \leqslant \max(\|p_1 - p^o\|_0, \|p_2 - p^o\|_0)$$

and for $p_2 = p^o$

$$\int_0^1 \|p_t - p^o\|_0\, dt = \tfrac{1}{2}\, \|p_1 - p^o\|_0^2$$

The theorem has been proved.

Now we shall call the inaccuracy of the model into considerations. Let us denote

$$\tilde{D}(x) = D_*(x) - D(x)$$

THEOREM 2. Let the assumptions of theorem 1 hold and let for $p, p_1, p_2 \in \Omega(p^o)$

(i) $$\|\tilde{D}(\bar{x}(p_1)) - \tilde{D}(\bar{x}(p_2))\|_0 \leqslant \tilde{L}\|p_1 - p_2\|_0$$

(ii) $$\|\tilde{D}(\bar{x}(p))\|_0 \leqslant a$$

We set

$$b_0 = \|E_0\|_0$$

$$h = ab_0^2 L$$

$$\xi = (1 - \alpha(\varepsilon))/\varepsilon$$

$$r_1 = \frac{\xi - \sqrt{\xi^2 - 2h}}{b_0 L}$$

$$r_2 = \frac{\xi + \sqrt{\xi^2 - 2h}}{b_0 L}$$

Let furthermore the following conditions be satisfied

(iii) $$\sqrt{b_0^2 \tilde{L}^2 + 2h} < \xi$$

(iv) $$\bar{S}(p^o; r_1) = \{p : \|p - p^o\|_0 \leqslant r_1\} \subset \Omega(p^o)$$

Then

1° The equation (9) has the solution $\tilde{p}$ in $\bar{S}(p^o; r_1)$.

2° The sequence (10) converges to $\tilde{p}$ and the following inequalities hold

$$\|p^{k+1} - \tilde{p}\|_0 \leqslant q\|p^k - \tilde{p}\|_0$$

$$\|p^{k+1} - \tilde{p}\|_0 \leqslant \frac{\varepsilon}{1-q}\|E_0 D_*(\bar{x}(p^k))\|_0$$

where

$$q = \alpha + \varepsilon b_0 L r_1 + \varepsilon b_0 \tilde{L} < 1.$$

3° The solution $\tilde{p}$ is unique in $\Omega(p^o) \cap S(p^o; r_2)$ and the sequence $\bar{p}^{k+1} = V_*(\bar{p}^k)$ converges to it for all $\bar{p}^o$ such that $r_3 = \|\bar{p}^o - p^o\|_0 < r_2$ and $\bar{S}(p^o; r_3) \subset \subset \Omega(p^o)$.

Proof. Let $p_1, p_2 \in \bar{S}(p^o; r_1)$. We have

$$\|V_*(p_1) - V_*(p_2)\|_0 \leqslant \|V(p_1) - V(p_2)\|_0 + \varepsilon\|E_0\|_0 \|\tilde{D}(\bar{x}(p_1)) - \tilde{D}(\bar{x}(p_2))\|_0. \tag{20}$$

The first constituent of the right hand side of the above inequality may be estimated according to theorem 1 by (15). Taking into account assumption (ii) we obtain from (20)

$$\|V_*(p_1) - V_*(p_2)\|_0 \leqslant (\alpha + \varepsilon b_0 L r_1 + \varepsilon b_0 \tilde{L})\|p_1 - p_2\|_0 \tag{21}$$

On the other hand, if $p \in \bar{S}(p^o; r_1)$ then

$$\|V_*(p) - p^o\|_0 \leqslant \|V_*(p) - V(p)\|_0 + \|V(p) - p^o\|_0.$$

It follows from the assumption (ii) and (15) that the following inequality holds

$$\|V_*(p)-p^o\|_0 \leqslant \varepsilon b_0\, a+\tfrac{1}{2}\,\varepsilon b_0\, L r_1^2+\alpha r_1\,. \tag{22}$$

Moreover, r_1 is a root of the trinomial

$$w(r)=\tfrac{1}{2}\,\varepsilon b_0\, L r^2-(1-\alpha)\, r+\varepsilon b_0\, a$$

and therefrom for $p \in \bar{S}(p^o; r_1)$

$$\|V_*(p)-p^o\|_0 \leqslant r_1\,.$$

We have demonstrated that in $\bar{S}(p^o; r_1)$

$$\|V_*(p_1)-V_*(p_2)\|_0 \leqslant q\|p_1-p_2\|_0$$

and that

$$V_*(\bar{S}(p^o; r_1)) \subset \bar{S}(p^o; r_1)\,.$$

Inequality $q<1$ results immediately from (iii). Applying the contraction mapping theorem we obtain the points 1° and 2° of our proposition. It remains only to prove the point 3°.

Let $r_1<\|p-p^o\|_0<r_2$. Then

$$\|V_*(p)-p^o\|_0<\|p-p^o\|_0 \tag{23}$$

because for $r_1<r<r_2$ the trinomial $w(r)$ attains negative values. Therefore p cannot be a fixed point of V_*. Let $r_1<\|\bar{p}^o-p^o\|_0=r_3<r_2$ and $\bar{S}(p^o; r_3) \subset \Omega(p^o)$. If for any i there is $\|\bar{p}^i-p^o\|_0 \leqslant r_1$ convergence will result from 2°. Otherwise it follows from the inequality (23) that the sequence $\{\|\bar{p}^k-p^o\|_0\}$ converges down to r_1. Then the sequence $\{\bar{p}^k\}$ remains in $\Omega(p^o)$ and we have

$$\|\tilde{p}-\bar{p}^{k+1}\|_0=\|V_*(\tilde{p})-V_*(\bar{p}^k)\|_0 \leqslant (\alpha+\varepsilon b_0\, L\|\bar{p}^k-p^o\|_0+ \\ +\varepsilon b_0\, \tilde{L})\,\|\tilde{p}-\bar{p}^k\|_0=\bar{q}\,\|\tilde{p}-\bar{p}^k\|_0$$

where

$$\bar{q}=q+\varepsilon b_0\, L(\|\bar{p}^k-p^o\|_0-r_1)\,.$$

It is obvious that for sufficiently large k it will be $\bar{q}<1$ and $\{\bar{p}^k\} \to \tilde{p}$.

The theorem has been proved.

Let us discuss briefly the meaning of the assumptions of theorem 2. The assumptions (i) and (ii) require from the difference $\tilde{D}$ between the system and the model to be bounded and Lipschitz continuous. The condition (iii) limits the upper bound a and the Lipschitz constant $\tilde{L}$. It is obvious that a condition of this kind must be included. If we want to coordinate the real system basing on the model we must have a sufficiently good model.

Let us note that convergence conditions for the above theorem are greatly similar to those for Newton method (see [2]). It is worth to bring out into relief that the equation under consideration is far more difficult to solve than those ones previously investigated. In particular there is no assumption of differentiability of any of the functions $D_*(\bar{x}(p))$ and $D(\bar{x}(p))$ with respect to p made in our considera-

tions. If the function $D(\bar{x}(\cdot))$ (obtained from the model) were differentiable it would be possible to apply the method suggested in [8] to the equation (9). The results obtained there are of the same nature as the conditions of theorem 2. In our case however the function $D(\bar{x}(\cdot))$ may be nondifferentiable which makes the task of solving the equation (9) more difficult. Perhaps it is possible to find a certain differentiable approximation of the function $D_*(\bar{x}(\cdot))$ and then apply the results of Zinchenko. But it seems rather difficult. We have chosen another way — taking advantage of specific properties of our problem.

7. Some computational results

In order to test the suggested algorithm the following example of a system has been considered:

subsystem 1:

$$y^1 = c_1^1 - c_2^1 + 2u^1 + A_1 (c_1^1)^2 + A_2 (c_1^1 + c_2^1 - 2) u^1$$

$$Q^1 (c_1, u^1) = (u^1 - 1)^2 + (c_1^1)^2 + (c_2^1 - 2)^2$$

$$CU^1 = \{(c_1^1, c_2^1, u^1) : c_1^1 + u^1 \leqslant 1.007\}$$

subsystem 2:

$$y_1^2 = c_1^2 - c_2^2 + u_1^2 - 3u_2^2$$

$$y_2^2 = 2c_2^2 - c_3^2 - u_1^2 + 2u_2^2$$

$$Q^2 (c^2, u^2) = 2 (c_1^2 - 2)^2 + (c_2^2)^2 + 3 (c_3^2)^2 + 4 (u_1^2)^2 + (u_2^2)^2$$

$$CU^2 = R^5$$

subsystem 3:

$$y^3 = c^3 - 4u^3 + A_3 c^3 u^3$$

$$Q^3 (c^3, u^3) = (c^3 + 1)^2 + (u^3 - 1)^2$$

$$CU^3 = \{(c^3, u^3) : c^3 + u^3 \geqslant -0.5\}$$

The structure of the above system is shown in Fig. 2. In a mathematical model at our disposal there were: $A_1 = 0$, $A_2 = 0$, $A_3 = 0$. Due to linearity of the model the

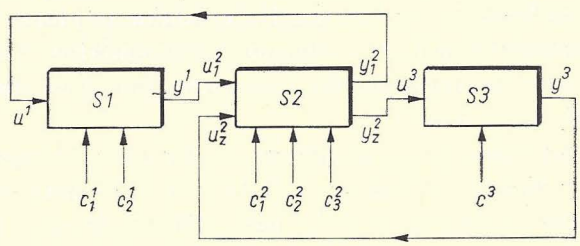

Fig. 2. The structure of tested systems

local minimizations (6) have been solved in an analytical way. The choice of $A=Q_{xx}^{-1}$ has been made (the detailed motivation of it in the linear case is contained in [4]). Various combinations of the values of A_1, A_2, A_3 in the real system have been tested. Representative are the six presented in Table 1. The introductory step of coordination — solution of the model equation $D(\bar{x}(p))=0$ — has been omitted, since in the linear model the derivatives $D_x(\bar{x}(p^o))$ and $G_x(\bar{x}(p^o))$ do not depend on p^o. The initial prices were $p^o=(0, 0, 0, 0)$, which corresponds to full decentralization. Numbers of iterations required to obtain the accuracy $\|\bar{u}(p)-u_*(\bar{c}(p))\leqslant 10^{-6}$ are presented in Table 1. Constraint in S3 was active in all iterations. Constraint in S1 has been chosen so as to be sometimes active and sometimes not. The process of norm minimizations is illustrated in Fig. 3.

Table 1. Number of iterations required to obtain the discoordination 10^{-6}

No.	A1	A2	A3	ε	Iterations	Constraint in S1
1	−0.5	0.5	−0.5	0.8	83	active in all steps
2	0.13	0.5	−0.5	0.8	16	active in first 11 steps
3	−0.5	−0.5	−0.5	0.8	24	active in all steps
4	0.5	0.5	0.5	0.8	16	inactive
5	0.13	0.5	−0.5	1.2	10	active in first 7 steps
6	0.13	0.5	−0.5	1.8	73	active in every second step up to the 23

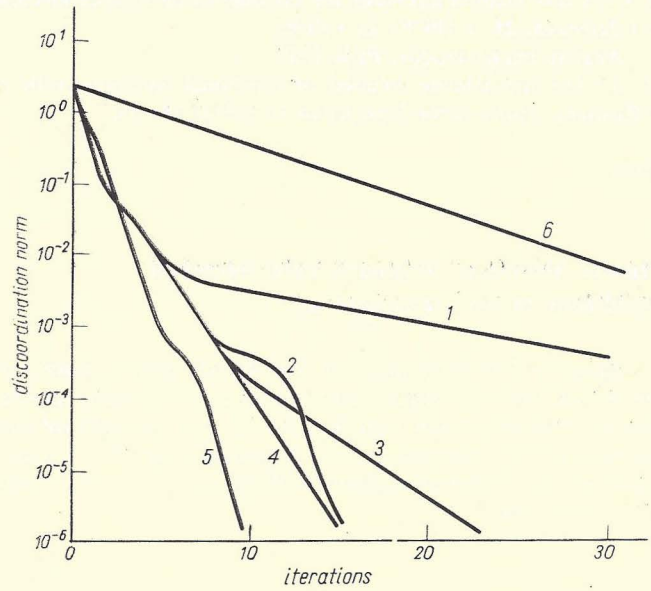

Fig. 3. The process of norm minimization under various disturbances

8. Final conclusions

The algorithm of real system coordination presented in this paper has several important features. It takes advantage of the measurements of real interactions, but its parameters ε and E_0 are constant and may be computed from the model. Although coordination equation is not differentiable convergence of the algorithm has been proved. The convergence is linear. The conditions obtained in the paper are greatly similar to those developed for Newton-like methods. Numerical experiments indicate that the algorithm is efficient.

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Warunki zbieżności algorytmu zrównoważenia interakcji opartego o przybliżony model matematyczny

W pracy przedstawiony jest nowy algorytm koordynacji procesu rzeczywistego. Algorytm oparty jest na modelach matematycznych podsystemów i wykorzystuje sprzężenie zwrotne od systemu rzeczywistego. Przy warunkach typu Kantorovica oraz pewnych założeniach ograniczających różnicę między modelem matematycznym a systemem rzeczywistym udowodniona została zbieżność tego algorytmu. Wszystkie rozważania prowadzone są w przestrzeniach Hilberta. Uzyskano zachęcające wyniki numeryczne.

Условия сходимости для алгоритма согласования взаимодействий основанного на приближенной математической модели

В статье представлен новой алгоритм координации действительного процесса. Алгоритм основан на математических моделях подсистем и использует обратную связь от реальной системы, При условиях типа Канторовича и некоторых предположенниях ограничивающих разность между математической моделю а действительной системой доказана сходимость этого алгоритма. Все рассуждения проводятся в гильбертовых пространствах. Получены обещающие численные результаты.