

On an extension of the method of minimally interconnected subnetworks

by

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In the paper, an extension of the method of minimally interconnected subgroups [5] is presented. The extension consists of the generalization of the above method to arbitrary weighted graphs, in particular to unigraphs with nonnegative real weights. For this case the properties given in [5] are reformulated and proved, moreover, many original new ones are stated and proved. The above extension is of use for solving many problems of graph partitioning type, consisting of partitioning the set of vertices into subsets, such that the mutual connections between vertices in a subset are stronger than their connections with vertices not belonging to that subset.

1. Introduction

The method of minimally interconnected subnetworks was introduced by R. Luccio and M. Sami [5] for solving the problem of decomposing a given electrical network into subnetworks, such that the number of interconnections (leads) between them is minimal under some specified conditions. In the setting of the graph theory, this method referred to the decomposition of a multigraph with edge weights being natural numbers.

The method mentioned above was generalized on arbitrary graphs, specifically on unigraphs with nonnegative real edge weights. It proved to be a relatively useful technique for solving problems of graph partitioning type consisting of partitioning the set of vertices into subsets, such that the mutual connections between vertices in a subset are stronger than their connections with vertices not belonging to that subset, as it was shown in two recent papers of authors et al., in [3] for the decomposition of the telephone interexchange network structure and in [4] for the decomposition of a group of enterprises into subgroups. These applications will be briefly described in the Appendix.

The goal of the paper is to present in a rigorous and systematic way the above extension of the method of minimally interconnected subnetworks. Apart of ex-

tending basic notions and properties given in [5], many original new ones will be stated and proved. The terminology will be in principle consistent with that in [5].

2. Basic notions and properties of minimal groups

Let us consider a graph G , complete, undirected and without loops. The set of its vertices will be denoted by V and the set of its edges — by E . A function:

$$w: E \rightarrow R_+ \cup \{0\} \quad (1)$$

is defined on the set of edges E , where R_+ is the set of positive real numbers. Let us denote in the sequel by $\langle G, w \rangle$ the above graph G with the function w given by (1), i.e. the weighted graph. The values of w can be represented in the form of a matrix W , $\dim W = |V| \times |V|$, $|V|$ — the cardinal number of V . The matrix W is evidently a symmetric one. The element w_{ij} of W is the weight of the edge beginning in the vertex i and ending in the vertex j ; $w_{ii} \stackrel{\text{df}}{=} 0$, $w_{ij} = w_{ji}$ for all $i, j = 1, \dots, |V|$.

DEFINITION 1. For a given $\langle G, w \rangle$, any subset $S \subset V$, taken with all the edges connecting each pair of its elements, will be called the group S .

In the sequel, the groups as well as the corresponding sets of vertices will be denoted by a capital Latin letter. All set — theoretic operations on groups are, therefore, performed on corresponding sets of vertices adding to the result obtained corresponding sets of edges, according to the definition 1. Thus, all symbols of set-theoretic properties or operations, such as e.g. inclusion (\subset), cardinal number ($|\cdot|$), union (\cup), difference (\setminus), intersection (\cap) etc. refer, if not otherwise indicated, to sets of vertices.

Remark. In order to simplify later notations, let us denote:

$$a(S, R) = \sum_{\substack{i \in S \\ j \in R}} w_{ij}, \quad (2)$$

where: $S, R \subset V$, $S \cap R = \emptyset$. In the case when R is the complement of S to V , $a(S, V \setminus S)$ will correspond to the group S and will be denoted by s .

DEFINITION 2. For a given $\langle G, w \rangle$, a nonempty group S , such that for every nonempty $R \subset S$, $R \neq S$, the inequality $r > s$ holds, will be called the minimal group. Moreover, every particular vertex of G is defined as the minimal group.

COROLLARY 1. If S is a minimal group in $\langle G, w \rangle$, then there does not exist any nonempty group $R \subset S$, such that $r = 0$.

The corollary results directly from Definition 2.

LEMMA 1. For a given minimal group S in $\langle G, w \rangle$ and every its proper subset $R \neq \emptyset$, the following inequality holds:

$$a(R, S \setminus R) > a(S \setminus R, V \setminus S). \quad (3)$$

Proof. Since $s > r$ by assumptions of Lemma and Definition 2, then:

$$s = a(R, V \setminus S) + a(S \setminus R, V \setminus S) < r = a(R, V \setminus S) + a(R, S \setminus R),$$

i.e.

$$a(R, S \setminus R) > a(S \setminus R, V \setminus S).$$

Q.E.D.

Lemma 1 shows one of the basic properties of minimal groups. It also indicates their usefulness for applications. Namely, the formula (3) can be interpreted in such a way that the entire dependence of a nonempty proper subset $Q = S \setminus R$ of S on the rest R of this group S is greater than the analogous parameter determined for, respectively, Q and $V \setminus S$. An example illustrating it is shown in Fig. 1.

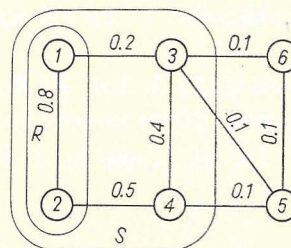


Fig. 1. The group S is minimal. $R \subset S$, $R \neq S$. Since $r = 0.7$ and $s = 0.3$, then $s > r$

LEMMA 2. Two minimal groups in $\langle G, w \rangle$ are either disjoint or one of them is contained in the other.

Proof. Let R and S be minimal groups in $\langle G, w \rangle$, such that $R \not\subset S$ and $S \not\subset R$. Let us denote $P = S \cap R$ and, moreover, $Z = S \setminus P$, $Q = R \setminus P$. Let us now assume that $P \neq \emptyset$. Then, by Lemma 1, we have:

$$a(P, V \setminus S) < a(Z, P), \quad (5)$$

$$a(P, V \setminus R) < a(Q, P). \quad (6)$$

Thus, from relation (2) we get:

$$a(P, V \setminus S) = a(P, V \setminus (S \cup R)) + a(Q, P), \quad (7)$$

$$a(P, V \setminus R) = a(P, V \setminus (S \cup R)) + a(Z, P). \quad (8)$$

Introducing (7) and (8) to, respectively, (5) and (6), we obtain:

$$a(P, V \setminus (S \cup R)) < 0, \quad (9)$$

which is impossible due to the form of formulae (1) and (2). Therefore, P must be empty. Q.E.D.

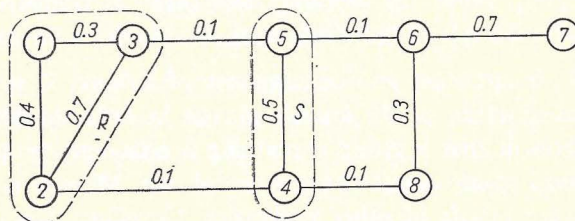


Fig. 2. The groups R and S are minimal. $R \not\subset S$, $S \not\subset R$. The group $R \cap S$ is empty

This lemma makes the inclusion relation order partially the set of all possible minimal groups for a given $\langle G, w \rangle$. It is of great importance for the construction of an efficient algorithm for determining minimal groups. An example illustrating Lemma 2 is shown in Fig. 2.

3. Additional properties of minimal groups

This part is devoted to the extension of the theory of minimal groups. There will be formulated and proved further properties of minimal groups aimed at the construction of an efficient algorithm for determining them.

COROLLARY 2. Let $Z_i, R_i, F, i \in I = \{1, 2, \dots, m\}$, be given nonempty pairwise disjoint groups in $\langle G, w \rangle$. Let $K_i = Z_i \cup R_i$ be a minimal group for every $i \in I$.

Let us denote by L the following group:

$$L = F \cup \bigcup_{i=1}^m R_i. \quad (10)$$

Then, $l > f$.

Proof. Let us assume that there is given an increasing sequence of groups, $(L_p)_{p \in I}$, defined by:

$$L_p = F \cup \bigcup_{i=1}^p R_i. \quad (11)$$

The proof will proceed by induction on p . Let $p=1$. It is easy to show that:

$$l_1 - f = a(R_1, V \setminus (L_1 \cup K_1)) - a(R_1, F) + a(R_1, Z_1) \geq a(R_1, Z_1) - a(R_1, F). \quad (12)$$

Since the group K_1 is minimal, then — as in the proof of Lemma 2 — we have:

$$k_1 = z_1 = a(R_1, V \setminus (L_1 \cup K_1)) - a(R_1, Z_1) + a(R_1, F) < 0, \quad (13)$$

and thus:

$$a(R_1, Z_1) > a(R_1, F). \quad (14)$$

Taking into account relation (12), it gives $l_1 > f$ and terminates the first induction step.

Let us now assume that $l_p > f$, $1 \leq p < m$. As in the first step of the proof, it is easy to show that $l_{p+1} > l_p$. Since the relation "less then" is transitive, then $l_{p+1} > f$, which — due to the principle of finite induction — completes the proof. Q.E.D.

Corollary 2 is an extension and consequence of Lemma 2, making it possible to use implicit enumeration of all possible groups in order to determine minimal ones. It follows from it that a group containing a nonempty proper part of even one another minimal group cannot be a minimal one. Moreover, this corollary is of importance in the proofs of other properties. Corollary 2 is illustrated by an example shown in Fig. 3.

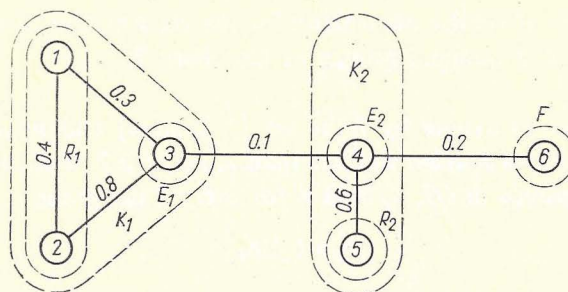


Fig. 3. The groups K_1 and K_2 are minimal. $L = R_1 \cup R_2$, $l = 0.2 + 0.6 + 1.1 = 1.9$, $f = 0.2$. Thus, $l > f$

Proposition 1. Let K_i , $i \in I = \{1, 2, \dots, m\}$, be pairwise disjoint minimal groups in $\langle G, w \rangle$ and R_i — respectively, their proper parts.

Let us denote:

$$P = \bigcup_{i=1}^m R_i. \quad (15)$$

Then, for $m \geq 2$ the following relation holds:

$$p > \max_{i \in I} \{k_i\}. \quad (16)$$

Proof. The proof will proceed by mathematical induction on m . Let us assume that $m=2$. Using twice Corollary 2, we get:

$$p > \max \{s_1, s_2\}, \quad (17)$$

there: $S_j = K_j \setminus R_j$, $j \in I$. Since K_1 and K_2 are minimal groups, then by Definition 2 and transitivity of the relation "greater then" we have:

$$p > \max \{k_1, k_2\}, \quad (18)$$

which terminates the first step of induction proof.

Let us now construct an increasing sequence of groups, $(P_j)_{j \in I}$, defined by:

$$P_j = \bigcup_{i=1}^j R_i. \quad (19)$$

Let us assume that the following inequality holds:

$$p_j > \max \{k_i : i \in I, i \leq j\}, \quad (20)$$

for $2 \leq j < m$. It is evident that $P_{j+1} = R_{j+1} \cup P_j$. From Corollary 2 it follows that $p_{j+1} > r_{j+1}$. Since K_{j+1} is a minimal group, then:

$$p_{j+1} > k_{j+1}. \quad (21)$$

Referring once more to Corollary 2 we get:

$$p_{j+1} > p_j, \quad (22)$$

which, together with the formula (21) terminates the second step of induction proof and, therefore, the whole proof. Q.E.D.

Proposition 1 is of similar importance for the construction of an efficient algorithm for determining minimal groups as Corollary 2.

THEOREM 1. Let us denote by I the set $\{1, 2, \dots, m\}$ and by J — its arbitrary subset with the cardinal number not less than 2, i.e. $|J| \geq 2$. If $K_i, i \in I$, are pairwise disjoint minimal groups in $\langle G, w \rangle$ and if for every J the group:

$$P_J = \bigcup_{i \in J} K_i \quad (23)$$

is not minimal, then the following inequality holds:

$$p_I \geq \min_{i \in I} \{k_i\}. \quad (24)$$

Proof. The proof will proceed by mathematical induction on m . Let $m=2$. Since P_I is not a minimal group, then there exists at least one such group $Q \subset P_I$, $Q \neq P_I$, that $p \geq q$. The following three cases may occur:

- 1) $Q \subseteq K_1$ or $Q \subseteq K_2$. From Definition 2 we get, respectively, $q \geq k_1$ or $q \geq k_2$.
- 2) $Q \supset K_1$, $Q \neq K_1$, or $Q \supset K_2$, $Q \neq K_2$. To be more specific, let us assume that the first variant holds, i.e. $Q \supset K_1$, $Q \neq K_1$. Let us denote: $H_2 = Q \setminus K_1$, $F_2 = K_2 \setminus H_2$. It is clear that $H_2, F_2 \neq \emptyset$. From Corollary 2 it follows that $q > k_2$. Respectively, in the second case (i.e. $Q \supset K_2$, $Q \neq K_2$) we get $q > k_1$.
- 3) $Q = H_1 \cup H_2$, where $H_1 \subset K_1$, $H_1 \neq K_1$, and $H_2 \subset K_2$, $H_2 \neq K_2$. Then, by Proposition 1 we have:

$$q > \max \{k_1, k_2\} \geq \min \{k_1, k_2\}.$$

Taking into account three cases mentioned above, it follows that the following relation holds:

$$p \geq q \geq \min \{k_1, k_2\}, \quad (25)$$

which terminates the first step of induction proof.

Let us now assume that for every J , $1 < |J| < m$, the following relation holds:

$$p_J \geq \min_{i \in J} \{k_i\}. \quad (26)$$

Let us denote:

$$J' = J \cup \{i: i = \min_{j \in J} j\}. \quad (27)$$

We will now discuss the properties of groups $P_{J'}$. From assumptions of the theorem it follows that none of them is a minimal one. Then, for each of them there exists — by Definition 2 — such a $Q \subset P_{J'}$, $Q \neq P_{J'}$, that $q \leq p_{J'}$.

As above, three following cases may occur:

- 1) $Q \subseteq K_i, i \in J'$. Then, by the definition of minimal group (Definition 1) we get immediately $q \geq k_i$.
- 2) $Q \supset K_i, Q \neq K_i, i \in J'$. The group Q may be, therefore, represented in the following form:

$$Q = P_{J_1} \cup \bigcup_{r \in J_2} H_r, \quad (28)$$

where: $J_1 \cap J_2 = \emptyset$, $J_1 \neq \emptyset$, $|J_1| < m$, $|J_2| < m$, $J_1 \cup J_2 \subset J'$, $H_r \subset K_r$, $H_r \neq K_r$, $r \in J'$. Taking advantage of Corollary 2 we get for $J_2 \neq \emptyset$: $q > p_{J_1}$. If $J_2 = \emptyset$, then $q = p_{J_1}$ and, thus, by the induction assumption we have:

$$q \geq \min_{i \in J_1} \{k_i\}.$$

3) Q is given by (28), but $J_1 = \emptyset$; all remarks from 2) concerning H_r are valid. Then, it is clear that $J_2 \neq \emptyset$. From Proposition 1 we have then:

$$q > \max_{i \in J_2} \{k_i\} \geq \min_{i \in J_2} \{k_i\}. \quad (29)$$

Thus, from the theorem on finite induction it can be stated that for every natural m the relation (24) holds, which completes the proof. Q.E.D.

An immediate consequence of Theorem 1 is the following corollary:

COROLLARY 3. Let the notation be the same as in Theorem 1. If for every $J \neq I$ the group P_J is not minimal and, moreover, the following inequality holds:

$$p_I < \min_{i \in I} \{k_i\}, \quad (30)$$

then P_I is a minimal group.

In fact, if P_I would not be a minimal group, then — by Theorem 1 — the inequality (24) would have to hold.

Corollary 3 is illustrated by an example shown in Fig. 4.

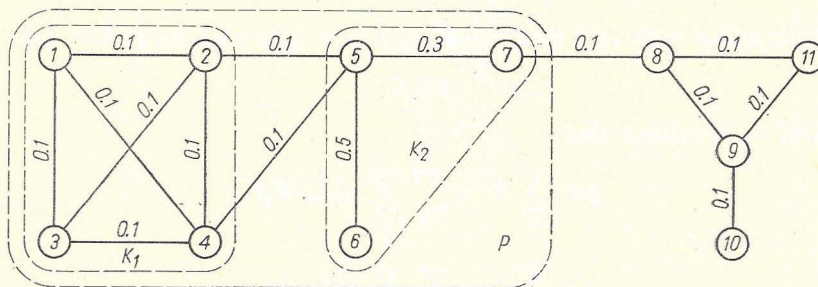


Fig. 4. The groups K_1 and K_2 are minimal, $k_1=0.2$, $k_2=0.3$, $p=0.1$. Thus, the group P is minimal as well

Corollary 2 and Proposition 1 refer above all to proper parts of groups. Theorem 1 and Corollary 3 make it possible to extend the properties of vertices of G (minimal groups as well) on all minimal groups. More precisely, it follows from them that every minimal group actually found can be considered as a vertex of a new weighted graph. Thus, if the process starts from $\langle G, w \rangle$, then, after having found a minimal group S , one can do the following. Namely, a new graph G^S is constructed by replacing all vertices of S by one and removing loops emerging in that way. Then, a function w^S is constructed:

$$w^S: E^S \rightarrow R_+ \cup \{0\}, \quad (31)$$

where: E^S — the set of all edges of G^S . The values of w^S (i.e. the matrix W^S) are defined by:

$$w_{ij}^S = \begin{cases} w_{ij} & \text{if } i, j \in V \setminus S, \\ a(S, \{j\}) & \text{if } j \in V \setminus S \text{ and } i \text{ is the vertex in } G^S \text{ replacing } S. \end{cases} \quad (32)$$

Thus, a new weighted graph $\langle G^S, w^S \rangle$ emerges. It is easy to note that there is a one — to — one mapping of any minimal group R^S found in that graph into the group:

$$R^S \setminus \{i^S\} \cup f(R^S \cap \{i^S\}) \quad (33)$$

in $\langle G, w \rangle$, where i^S denotes the vertex of G^S emerging from S and the function f :

$$f: \{\emptyset, \{i^S\}\} \rightarrow \{\emptyset, S\} \quad (34)$$

is defined by the following formula:

$$f(x) = \begin{cases} \emptyset & \text{if } x = \emptyset \\ S & \text{if } x = \{i^S\}. \end{cases} \quad (35)$$

Proposition 2. Let $R_i, E_i, i \in I = \{1, 2, \dots, m\}$, be pairwise disjoint groups, $R_i \neq \emptyset$. Let $K_i = R_i \cup E_i$ be a minimal group in $\langle G, w \rangle$ for every $i \in I$. Let us denote:

$$P = \bigcup_{i \in I} K_i, \quad F = \bigcup_{i \in I} R_i.$$

If there exists such $i \in I$ that $E_i \neq \emptyset$, then the following inequality holds:

$$p < f. \quad (36)$$

Proof. It is evident that:

$$p = \sum_{i \in I} k_i - \sum_{i \in I} \sum_{\substack{j \in I \\ j > i}} a(K_i, K_j), \quad (37)$$

$$f = \sum_{i \in I} r_i - \sum_{i \in I} \sum_{\substack{j \in I \\ j > i}} a(R_i, R_j). \quad (38)$$

For every $i, j \in I, i \neq j$, the following relation holds:

$$a(R_i, R_j) = a(K_i, K_j) - a(E_i, E_j). \quad (39)$$

Thus:

$$\sum_{i \in I} \sum_{\substack{j \in I \\ j > i}} a(K_i, K_j) \geq \sum_{i \in I} \sum_{\substack{j \in I \\ j > i}} a(R_i, R_j). \quad (40)$$

Since there exists such $i \in I$ that $E_i \neq \emptyset$, then by Definition 2 — it must be:

$$\sum_{i \in I} k_i < \sum_{i \in I} r_i, \quad (41)$$

i.e. $p < f$.

Q.E.D.

4. On some specific groups

The properties of groups given in the precedent part of the paper are quite sufficient for the construction of an efficient algorithm for determining minimal groups. The computational procedure can be, however, not efficient enough for high dimensional problems. Relations given below can cause a substantial increase of algorithms' speed. Moreover, they will be of use for the estimation of expected results.

Proposition 3. If in $\langle G, w \rangle$ there exists such $Q \subseteq V$, $|Q| \geq 3$, that for every $i, j \in Q$ $w_{ij} = w = \text{const.}$, then every group $S \subset Q$, $S \neq Q$, $|S| > 1$, is not minimal.

Proof. Let us remark that:

$$u = a(\{x\}, V \setminus \{x\}) = a(\{x\}, V \setminus Q) + w(|Q| - 1), \quad (42)$$

where $x \in S$. Moreover:

$$s = a(S, V \setminus Q) + w|S||Q - S|. \quad (43)$$

It is easy to note that by the assumptions taken:

$$|S||Q - S| \geq |Q| - 1 \quad (44)$$

and

$$a(S, V \setminus Q) \geq a(\{x\}, V \setminus Q). \quad (45)$$

From (44) and (45) it follows that $u \leq s$, which completes the proof. Q.E.D.

The proposition mentioned above makes it possible to eliminate from considerations the subsets of sets of vertices connected by edges with the same weight. It is illustrated by an example shown in Fig. 5.

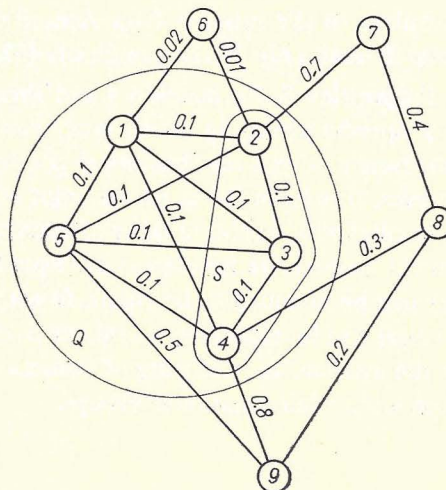


Fig. 5. For every duples $i, j \in Q$, $w_{ij} = 0.1$. The group $S \subset Q$ is not minimal

Proposition 4. Let for a given $\langle G, w \rangle$ there exists such $Q \subseteq V$, $|Q| \geq 2$, that for every $i, j \in Q$ the following relation holds:

$$w_{ij} = w = \max_{r, p \in V} \{w_{rp}\}. \quad (46)$$

Let $R \subset Q$, $R \neq Q$, $P \subset V \setminus Q$, $P \neq V \setminus Q$. If the inequality:

$$|Q| \geq |R| + |P| \quad (47)$$

holds, then the group $H = R \cup P$ is not minimal.

Proof. It is clear that:

$$h = p + w|R|(|Q| - |R|) + a(R, V \setminus (Q \cup P)) - a(R, P). \quad (48)$$

From (47) it follows that

$$w|R|(|Q| - |R|) - a(R, P) \geq 0. \quad (49)$$

Thus, $h \geq p$.

Q.E.D.

Proposition 5. Let for a given $\langle G, w \rangle$ there exists such $Q \subseteq V$, $|Q| \geq 2$, that for every $i, j \in Q$ the inequality (46) holds and for other pairs $s, t \in V$, $w_{st} \neq w$. Then, the necessary and sufficient condition for Q to be a minimal group is that for every $x \in Q$:

$$w(|Q| - 1) > a(Q \setminus \{x\}, V \setminus Q). \quad (50)$$

Proof. From the proof of Proposition 3 it follows that for every $R \subset Q$, $R \neq Q$ and for every $x \in Q$ we have:

$$r \geq a(\{x\}, V \setminus \{x\}) = u. \quad (51)$$

For proving the proposition it is, therefore, sufficient to consider the inequality:

$$u > q, \quad (52)$$

equivalent to the relation (50). According to the inequality (51), Q is a minimal group if and only if the condition (52) holds. Q.E.D.

Proposition 3, Proposition 4 and Proposition 5 formulated in this part concern some specific group, in which every two vertices are connected by edges with the same weight. From the theoretical point of view, it is a rear case. The same is not, however, true from the practical point of view. In fact, in applications there often occur cases with a great number of approximately equal weights. Taking into account that the parameters mentioned are usually approximated, or even estimated values, they can be assumed to be equal. In such a case, the properties of groups given in this part can be especially useful for the initial estimation of results and, therefore, for the eventual renumbering of vertices in order to increase the speed of the algorithm determining minimal groups.

5. An algorithm for determining minimal groups

In this part, an outline of the algorithm for determining minimal groups will be presented. A simplified flow diagram of the procedure is presented in Fig. 6. Some additional comments are given below. The procedure makes use of four

working lists — B_1, B_2, B_3 and B_4 . At the end of the process, all the minimal groups determined except the single vertices are contained in B_4 .

Let us describe briefly some of more important steps of the algorithm. Numbers placed before them refer to those in the flow diagram.

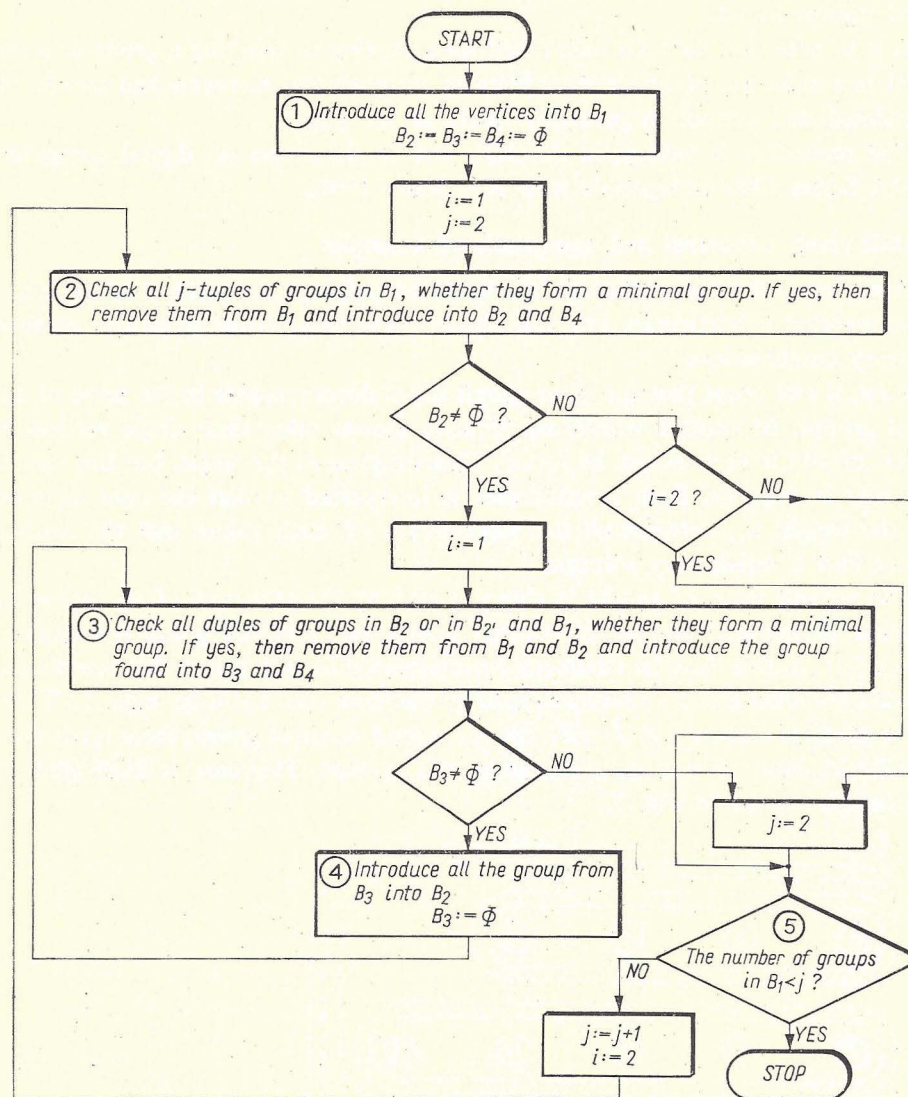


Fig. 6. Flow diagram of the algorithm for determining minimal groups

1. START. All the single vertices, i.e. initial minimal groups, are introduced into B_1 . All other lists, i.e. B_2, B_3 and B_4 are empty.

2. J — tuples of groups in B_1 are checked, whether they form a minimal group. If it is the case, then they are removed from B_1 and introduced into B_2 and B_4 .

3. Duples of groups in B_2 or in B_1 and B_2 are checked, whether they form a minimal group. If it is the case, then the appropriate groups are removed from B_1 and B_2 and the group actually found is introduced into B_3 and B_4 .

4. All the groups from B_3 are introduced into B_2 and $B_3 = \emptyset$.

5. If the number of groups is less than j , then END and B_4 contains all the minimal groups found.

It is to note that the tests mentioned for accepting or rejecting a group as a minimal one make use of appropriate lemmas, propositions, theorems and corollaries formulated and proved in precedent parts of the paper.

The procedure is written in ALGOL and implemented on digital computers Odra Series 1300 (compatible with ICL Series 1900).

6. Basic result structures and computational examples

Applying to the decomposition of a weighted unigraph the method of minimally interconnected subnetworks, one can obtain three basic result structures or their arbitrary combinations.

First, it can occur that the given graph is not decomposable in the sense of minimal groups. In another words, no minimal groups other than single vertices or the set of all the vertices can be found. This structure is the worst, but this case is very rear in practice. Such a result can be interpreted so that the data assumed (i.e. the weight w_{ij} between all the vertices) are of such values that the method cannot find a satisfactory solution.

The second form of results is characterized by the existence of an increasing sequence of minimal groups. In another words, one minimal group is contained in an other. In this case the result can be interpreted in such a way that one should choose a minimal group and assume it to be the basic one. Let it be, e.g., Z_1 . Then, a next group is chosen, e.g. $Z_2 \supset Z_1$. As the second minimal group, there is assumed $Z_2 = Z_1 \cup Z$. Next groups are constructed analogously. This case is illustrated by an example shown in Fig. 7.

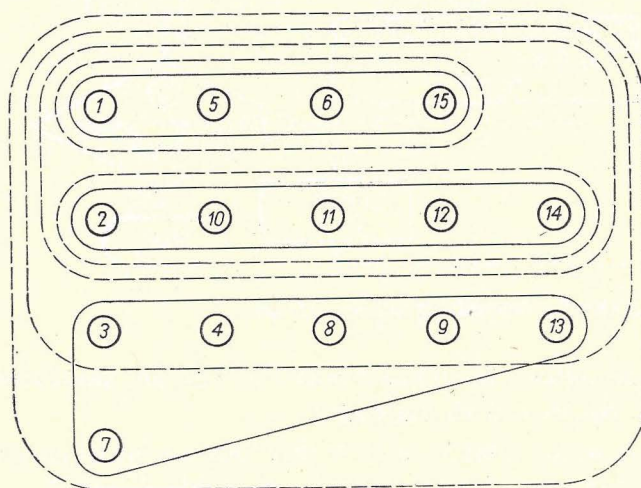


Fig. 7. The case of an increasing sequence of minimal groups. The groups consecutively found are shown by dashed lines and the solution — by continuous ones. For simplicity, edges and their weights are omitted

Table 1

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	10	0.1	0.2	0.9	2.8	1.6	0.2	0.1	0	0	0	0.1	0.1	1.2	0.8
2	0.1	10	0.1	0	0.1	2.8	0	0.1	0.1	7.5	2.2	0.1	0	5.7	0
3	0.1	0	10	0.8	0	0.1	1.3	4.3	5.3	0.2	0.1	0	0.7	0.1	0.2
4	1.1	0	1.2	10	0.1	0	0	1.4	1.7	0.4	0.2	0.3	0	0	0
5	3.2	0.2	0.1	0.1	10	5.9	0	0	0.3	0.6	0	0	0	0.2	3.1
6	5.4	0.2	0.2	0.1	3.1	10	1	0	0	0	0	0	0	0.2	2.4
7	0.2	0.1	1.7	0	0	1	10	2.7	0.2	0.1	0	0	0	0	0
8	0	0.1	4.7	1.6	0	0	1.3	10	8.7	0	0.3	0	0	0.1	0
9	0	0.2	3.7	1.3	0.2	0	0.1	0.3	10	0	0	0.1	1.8	0	0
10	0	0.5	0.2	0.2	2.4	0	0	0	0	10	6.6	4.1	0.1	0	0.1
11	0.1	4.8	0.1	0.1	0	0.1	0.1	0.1	0	1.4	10	8.2	1	7.6	1.1
12	0.1	0.3	0	0.2	0	0.1	0.1	0	0.1	5.9	1.8	10	0.5	3.2	0
13	0.2	0	2.3	0	0	0	0	0	1.2	2	1	1.5	10	0	0
14	0.8	0.3	0.1	0	0.1	0	0	0.5	0	0	1.4	1.8	0	10	0
15	4.2	0.1	0.2	0	4.9	1.6	0	0	0	0	0.9	0	0	0	10

In the third case — the best one — one obtains minimal groups, e.g. $Z_i, i=1, \dots, s$, satisfying the conditions:

$$Z_i \cap Z_j \neq \emptyset, \text{ for every } i, j \in \{1, \dots, s\}, \quad (52)$$

$$\bigcup_{k=1}^s Z_k = V. \quad (53)$$

One gets, therefore, directly the desired partition of the graph. Moreover, one can assume the solutions to be set — theoretic sums of the determined minimal groups.

For illustrating the method developed in the paper, let us now present a computational example.

Let us consider the problem of decomposing a graph consisting of 15 vertices, i.e. $V = \{1, \dots, 15\}$. The weights w_{ij} are shown in Table 1.

The method gives the following solution. As the first minimal group there is found the group $\{1, 5, 6, 15\}$, as the second one — $\{2, 10, 11, 12, 14\}$, as the third one — $\{1, 5, 6, 15, 2, 10, 11, 12, 14\}$, as the fourth one — $\{1, 5, 6, 15, 2, 10, 11, 12, 14, 3, 4, 8, 9, 13\}$ and as the fifth one — $\{7, 1, 5, 6, 15, 2, 10, 11, 12, 14, 3, 4, 8, 9, 13\}$. The results obtained form an increasing sequence of groups. One can, therefore, assume that the solution is as follows: $\{1, 5, 6, 15\}$, $\{2, 10, 11, 12, 14\}$, $\{3, 4, 8, 9, 13\}$ and $\{7\}$. Since one — element minimal groups are usually not interesting, then the last minimal group, i.e. $\{7\}$, should be inserted to another group. Preferably, it should join the group $\{3, 4, 8, 9, 13\}$, which was determined on the lowest level, i.e. as a part of the group found as the fourth one. Thus, the final result is:

$$\{1, 5, 6, 15\}, \{2, 10, 11, 12, 14\}, \{3, 4, 7, 8, 9, 13\}. \quad (54)$$

It is shown in Fig. 7.

7. Conclusions

In the paper, the method of minimally interconnected subnetworks as developed in [5] is extended on arbitrary graphs, in particular on unigraphs with nonnegative real weights. Except the reformulation and proving of properties given in [5], many original new ones are stated and proved. Moreover, some properties of specific groups are considered.

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APPENDIX

In the appendix, a short description of some applications of the method considered in the paper is given.

1. Application to electrical network [5]

An electrical network consists of a number of passive or active elements and links joining them. Links can cross one another. Moreover, some junctions are short and other ones are very long. Thus, the minimization of the crossing number and the decrease of links length is of great importance for increasing the network reliability and decreasing its cost. For solving this problem, it is necessary to compute the degree of connectivity between any two sets of elements. For two single elements, this measure is equivalent to the number of links joining them. It is obvious that elements (sets of elements) having a relatively high degree of connectivity should be situated in the vicinity. It decreases the cost of electrical network and increases its reliability.

2. Application for partitioning a group of enterprises into subgroups [4]

Let us denote by Z the group of enterprises, $Z = \{1, 2, \dots, n\}$. The set Z is to be partitioned into subsets Z_1, Z_2, \dots, Z_s , such that:

$$Z_i \cap Z_j = \emptyset \text{ for } i, j = 1, 2, \dots, s; i \neq j, \quad (\text{A1})$$

and

$$\bigcup_{i=1}^s Z_i = Z. \quad (\text{A2})$$

The partitioning criterion is as follows:

$$\sum_{\substack{h \in Z_i \\ m \in Z_i}} w_{hm} > \sum_{\substack{h \in Z_i \\ m \in Z \setminus Z_i}} w_{hm} \quad (\text{A3})$$

for every $Z_i \subset Z$, i.e. the strength of interconnections between the enterprises belonging to a given subgroup Z_i is greater than the strength of interconnections between those belonging to it and those not belonging to it.

The weights w_{ij} , $i, j = 1, 2, \dots, n$, represent the strength of interconnection between the i -th and the j -th enterprise and are defined by:

$$w_{ij} = \frac{1}{2} (q_{ij} + q_{ji}) \quad (\text{A4})$$

where:

$$q_{ij} = \alpha_1 a_{ij} + \alpha_2 b_{ij} + \alpha_3 k_{ij} + \alpha_4 p_{ij} + \alpha_5 g_{ij} + \alpha_6 t_{ij} + \alpha_7 r_{ij}. \quad (\text{A5})$$

Parameters in formula (A5) have the following meanings:

$\alpha_1, \alpha_2, \dots, \alpha_7$ — constants,

a_{ij} — coefficient of assortment similarity,

b_{ij} — coefficient of branch similarity,

k_{ij} — coefficient of cooperation intensivity,

p_{ij} — coefficient of interconnections in the field of raw materials and semiproducts,

g_{ij} — coefficient of geographical compactness (normed distance),

t_{ij} — coefficient of production — technological similarity,

r_{ij} — coefficient of economic system similarity.

3. Application for the decomposition of the telephone inter-exchange network [2, 3].

The process of the telephone network structure design is very complex. Initial decomposition of the network structure leads to a high simplification of the problem. It makes possible to obtain quickly a suboptimal solution. The partitioning criterion is given similarly to that in the case 2 by formulae (A1), (A2), (A3), where Z is now the set of telephone exchanges. The interpretation is also similar to that mentioned in case 2. The only difference is in the definition of the edge — weight w_{ij} describing the strength of interdependence between, respectively, the i -th and j -th exchanges. There is proposed a following one [2, 3]:

$$w_{ij} = \frac{a_1 (A_{ij} + A_{ji})}{b_0 + (b_1 + b_2 m) \mu(l_{ij}) l_{ij}} \quad (\text{A6})$$

where:

A_{ij} — traffic generated in the i -th and directed to the j -th exchange in busy hour,

l_{ij} — length of probable mutual junction for the i -th and j -th exchange,

b_0 — cost of the above mentioned junction independent of its length and capacity,

b_1 — the lowest cost of the considered junction by the unit of distance,

b_2 — the lowest average cost of a single trunk in junction connecting the i -th exchange by the unit of distance,

m — capacity of junction in number of trunks, dependent on A_{ij} and A_{ji} ,

μ — a nondecreasing function,

a_1 — a constant.

The function μ represents the increasing of distance unit cost in dependence on the connection length. Let the lowest cost (for the unit of length) be b . Further, let the cost of junction (for the unit of distance) be b_r for the junction length r . Then:

$$\mu(r) = \frac{b_r}{b}. \quad (A7)$$

The function μ is introduced in order to represent effects of the decrease of reference equivalent.

О уогólnieniu metody zespołów minimalnych

Zaprezentowano uogólnienie metody zespołów minimalnych [5]. Polega ono na takim przedstawieniu wspomnianej metody, że można ją zastosować do dowolnego grafu ważonego, zwłaszcza zaś do unigrafu z wagami nieujemnymi i rzeczywistymi. Na nowo sformułowano i udowodniono rezultaty uzyskane w pracy [5]. Ponadto podano i udowodniono nowe zależności. Wspomniane uogólnienie jest użyteczne przy rozwiązywaniu wielu problemów związanych z podziałem grafu. Polegają one na dekompozycji zbioru wszystkich wierzchołków na takie podzbiory, że sumaryczne związki między elementami tych podzbiorów są silniejsze niż analogiczne zależności między nimi a wierzchołkami spoza wspomnianych podzbiorów.

О обобщения метода минимальных взаимно-связанных подграфов

В статье предложено обобщение метода минимальных взаимно-связанных подграфов [5]. Оно состоит в таком представлении предлагаемого метода, что есть возможность применять его к всем графам с весами. Особенно он относится к униграфам, в которых веса это неотрицательные действительные числа. Еще раз сформулировано и доказано результаты получены в [5]. Кроме того предложено и доказано новые зависимости. Предлагаемое обобщение полезно для разрешения многих проблем разделения графа. Они состоят в том, чтобы декомпоновать множество всех вершин на такое подмножество, что итоговые взаимосвязи между элементами этих подмножеств были сильнее чем их связи с остальными вершинами.

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