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External approximation of a parametric optimization problem for parabolic equations

by

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An optimal control problem for a system described by linear parabolic equation with mixed boundary conditions is considered. The coefficients of the equation depend on control function while the cost functional depends on the terminal state.
To find an approximate solution of this problem an external approximation (finite difference) method is used. It is shown that this approximation is convergent.
Theoretical results are illustrated by numerical examples.

Introduction

In the paper the approximation of a parametric optimization problem is considered [1]. In the problem coefficients of linear partial differential equation of parabolic type-state equation-depend on control function. Problems of such type appear in technology of solid state devices [6] and in identification theory [1] and were investigated by Bensoussan [1], Chavent [3], Sokołowski [9], [10], [11].

In the paper we prove convergence, in some sense, of external approximation for parametric optimization problem and then we give a numerical example.

In section 1 we recall some existence results concerning weak solutions of linear parabolic equations.

In section 2 we formulate parametric optimization problem and recall an existence result.

In section 3, 4 we define some spaces of steps functions over discretized domains and we recall some regularity results for discretization of domain due to Cea [2].

In section 5 we define external approximation of linear parabolic equation.

In section 6 we show that external approximation of parametric optimization problem is convergent in some sense.

In section 7 a numerical example is given.

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1. Preliminaries

Let Ω be given domain in R^n with enough regular boundary $\Gamma=\partial\Omega$.

By $V=H^1(\Omega)$ we denote Sobolew space of functions (class of functions) $y\in L^2(\Omega)$, such that $\frac{\partial y}{\partial x_i}\in L^2(\Omega)$, $i=1, \dots, n$, where $H=L^2(\Omega)$ is the usual Hilbert space of equivalence class of real-valued, square-integrable functions on Ω. By $V'=(H^1(\Omega))'$ we denote dual space to $H^1(\Omega)$.

Let E be a given Hilbert space, $L^2(0, T; E)$ is the Hilbert space of equivalence classes of E-valued, square-integrable functions on $[0, T]$.

By $W(0, T)$ we denote linear subspace of $L^2(0, T; V)$:

$$W(0,T)=\left\{f\in L^2(0,T;V)\ \middle|\ \frac{df}{dt}\in L^2(0,T;V')\right\}. \tag{1.1}$$

$W(0, T)$ is a Hilbert space [4] under the scalar product:

$$(y,z)_{W(0,T)}=\int_0^T\left[(y(s),z(s))_V+\left(\frac{dy}{dt}(s),\frac{dz}{dt}(s)\right)_{V'}\right]ds. \tag{1.2}$$

It is known [4], that imbedding

$$W(0,T)\subset C(0,T;H) \tag{1.3}$$

is continuous, where by $C(0, T; H)$ we denote the Banach space, of continuous mappings $[0, T]\mapsto H$ with usual sup norm.

On the space $H^1(\Omega)$ we define a bilinear form $(y, z)\mapsto a_r(t; y, z)$ depending on a real parameter r setting:

$$a_r(t;y,z)=\sum_{i=1}^n\int_\Omega a_i(r,x,t)\frac{\partial y}{\partial x_i}\frac{\partial z}{\partial x_i}\,d\Omega+\int_\Gamma g(r,x,t)\,yz\,d\Gamma,$$
$$\forall y,z\in H^1(\Omega),\qquad r\in R. \tag{1.4}$$

LEMMA 1. *Assume:*

$$\left.\begin{aligned}0<\alpha_0\leqslant a_i(r,x,t)\leqslant M\\ 0\leqslant g(r,x,t)\leqslant M\end{aligned}\right\}\ \begin{aligned}&\forall r\in[0,1]\\ &\forall(x,t)\in Q=\Omega\times]0,T[\end{aligned} \tag{1.5}$$

then

(i) $|a_r(t;y,z)|\leqslant C\|y\|_V\|z\|_V,\quad \forall y,z\in V=H^1(\Omega)\quad \forall t\in[0,T],\quad \forall r\in[0,1];$ (1.6)

(ii) $a_r(t;y,y)+\|y\|_H^2\geqslant\alpha\|y\|_V^2,\quad \alpha>0\quad \forall y\in V,\quad \forall r\in[0,1],\quad \forall t\in[0,T].$ (1.7)

Proof of the above lemma is elementary. Let there be given functions:

$$
\begin{aligned}
& u=u(x,t),\ 0\leqslant u(x,t)\leqslant 1 \text{ a.e. in } Q \\
& u\in L^\infty(Q) \\
& f\in L^2(0,T;V') \qquad (1.8) \\
& \varphi\in L^2(0,T;S'),\ S=H^{-1/2}(\Gamma) \\
& y_0\in H
\end{aligned}
$$

then [4] there exists the unique solution

$$y=y_u\in W(0,T) \tag{1.9}$$

of the following parabolic equation:

$$\left(\frac{dy}{dt}(t),z\right)_{V'V}+a_u(t;y(t),z)=(f(t),z)_{V'V}+(\varphi(t),\gamma z)_{S'S},\quad \forall z\in V$$

$$\text{a.e. in } [0,T] \tag{1.10}$$

$$y(0)=y_0 \tag{1.11}$$

where by $\gamma z\in H^{1/2}(\Gamma)=S$ we denote trace [4] of $z\in H^1(\Omega)$.

Remark 1. Formally problem (1.10), (1.11) can be interpreted as follows:

$$
\begin{aligned}
& \frac{\partial y}{\partial t}-\sum_{i=1}^{n}\frac{\partial}{\partial x_i}\left(a_i\frac{\partial y}{\partial x_i}\right)=f, && (x,t)\in Q \\
& \frac{\partial y}{\partial n}+gy=\varphi, && (x,t)\in\Sigma \qquad (1.12) \\
& y(x,0)=y_0(x), && x\in\Omega
\end{aligned}
$$

Remark 2. If we assume more regularity for function φ, that is $\varphi\in L^2(\Sigma)$, then duality pairing $(\cdot,\cdot)_{S'S}$ coincides with scalar product of $L^2(\Gamma)$.

Let us denote by $\bar{U}$ a subset of the Banach space $L^\infty(Q)$, given by

$$\bar{U}=\{u\in L^\infty(Q)|0\leqslant u(x,t)\leqslant 1,\ \text{a.e. in } Q\} \tag{1.13}$$

Lemma 2. *Assume mappings*

$$[0,1]\ni r\mapsto a_i(r,\cdot,\cdot)\in L^\infty(Q),\ i=1,\dots,n \tag{1.14}$$

$$[0,1]\ni r\mapsto g(r,\cdot,\cdot)\in L^\infty(\Sigma), \tag{1.15}$$

to be Lipschitzian, then mapping generated by (1.10) (1.11)

$$L^\infty(Q)\supset\bar{U}\ni u\mapsto y_u\in W(0,T) \tag{1.16}$$

is continuous.

Proof is given in [11].

REMARK. If mappings (1.14), (1.15) are differentiable, then the mapping (1.16) is differentiable [11].

2. Parametric optimization problem

Let there be given a Hilbert space U of controls (parameters) such that

$$U \subset L^{\infty}(Q) \tag{2.1}$$

with continuous imbedding

Example. If $u=u(t)$ depends only on time variable $t \in [0, T]$, then $U=H^1(0, T)$ may be chosen and (2.1) is satisfied.

We assume, that the set of admissible controls $U_{ad} \subset U$ is a bounded, convex and closed subset of U.

Let $z_d \in L^2(\Omega)$ be given function.

Our problem of parametric optimization is the following:

(P) *minimize cost functional*

$$J(u)=\frac{1}{2}\int_{\Omega}(y_u(x, T)-z_d(x))^2\,dx+\frac{\nu}{2}\|u\|_U^2, \quad \nu>0 \tag{2.2}$$

subject to $u \in U_{ad}$ *and the state equation* (1.10), (1.11).

LEMMA 3. *Assume set* U_{ad} *to be a compact subset of* $L^{\infty}(Q)$, *then there exists a solution* $\hat{u} \in U_{ad}$ *to* (P).

Proof. Lemma 3 follows from Lemma 2 and Weierstrass theorem. In the sequal for the sake of simplicity we assume that the set U_{ad} has the following form:

$$U_{ad}=\{u \in H^1(0, T) | 0 \leqslant u(t) \leqslant 1, \left|\frac{du}{dt}\right| \leqslant 1 \quad \text{a.e. in } [0, T]\}. \tag{2.3}$$

LEMMA 4. *Set* (2.3) *is a compact subset of Banach space* $C[0, T]$.

Proof. Set U_{ad} being convex, bounded and closed subset of Hilbert space $H^1(0, T)$ is weakly compact. By imbedding theorem [4] it is a compact subset of Sobolew space

$$H^{1-\varepsilon}(0, T), \ \forall \varepsilon>0,$$

Taking advantage of the imbedding $H^{1/2+\delta}(0, T) \subset C(0, T)$ which is continuous $\forall \delta>0$ [4], we obtain required result. From Lemmas 3 and 4 we get,

THEOREM 1. *For* U_{ad} *of the form* (2.3) *there exists an optimal control* $\hat{u} \in U_{ad}$ *to the problem* (P).

REMARK. Theorem 1 is valid in general case, for problem (P) with cost functional of the form:

$$J(u)=I(u, y_u), \ u \in U_{ad}$$

where functional

$$I(\cdot, \cdot): L^{\infty}(0, T) \times W(0, T) \mapsto R_{+}$$

is assumed to be continuous.

3. External approximation — discretization of the domain

In this section we introduce the clasical concept of discretization of domains Ω and Q in order to define some spaces of step functions over discretized domain Ω_h, Q_τ where $\tau = (h, \Delta t)$ denotes vector parameter of discretization; h, Δt correspond to discretization of space and time variables respectively.

We also shortly describe some regularity properties of discretization of the domain Ω, which were introduced by Cea [2]. These regularity properties are satisfied provided that domain Ω has piecewise Lipschitzian boundary Γ. They ensure existence of limits, in some sense, of sequences of traces of steps functions.

We denote

$$h = (h_1, \ldots, h_n), \; h_i > 0$$

$$|h|^2 = \sum_{i=1}^{n} h_i^2$$

$$|\tau|^2 = |h|^2 + (\Delta t)^2, \text{ where } \tau = (h, \Delta t).$$

By $\tau \to 0$ (resp. $h \to 0$) we mean $|\tau|^2 \to 0$ (resp. $|h|^2 \to 0$).

For given parameter h we introduce mesh R_h of nodal points M, where $M = (j_1, h_1, \ldots, j_n, h_n) \in R^n$, j_i, i — integers.

For given parameter h, mesh R_h corresponds to discretization of space variable x.

Let $M = (m_1, \ldots, m_n)$ be given point of R_h, we define two sets:

$$0_h(M; 0) = \prod_{i=1}^{n} \left[m_i - \frac{h_i}{2}, m_i + \frac{h_i}{2} \right[\tag{3.1}$$

$$0_h(M. 1) = \bigcup_{|j| \leq 1} 0_h\left(M + \tfrac{1}{2}(j_1 \vec{h}_1 + \ldots + j_n \vec{h}_n); 0\right) \tag{3.2}$$

where $|j| = j_1 + \ldots + j_n$

$$\vec{h}_1 = (h_1, 0, \ldots, 0)$$

$$\cdots\cdots\cdots$$

$$\vec{h}_n = (0, \ldots, 0, h_n).$$

We denote by $\Omega(h)$ subset of mesh R_h, corresponding, in some sense, to the domain Ω:

$$\Omega(h) = \{M \in R_h | 0_h(M; 1) \cap \Omega \neq \emptyset\} \tag{3.3}$$

and we set:

$$\Omega_h = \bigcup_{M \in \Omega(h)} 0_h(M; 0). \tag{3.4}$$

Let there be given a subset $\Gamma_v \subseteq \Gamma$.

DEFINITION 1. Collection of points $\bar{M} \in \Omega(h)$ such that

$$0_h(\bar{M}; 0) \cap \Gamma_v \neq \emptyset \tag{3.5}$$

will be called "the set of boundary points of $\Omega(h)$ with respect to Γ_v".

For given point $M \in \Omega(h)$, we denote

$$\Omega_M = \Omega \cap 0_h(M; 0) \tag{3.6}$$

$$\Gamma_M = \Gamma \cap 0_h(M; 0). \tag{3.7}$$

DEFINITION 2. Family $\{\Gamma_v, R_h\}_{h>0}$ is called "pre-regular" if for any given point $M \in \Omega(h)$ which belongs to the set of boundary points of $\Omega(h)$ with respect to Γ_v, the following conditions are satisfied:

(i) there exists an integer $i = i(M)$, $i = 1, \dots, n$, which will be called "Index of regularity of point M" such that:

$$h_i |\Gamma_M| \leqslant C |\Omega_M| \tag{3.8}$$

where

$$|\Gamma_M| = \int_{\Gamma_M} d\Gamma, \quad |\Omega_M| = \int_{\Omega_M} d\Omega;$$

(ii) at least one of the following inclusions takes place:

$$\delta_l^+ \Gamma_M \subset \Omega \text{ or } \delta_l^- \Gamma_M \subset \Omega \tag{3.9}$$

where

$$\delta_l^+ \Gamma_M = \{x = (x_1, \dots, x_n) \mid (x_1, \dots, x_{i-1}, x_i - \rho h_i, x_{i+1}, \dots, x_n) \in \Gamma_M \tag{3.10}$$
$$\rho > 0,\ 0 < \rho h_i < l\},\ l > 0;$$

$$\delta_l^- \Gamma_M = \{x = (x_1, \dots, x_n) \mid (x_1, \dots, x_{i-1}, x_i + \rho h_i, x_{i+1}, \dots, x_n) \in \Gamma_M, \rho > 0,$$
$$0 < \rho h_l < l\}, \quad l > 0; \tag{3.11}$$

(iii) constants C, l do not depend on h.

DEFINITION 3. Family $\{\Gamma, R_h\}_{h>0}$ is called "regular" if there exists a finite cover $\{\Gamma_v\}_{v \in A}$ of boundary $\Gamma = \partial\Omega$, such that:

(i) all boundary points $\bar{M} \in \Omega(h)$ corresponding to given set Γ_v have the same index of regularity $i(\bar{M}) = i(v)$ and family $\{\Gamma_v, R_h\}_{h>0}$ is pre-regular;

(ii) at least one of the following inclusions holds:

$$\delta_l^+ \Gamma_v \subset \Omega \text{ or } \delta_l^- \Gamma_v \subset \Omega \tag{3.12}$$

where

$$\delta_l^+ \Gamma_v = \bigcup_{\substack{M \in \Omega(h) \\ \Gamma_v \cap 0_h(M; 0) \neq \emptyset}} \delta_l^+ \Gamma_M \tag{3.13}$$

$$\delta_l^- \Gamma_v = \bigcap_{\substack{M \in \Omega(h) \\ \Gamma_v \cap 0_h(M; 0) \neq \emptyset}} \delta_l^- \Gamma_M \tag{3.14}$$

(iii) for h small enough, at least one of the following inclusions holds:

$$\delta_i \Gamma_v \subset \delta\Omega_i^+ \text{ or } \delta_i \Gamma_v \subset \delta\Omega_i^-, \; i=i(v) \tag{3.15}$$

where

$$\delta_i \Gamma_v = \delta_i^+ \Gamma_v \cup \delta_i^- \Gamma_v \tag{3.16}$$

$$\delta\Omega_i^+ = \Omega_i^r \setminus \Omega_i^r \cap \Omega_i^l \tag{3.17}$$

$$\delta\Omega_i^- = \Omega_i^l \setminus \Omega_i^r \cap \Omega_i^l \tag{3.18}$$

and

$$\Omega_i^r = \{x=(x_1, \dots, x_n) \in R^n | (x_1, \dots, x_{i-1}, x_i - \tfrac{1}{2} \rho h_i, x_{i+1}, \dots, x_n) \in \Omega, \; 0<\rho<1\} \tag{3.19}$$

$$\Omega_i^l = \{x=(x_1, \dots, x_n) \in R^n | (x_1, \dots, x_{i-1}, x_i + \tfrac{1}{2} \rho h_i, x_{i+1}, \dots, x_n) \in \Omega, \; 0<\rho<1\};$$

(iv) above conditions (i), (ii), (iii) are satisfied $\forall v \in A$.

We define also a mesh on the space R^{n+1}, which we use to approximate some elements of $L^2(Q)$, $Q=\Omega x]0, T[\subset R^{n+1}$ we put

$$L=(M, j\Delta t), \; M \in R_h, \; j \text{ — an integer} \tag{3.20}$$

and we denote by R_τ collection of such nodal points L.

In the same way as before we put

$$Q(\tau) = \{L \in R_\tau | L=(M, j\Delta t), \; M \in \Omega(h), j\Delta t \in [0, T+\Delta t[\} \tag{3.21}$$

and we define

$$Q_\tau = \bigcup_{L \in Q(\tau)} 0_h(M, 0) \times [j\Delta t, (j+1)\Delta t[. \tag{3.22}$$

For given $\tau>0$, set Q_τ is considered to be an approximation of the set Q.

Let Σ_0 be a given subset of $\Sigma=\Gamma \times]0, T[$, assume $\Sigma_0=\Gamma_0 \times]0, T[$, $\Gamma_0 \subseteq \Gamma$.

Our object now is to give suitable hypotheses on the set Σ_0, and family R_τ in order to ensure regularity of the family $\{\Sigma_0, R_\tau\}$. The following lemma holds:

LEMMA 5. *If the family $\{\Gamma_0, R_h\}$ is regular then the family $\{\Sigma_0, R_\tau\}$ is also regular.*

Proof. By regularity of family $\{\Gamma_0, R_h\}$ there exists a finite cover $\{\Gamma_v\}_{v \in A}$ of the set Γ_0. Define $\Sigma_v = \Gamma_v \times]0, T[$, hence the family $\{\Sigma_v\}_{v \in A}$ is a finite cover of the set Σ_0. It is easy to see that the family $\{\Sigma_v\}_{v \in A}$ satisfy all requirments of the definition 3 of regularity of family $\{\Sigma_0, R_\tau\}$.

4. Some properties of steps functions

Steps functions are not differentiable, hence in order to approximate derivatives of given function $y=y(x)$, or $z=z(x, t)$ we use so called finite differences.

We put

$$(\partial_i y)(\cdot, x_i) = \frac{1}{h_i} [y(\cdot, x_i + \tfrac{1}{2} h_i) - y(\cdot, x_i - \tfrac{1}{2} h_i)] \tag{4.1}$$

$$(\partial_0 z)(\cdot, t) = \frac{1}{\Delta t} [z(\cdot, t+\Delta t) - z(\cdot, t)] \tag{4.2}$$

where ∂_i (resp. ∂_0) is finite difference operator with respect to space variable x_i (resp. time variable t).

REMARK. Let us observe, that for a function defined on Ω_h, its finite differences with respect to space variables are well defined on Ω.

We denote by V_h the linear space of step functions defined on Ω_h. Elements of V_h have form:

$$z_h(x)=\sum_{M\in\Omega(h)} z(M)\, W_M(x),\ x\in\Omega_h \tag{4.3}$$

where

$$z(M)\in R$$

$$W_M(x)=\begin{cases}1 & \text{for } x\in 0_h(M,0)\\ 0 & \text{for } x\notin 0_h(M,0).\end{cases}$$

We define two norms $|\cdot|_h$, $\|\cdot\|_h$ in V_h, namely

$$|z_h|_h^2=h_1\cdot\ldots\cdot h_n \sum_{M\in\Omega(h)} z^2(M) \tag{4.4}$$

$$\|z_h\|_h^2=|z_h|_h^2+\sum_{i=1}^{n}\|\partial_i z_h\|_{L^2(\Omega)}^2. \tag{4.5}$$

These norms can be considered as discrete versions of norms in $L^2(\Omega)$ and $H^1(\Omega)$ respectively.

We denote by E_τ the linear space of step functions defined on set Q_τ corresponding to mesh $Q(\tau)$, we consider elements of E_τ as mappings:

$$z_\tau\colon [0, T+\Delta t[\mapsto V_h. \tag{4.6}$$

It is clear, that for a given element $z_\tau\in E_\tau$ there exists a sequence $\{z_i\}$, $i=0, 1, \ldots$ $\ldots, N$, $N\Delta t=T$, such that

$$z_\tau=z^i,\ t\in[i\Delta t,(i+1)\Delta t[,\ z^i\in V_h. \tag{4.7}$$

We define a norm $\|\cdot\|_\tau$ in E_τ which corresponds to the norm in $L^2(0, T; H^1(\Omega))$, that is:

$$\|z_\tau\|_\tau^2=\sum_{i=1}^{N}\Delta t\,\|z^i\|_h^2. \tag{4.8}$$

Elements of V_h (resp. E_τ) are well defined on the set $\bar\Omega$ (resp. $\bar Q$) hence we can define trace

$$\gamma y_h=y_h|_\Gamma,\quad \forall y_h\in V_h \tag{4.9}$$

resp.

$$\gamma z_\tau=z_\tau|_\Sigma,\quad \forall z_\tau\in E_\tau.$$

Moreover, some a priori estimations hold:

LEMMA 6. *Assume family $\{\Gamma, R_h\}$ to be regular, denote by $i=i(\nu)$ index of regularity of a set $\Gamma_\nu \subseteq \Gamma$, then*

(i) $$\|\gamma y_h\|^2_{L^2(\Gamma_\nu)} \leqslant C_1 \left(\|y_h\|^2_{L^2(\Omega)} + \|\partial_i y_h\|^2_{L^2(\Omega)}\right),\ i=i(\nu),\ \forall y_h \in V_h,\ \forall h>0; \tag{4.10}$$

(ii) $$\|\gamma z_\tau\|^2_{L^2(\Sigma_\nu)} \leqslant C_2 \left(\|z\|^2_{L^2(Q)} + \|\partial_i z_\tau\|^2_{L^2(Q)}\right),\ i=i(\nu),\ \forall z_\tau \in E_\tau,\ \forall \tau>0; \tag{4.11}$$

where constant C_1 (resp. C_2) does not depend on h (resp. τ).

Proof of (i) is given in Cea [2], and of (ii) in [11].

REMARK. We derive some properties of step functions, which can be verified by using of above lemma, provided that the family $\{\Gamma, R_h\}$ is regular:

(i) given sequence $y_h \in V_h$, let for $h \to 0$:

$$y_h|_\Omega \to y \text{ in } L^2(\Omega) \quad \text{strongly (resp. weakly)} \tag{4.12}$$

$$\partial_i y_h|_\Omega \to \frac{\partial y}{\partial x_i} \text{ in } L^2(\Omega) \quad \text{strongly (resp. weakly)} \quad i=1, \ldots, n$$

for some $y \in H^1(\Omega)$, then

$$\gamma y_h \to \gamma y \text{ in } L^2(\Gamma) \text{ strongly (resp. weakly);}$$

(ii) given sequence $z_\tau \in E_\tau$, let for $\tau \to 0$:

$$z_\tau|_Q \to z \text{ in } L^2(Q) \text{ strongly (resp. weakly)}$$

$$\partial_i z_\tau|_Q \to \frac{\partial z}{\partial x_i} \text{ in } L^2(Q) \quad \text{strongly (resp. weakly)} \quad i=1, \ldots, n \tag{4.13}$$

for some $z \in L^2(0, T; H^1(\Omega))$, then

$$\gamma z_\tau \to \gamma z \text{ in } L^2(\Sigma) \text{ strongly (resp. weakly).}$$

For given element $y_h \in V_h$ (resp. $z_\tau \in E_\tau$) we put

$$P_h y_h = (y_h|_\Omega, \partial_1 y_h|_\Omega, \ldots, \partial_n y_h|_\Omega, \gamma y_h) \in F \tag{4.14}$$

$$F = [L^2(\Omega)]^{n+1} \times L^2(\Gamma)$$

(resp. $P_\tau z_\tau = (z_\tau|_Q, \partial_1 z_\tau|_Q, \ldots, \partial_n z_\tau|_Q, \gamma z_\tau) \in L^2(0, T; F)$).

It is easy to see, that $P_h \in \mathscr{L}(V_h; F)$ (resp. $P_\tau \in \mathscr{L}(E_\tau; L^2(0, T; F))$).

We denote by $\omega_1 \in \mathscr{L}(L^2(0, T; H^1(\Omega)); L^2(0, T; F))$ a linear mapping

$$H^1(\Omega) \ni y(t) \mapsto \omega y(t) \in F, \text{ a.e. in } [0, T] \tag{4.15}$$

where $\omega \in \mathscr{L}(H^1(\Omega); F)$ is defined as follows:

$$\omega: H^1(\Omega) \ni y \mapsto \omega y = \left(y, \frac{\partial y}{\partial x_1}, \ldots, \frac{\partial y}{\partial x_n}, \gamma y\right) \in F. \tag{4.16}$$

LEMMA 7. *Assume $\|y_h\|_h \leqslant C$ for $h \to 0$, then there exist an element $\bar{y} \in H^1(\Omega)$, and a subsequence $h' \to 0$, such that*

$$P_{h'} y_{h'} \rightharpoonup \omega \bar{y} \text{ weakly in } F.$$

LEMMA 8. *Assume* $\|z_\tau\|_\tau \leqslant C$ *for* $\tau\to 0$, *then there exists an element* $\bar{z}\in L^2(0, T; H^1(\Omega))$ *and a subsequence* $\tau'\to 0$, *such that*

$$P_\tau z_\tau \rightharpoonup \omega_1 \bar{z} \text{ weakly in } L^2(0, T; F).$$

Lemma 7 is proved in [2], lemma 8 in [12]. To define finite dimensional approximation of the problem (P) we have to have:

(i) approximation of the set U_{ad};

(ii) approximation of the state equation (1.10), (1.11).

First we define an approximation of the set U_{ad}.

For given $\tau>0$ we set

$$U_{ad}^\tau=\{u_\tau\in H^1(0, T)|0\leqslant u(t)\leqslant 1, \tag{4.17}$$

$$\frac{du_\tau}{dt}(t)=\sum_{i=1}^{N} v_i W_i(t), N\Delta t=T, 0\leqslant v_i\leqslant 1, l=1, ..., N\}$$

where

$$W_i(t)=\begin{cases} 1, t\in[i\Delta t, (i+1)\Delta t[\\ 0, t\neq\notin[i\Delta t, (i+1)\Delta t[. \end{cases}$$

REMARK. Set U_{ad}^τ depends in fact only on Δt.

Family $\{U_{ad}^\tau\}$ constitute an approximation of the set U_{ad}.

LEMMA 9. *Given sequence* $u_\tau\in U_{ad}^\tau$, $\Delta t>0$, *then there exists subsequence* $\Delta t'\to 0$ *and an element* $\bar{u}\in U_{ad}$ *such that* $u_{\tau'}\xrightarrow[\Delta t'\to 0]{}\bar{u}$ *weakly in* $H^1(0, T)$ *and also* $u_{\tau'}\xrightarrow[\Delta t'\to 0]{}\bar{u}$ *strongly in* $L^\infty(0, T)$.

Proof. Set U_{ad} is bounded in $H^1(0, T)$ and $U_{ad}^\tau\subset U_{ad}$, for every $\tau>0$, hence

$$\|u_\tau\|_{H^1(0,T)}\leqslant C, \ \forall\tau>0,$$

then for some $\chi\in H^1(0, T)$ and some subsequence $\tau'\to 0$ we have

$$u_{\tau'}\rightharpoonup\chi \text{ weakly in } H^1(0, T)$$

but U_{ad} is weakly compact hence $\chi=\bar{u}\in U_{ad}$.

By compactness of the imbedding $H^1(0, T)\subset L^\infty(0, T)$ we have

$$u_{\tau'}\to\bar{u} \text{ strongly in } L^\infty(0, T).$$

In order to extend in some sense bilinear form (1.4) to the space V_h, $h>0$ we put

$$a_u(t; \omega y, \omega z)=a_u(t; y, z) \ \forall y, z\in H^1(\Omega)$$

$$\forall u\in U_{ad}, \ \forall t\in[0, T]. \tag{4.18}$$

Bilinear form

$$a_u^h(\cdot, \cdot): V_h\times V_h\ni(y_h, z_h)\mapsto a_u(\cdot; P_h y_n, P_h z_h)\in L^\infty(0, T) \tag{4.19}$$

is then well defined.

For given element $u_\tau \in U_{ad}^\tau$ we put

$$a_{u_\tau}^\tau(t; y_h, z_h) = \frac{1}{\Delta t} \sum_{i=0}^{N-1} \left(\int_{i\Delta t}^{(i+1)\Delta t} a_{u_\tau}^h(t; y_n, z_h)\, dt \right) W_i(t) \quad \forall y_h, z_h \in V_h. \tag{4.20}$$

LEMMA 10. *Assume:*

$$P_\tau y_\tau \xrightarrow[\tau\to 0]{} \omega_1 \bar{y} \text{ strongly in } L^2(0, T; F)$$

$$P_\tau z_\tau \xrightarrow[\tau\to 0]{} \omega_1 \bar{z} \text{ weakly in } L^2(0, T; F) \tag{4.21}$$

$$u_\tau \rightharpoonup \bar{u} \text{ weakly in } H^1(0, T)$$

then:

$$a_{u_\tau}^\tau(\cdot\,; y_\tau, z_\tau) \rightharpoonup a_u(\cdot\,; \bar{y}, \bar{z}) \text{ weakly in } L^2(0, T). \tag{4.22}$$

Proof. Let us define

$$A_u^h \in L^\infty(0, T; \mathscr{L}(V_h, V_h)) \tag{4.23}$$

by equality

$$a_u(\cdot\,; P_h y_h, P_h z_h) = (A_h(\cdot)\, y_h, z_h)_h. \tag{4.24}$$

It can be checked that if for $h\to 0$

$$P_h y_h \to \omega y \text{ strongly in } F,\ y \in H^1(\Omega)$$

$$P_h z_h \rightharpoonup \omega z \text{ weakly in } F,\ z \in H^1(\Omega)$$

$$u_h \to u \text{ strongly in } L^\infty(0, T)$$

then

$$(A_{u_h}^h(\cdot)\, y_h, z_h)_h \to a_u(\cdot\,; y, z) \text{ strongly in } L^\infty(0, T). \tag{4.25}$$

Formula (4.20) gives us in fact an approximation of the sequence

$$A_{u_\tau}^h(\cdot) \in L^\infty(0, T; \mathscr{L}(V_h, V_h)) \tag{4.26}$$

by means of steps functions

$$A_\tau(\cdot) \in L^p(0, T; \mathscr{L}(V_h, V_h)),\ p \geqslant 1 \tag{4.27}$$

where

$$A_\tau(t) = \frac{1}{\Delta t} \int_{i\Delta t}^{(i+1)\Delta t} A_{u_\tau}^h(s)\, ds = A_\tau^i,\ t \in [i\Delta t, (i+1)\Delta t[. \tag{4.28}$$

It can be checked [12] that if for $\Delta t\to 0$

$$u_\tau \to \bar{u} \text{ strongly in } L^\infty(0, T) \tag{4.29}$$

then

$$A_\tau(\cdot) \to A_u^h(\cdot) \tag{4.30}$$

strongly in $L^p(0, T; \mathscr{L}(V_h, V_h))$, $\forall p \in [1, \infty[$.

Combining (4.25), (4.30) and some results of Temam [12] (p. 245) we obtain the required result.

5. External approximation of linear parabolic equation with mixed Neumann boundary conditions

Now we are in the position to define a finite dimensional approximation of state equation (1.10), (1.11).

Let us assume, that domain Ω has extension property of order 1, that is there exists linear bounded mapping

$$P \in \mathscr{L}\big(H^1(\Omega): H^1(R^n)\big) \cap \mathscr{L}\big(L^2(\Omega); L^2(R^n)\big) \tag{5.1}$$

such that

$$(Py)(x) = y(x) \text{ a.e. in } \Omega, \ \forall y \in H^1(\Omega). \tag{5.2}$$

In order to approximate elements of functional spaces $L^2(\Omega)$, $H^1(\Omega)$, $L^2(\Gamma)$, $L^2(\Sigma)$, $L^2(Q)$, we introduce the following restriction operators:

(i) restriction operator $r_h \in \mathscr{L}\big(H^1(\Omega); V_h\big)$:

$$(r_h y)(x) = (h_1 \cdot \ldots \cdot h_n)^{-1} \int\limits_{0_h(M;0)} (Py)(x)\, dx, \ \forall y \in H^1(\Omega), \tag{5.3}$$

$$x \in 0_h(M; 0),\ M \in \Omega(h),\ h_i > 0;$$

(ii) restriction operator $r_h^o \in \mathscr{L}\big(L^2(\Omega); V_h\big)$:

$$(r_h^o y)(x) = (h_1 \cdot \ldots \cdot h_n)^{-1} \int\limits_{0_h(M;0)} \tilde{y}(x)\, dx, \ \forall y \in L^2(\Omega), \tag{5.4}$$

$$x \in 0_h(M; 0),\ M \in \Omega(h),\ h_i > 0;$$

(iii) restriction operator $\rho_h \in \mathscr{L}\big(L^2(\Gamma); V_h\big)$:

$$(\rho_h \psi)(x) = (h_1 \cdot \ldots \cdot h_n)^{-1} \int\limits_{0_h(M;0)\cap\Gamma} \psi(x) \big(\gamma W_M(x)\big)\, d\Gamma, \tag{5.5}$$

$$\forall \psi \in L^2(\Gamma),\ x \in 0_h(M; 0),\ M \in \Omega(h),\ h_i > 0,$$

where

$$\tilde{z}(x) = \begin{cases} z(x), & x \in \Omega \\ 0 & , x \notin \Omega \end{cases}.$$

The above operators have the following properties [2], [8], [12]:

(i)
$$p_h r_h y \xrightarrow[h\to 0]{} \omega y \text{ strongly in } F, \tag{5.6}$$
$$\forall y \in H^1(\Omega);$$

(ii)
$$\widetilde{r_h^o y} \xrightarrow[h\to 0]{} \tilde{y} \text{ strongly in } L^2(R^n) \tag{5.7}$$
$$\forall y \in L^2(\Omega);$$

(iii)
$$(\rho_h \psi, y_h)_h = \int\limits_{\Omega_h} \rho_h \psi \cdot y_h\, d\Omega \xrightarrow[h\to 0]{} \int\limits_{\Gamma} \psi \gamma \bar{y}\, d\Gamma, \tag{5.8}$$
$$\forall \psi \in L^2(\Gamma), \text{ if } P_h y_h \rightharpoonup \omega y \text{ weakly in } F.$$

We denote by f_τ, φ_τ finite dimensional approximations of $f \in L^2(Q)$ and $\varphi \in L^2(\Sigma)$ respectively, that is:

$$f_i = f_\tau(t) = \frac{1}{\Delta t} \int\limits_{i\Delta t}^{(i+1)\Delta t} r_h^o f(s)\, ds,\ t \in [i\Delta t, (i+1)\, \Delta t[, \qquad i=0, 1, \ldots, N, \tag{5.9}$$

$$\varphi_i = \varphi_\tau(t) = \frac{1}{\Delta t} \int\limits_{i\Delta t}^{(i+1)\Delta t} \rho_h\, \varphi(s)\, ds,\ t \in [i\Delta t, (i+1)\, \Delta t[, \qquad i=0, 1, \ldots, N \tag{5.10}$$

For given parameter $\tau > 0$ we denote by

$$y_\tau(t) = y_\tau(u_\tau; t) \in E_\tau \tag{5.11}$$

the solution of the following equation

$$(\partial_0\, y_\tau(t), v_h)_h + a^\tau_{u_\tau}(t;\, w_\tau(t), v_h) = (f_\tau(t), v_h)_h + (\varphi_\tau(t), v_h)_h, \qquad \forall v_h \in V_h \tag{5.12}$$

$$\text{a.e. in } [0, T]$$

$$y_\tau(0) = r_h^o\, y_0 \tag{5.13}$$

where $w_\tau(t) = y_\tau(t + \Delta t)$.

Remark. (5.12), (5.13) is an implicit difference scheme for the problem (1.10), (1.11), and it is well known that the following result takes place:

Lemma 11. *Assume that there exists constant* $\alpha > 0$ *such that*

$$a^h_{u_\tau}(t;\, P_h\, y_h, P_h\, y_h) \geqslant \alpha\, \|y_h\|_h^2 \tag{5.14}$$

then $\forall \tau > 0$ *there exists the unique solution to the problem* (5.12), (5.13).

The following theorem is a variant of general theorem which is given in [12].

Theorem 2. *Assume*:

$$u_\tau \rightharpoonup \bar{u} \text{ weakly in } H^1(0, T),\ u_\tau, \bar{u} \in U_{ad} \tag{5.15}$$

then: *the sequence of solutions of problem* (5.12), (5.13) *converges to the solution* y_u *of the problem* (1.10), (1.11) *with* $u = \bar{u}$ *in the following sense*:

$$P_\tau\, w_\tau \xrightarrow[\tau \to 0]{} \omega_1\, y_{\bar{u}} \text{ strongly in } L^2(0, T; F) \tag{5.16}$$

$$\widetilde{y_\tau}(T) \to \widetilde{y_{\bar{u}}}(T) \text{ strongly in } L^2(R^n). \tag{5.17}$$

6. Approximation of parametric optimization problem

For given parameter $\tau > 0$ we define optimization problem:
(P_τ): *minimize*

$$J_\tau(u_\tau) = \frac{\nu}{2}\, \|u_\tau\|^2_{H^1(0,T)} + \frac{1}{2} \int\limits_{\Omega_h} (y_\tau(T, x) - (r_h^o z_d)(x))^2\, dx \tag{6.1}$$

subject to (5.12), (5.13) *and*

$$u_\tau \in U_{ad}^\tau \tag{6.2}$$

REMARK. Problem (P_τ) is finite dimensional for each $\tau>0$. In (6.1) second integral may be taken either over Ω_h or over Ω, it has no importance.

Let us denote by $\hat{u}_\tau$ a solution of the problem (P_τ).

THEOREM 3. *Let $\bar{u} \in U_{ad}$ be a weak accumulation point of the sequence $\{\hat{u}_\tau\}$, that is for some subsequence $\tau' \to 0$, $\hat{u}_{\tau'} \rightharpoonup \bar{u}$ weakly in $H^1(0, T)$, then:*

(i) $\hat{u}_{\tau'} \xrightarrow[\tau' \to 0]{} \bar{u}$ *strongly in* $L^\infty(0, T)$;

(ii) $\bar{u}$ *is a solution of* (P);

(iii) $J_\tau(\hat{u}_{\tau'}), \xrightarrow[\tau \to 0]{} J(\bar{u})$.

Proof. By lemma 9 we have (i). We may use theorem 2, hence

$$\widetilde{y_{\tau'}(T, \cdot)} \xrightarrow[\tau' \to 0]{} \widetilde{y_{\bar{u}}(T)}$$

strongly in $L^2(R^n)$, then

$$\int_{\Omega_h} (y_{\tau'}(T, x) - (r_h^o z_d)(x))^2 \, dx \to \int_{\Omega} (y_{\bar{u}}(T, x) - z_d(x))^2 \, dx.$$

Let $\{v_{\tau'}\}$ be any sequence such that, $v_{\tau'} \in U_{ad}^{\tau'}$ and

$$v_{\tau'} \xrightarrow[\tau' \to 0]{} \hat{u} \text{ strongly in } H^1(0, T)$$

then

$$J(\hat{u}) = \lim J_\tau(v_{\tau'}) \geqslant \lim J_\tau(\hat{u}_{\tau'}) = J(\bar{u})$$

hence $\bar{u}$ is a solution to the problem (P) and

$$J(\bar{u}) = J(\hat{u})$$

which implies (iii). Q.E.D.

7. Numerical example

Let us consider the following example of parametric optimization problem: (P_1) minimize

$$J(v) = \frac{\varepsilon}{2} \int_0^5 v^2(s) \, ds + \int_0^1 [y_v(x, 5) - 5(x-1)^2]^2 \, dx, \qquad \varepsilon > 0$$

subject to $v \in L^2(0, T)$ and state equation:

$$\frac{\partial y}{\partial t}(x, t) - F\left((Lv)(t)\right) \frac{\partial^2 y}{\partial x^2} = (x-1)^2 - 2t, (x, t) \in]0, 1[\times]0, 5[$$

$$-\frac{\partial y}{\partial x}(0, t) + g\left((Lv)(t)\right) y(0, t) = 98\, t, x=0, \; t \in]0, 5[$$

$$-\frac{\partial y}{\partial x}(1, t)+g\big((L v)(t)\big)\, y(1, t)=0$$

$$x=1, \quad t \in]0, 5[$$

$$y(x, 0)=0, \quad x \in]0, 1[$$

where

$$(Lv)(t)=e^{-t} \int_0^t e^s v(s)\, ds, \qquad \forall v \in L^2(0, 5)$$

$$F(u)=1+u^4$$

$$g(u)=1/(0.01+u^2).$$

Let us observe, that for any $\varepsilon>0$

$$J(v)=0 \Leftrightarrow v=\hat{v}=0$$

hence problem (P_1) has the unique solution $\hat{v}=0$.

There were used a gradient method of Polak—Ribiere [7] to minimize discrete cost functional. For starting point

$$v_\tau=\begin{cases} 20 & t \in [0; 2.5[\\ -5 & t \in [2.5; 5] \end{cases}$$

and parameters $h=0.1$, $\Delta t=1$ there were obtained the following numerical results:

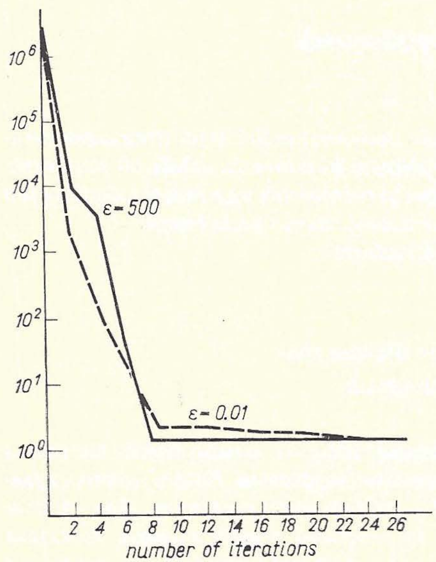

Fig. 1. Cost functional $J_\tau(v_\tau)$

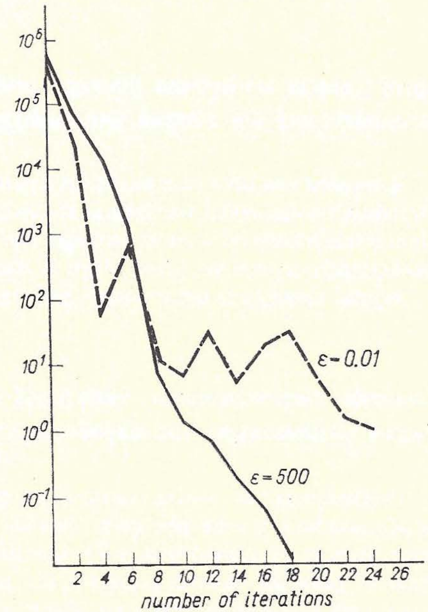

Fig. 2. Norm of the gradient of the cost functional $J_{\tau'}(v_\tau)$

Conclusions. For the above example numerical results were good. But it seems that such results cannot be expected in general case because for small values of parameter ε the cost functional is not convex.

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Aproksymacja zewnętrzna pewnego problemu optymalizacji parametrycznej dla równań parabolicznych

Rozważono zadanie sterowania optymalnego układu opisywanego liniowym równaniem parabolicznym z mieszanymi warunkami brzegowymi. Współczynniki równania zależą od sterowania a funkcjonał jakości od stanu końcowego. Dla przybliżonego rozwiązania tego zadania zastosowano aproksymację zewnętrzną (różnicową). Wykazuje się zbieżność takiej aproksymacji.

Wyniki teoretyczne zilustrowano przykładem numerycznym.

Внешняя аппроксимация некоторой задачи параметрической оптимизации для параболических уравнений

Рассматривается задача оптимального управления системы описываемой линейным параболическим уравнением со смешанными граничными условиями. Коэффициенты уравнения зависят от управления, а функционал качества от конечного состояния, Для приближённого решения задачи используется внешняя (разностная) аппроксимация, Показана сходимость такой аппроксимации. Теоретические результаты иллюстрируются на численном примере.