Control
and Cybernetics
VOL. 6 (1977) No. 1

Optimum investment allocation in a model of economic growth

by

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The paper develops a model of economic growth designed to provide framework for optimal allocation of resources. The resources are assumed to be the result of production accumulated over a time interval $[0, T]$. The allocation takes place at the Decision Center into several categories of resources for labour, productive investments and government expenditures for public consumption and services. These resources are then assigned to production and public goods production sectors. The disposable part of production comes from two sources: depreciation allowances and household saving.

Mathematically the problem is characterized by a nonlinear dynamic system. The objective of the system is to maximize the net national product per capita over $[0, T]$. The problem possesses a unique global optimal solution expressible in exogenous variables.

An extension of the model is possible which provides a framework for dealing with optimal selection of prices.

I. Introduction

This paper presents an attempt to develop a model of economic growth designed to provide a framework for dealing with the problem of optimal allocation of resources (investments). The investments are assumed to be carried out by a decision center out of "savings" from two sources: depreciation allowances and household saving. Depreciation allowances are determined in accordance with a specific depreciation policy d which specifies the amount $d(t)\,dt$ commited for allocation (reinvestment) during the period $(t, t+dt]$ following the original investment at time 0 in production and public goods production sectors. At any moment, the difference between gross national product and the total rate of reinvestment (depreciation expense) is paid out to household and constitutes their net income, out of which a constant fraction is instantaneously saved and partly reinvested. The remaining part of the net national income yields the value of individual consumption.

The gross product is obtained from production and public goods production sectors. The production process with its dynamics (inertia and delays) is approximated by a dynamic, nonlinear operator.

A part of the net national product accumulated over a given time interval is allocated to several categories of resources for individual consumption, production investments and other government expenditures for public consumption and services. These resources are then assigned to the n production sectors. Individual savings are partly being used for the purchase of durable consumer goods, to acquire equity in houses and to accelerate the development of agriculture.

The amount of resources to be allocated are given exogenously while the resources in each category of government expenditures are selected based on a strategy yielding optimum of a utility function subject to budget constraints.

A dynamic problem of optimum allocation of investment is formulated as the maximization of a total net product per capita over a given time interval subject to accumulated "investments" constraints. The optimal solutions depend only on exogenous variables.

The presented model provides also framework for dealing with the optimal selection of prices assuring the satisfaction of all production sectors demands for labour, productive investments and government expenditures for public consumption and services.

For a single, homogeneous commodity that does duty as input, output, consumption good and capital good a similar model for optimal selection of investment projects was used in [4] by J. Chipman. The idea of using Hölder and Minkowski inequalities in the proof of Theorem 1 was taken from R. Kulikowski [5] where a similar optimization problem for $m=1$, $n=1$ was formulated.

II. Problem Description

Suppose there are n production and public goods production sectors in the considered economy. Each sector produces a given product and cooperates with the remaining sectors as shown in Figure 1[1]. Besides, each sector has to reinvest part of its production in order to increase the production capacity or at least to slow the rate of production decline. This reinvestment is usually called the maintenance. Without the maintenance, as shown in Figure 2, the production sector i would suffer a decline, the output of the sector would gradually decline through use and age of the machines and technology. Maintenance can increase the output level but on the other hand it must be subject to decreasing returns to scale. Therefore, the additional production due to increased reinvestment must be balanced against the additional expenses for maintenance. The maintenance policy should be selected as to maximize the discounted net product consisting of gross product minus maintenance expense. Let us now turn to a specific formulation of the fol-

[1] Cooperation between sectors will not be discussed in the paper. It has been discussed e.g. in [3] and [6].

lowing problem. Suppose there exists a vector $(z_1, ..., z_m)$ of commodity goods, where z_1 is interpreted as a consumption good (labour)[2], z_2 as a capital good (productive investments) and $z_3, ..., z_m$ as capital goods which correspond to government expenditures for education, research and development, medical care, administration etc. These resources for productive investments and both individual

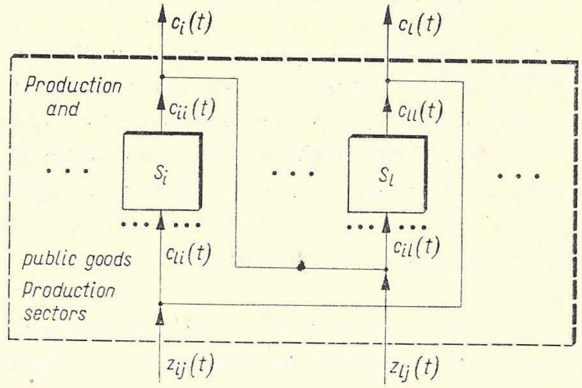

Fig. 1. Cooperation between production sectors (i and l)

and public consumptions are assigned to the production sectors of the economy by means of a matrix (z_{ij}), $i=1, ..., n$; $j=1, ..., m$, where the element z_{ij} is the j-th commodity good assigned to production sector i. Thus, the vector $z^i(\tau)=(z_{i1}(\tau), ...$ $..., z_{im}(\tau))$ denotes the intensities of different commodity goods that are assigned to the production sector i and in the centralized economy they are labour, capital,

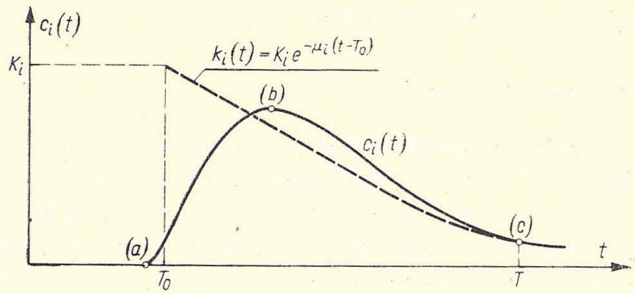

Fig. 2. Production operator $c_i(t)$ with $\psi_i(t)=1(t)$ and typical $k_i(t)$

individual and public consumption goods of sector i, allocated by the Decision Center at time instant t. Let the function $\psi_i(z^i(\tau))$, $i=1, ..., n$, be the output-result of a transformation which assigns the above commodity goods (labour and capital),

[2] Labour is assumed to be homogeneous given by the logistic growth model $\dot{z}_1(t)=c_i z_1(t)$ $[Z_1-z_1(t)]$, where Z_1 is the maximum possible labour force at the end of planning interval and $z_1(t)$ is the sustainable labour at time t.

to the production sector i at time τ. This instantaneous function may be assumed in the form of a constant elasticity of substitution (C.E.S.) function

$$\psi_i\left(z^i(\tau)\right)=\left[\sum_{j=1}^{m} \delta_j \left(z_{ij}(\tau)\right)^{-\nu}\right]^{-\frac{r}{\nu}}, \quad i=1, \ldots, n, \tag{1}$$

where δ_j, $-\nu$, r are given positive numbers, $\sum_{j=1}^{m} \delta_j=1$, $\nu \in (-1, 0]$ and $z_{ij}(\tau)$, $i=1, \ldots, n$; $j=1, \ldots, m$, are the input costs of this transformation at a given point in time τ. In order to take into account the dynamics of the production process (inertial phenomena and delays) the process will be approximated by an integral operator. Thus, *the aggregated*, over different vintage "investments"[3], *gross output* of the i-th production sector is determined by an integral equation of the form

$$c_i(t)=c_i\left(z^i(t)\right)=\int_0^t \tilde{k}_i(\tau)\, \psi_i\left(z^i(\tau)\right) d\tau+c_{0i}(t), \quad i=1, \ldots, n, \tag{2}$$

where $\tilde{k}_i(\tau)$, $i=1, \ldots, n$, are given, positive, continuous functions, $t \in [0, T]$ and $c_{0i}(t)$ is an exogenous term which may be interpreted as consisting of returns at time t from investments made prior to calendar time 0.

An "investment" $\tilde{k}_i$ is a function defined on $(0, \infty)$ indicating the return $\lambda \tilde{k}_i(\tau)\, d\tau$ during the interval $(t_0+\tau, t_0+\tau+d\tau]$ from the initial investment of λ units at time $t_0 \geqslant 0$.

If $\psi_i\left(z^i(\tau)\right)$ approximates an unitary pulse and $c_{0i}(t)=0$, then $c_i(t)$ changes in a manner similar to that shown in Figure 2. From the moment of investment (calendar time 0) up to stage (a) no production can be obtained. The interval $[0, (a)]$ corresponds to an investment delay (gestation lag). An increase of production occurs over interval $((a), (b)]$, followed by a slow depreciation of investment resulting in the sector production decrease.

Collapsibility of Production Function

In the above model the quantity of capital must be given a consistent meaning. As described by Solow[4] and Leontief[5] only in a narrow class of cases the various capital inputs can be summed up in a single index-figure so that the production function can give output as a function of inputs of labour (assumed here homogenous) and services of several capital goods treated as the overall index of capital.

Proposition I

A necessary and sufficient condition for the collapsibility of the production function $\psi(L, C_1, \ldots, C_m)$, with m distinct kinds of capital, to the production function

[3] The term investments refers here to all capital expenditures of the government z_j, $j=1, \ldots, m$, with labour included.

[4] See *Review of Economic Studies* **23** (1955–1956) 101–108: The production function and the theory of capital.

[5] See *Econometrica* **15**, 4 (1947) 364: Proposition I.

$\varphi(L, K)$ with the single index of the quantity of capital is that the marginal rate of substitution of one kind of capital good for another must be independent of the amount of labour in use, i.e.

$$\frac{\partial\psi/\partial C_l}{\partial\psi/\partial C_j}=\frac{\dfrac{\partial\varphi}{\partial K}\cdot\dfrac{\partial\Phi}{\partial C_l}}{\dfrac{\partial\varphi}{\partial K}\cdot\dfrac{\partial\Phi}{\partial C_j}}=\frac{\partial\Phi/\partial C_l}{\partial\Phi/\partial C_j},\quad l\neq j,\ l, j=1, \ldots, m,$$

where $K=\Phi(C_1, \ldots, C_m)$.

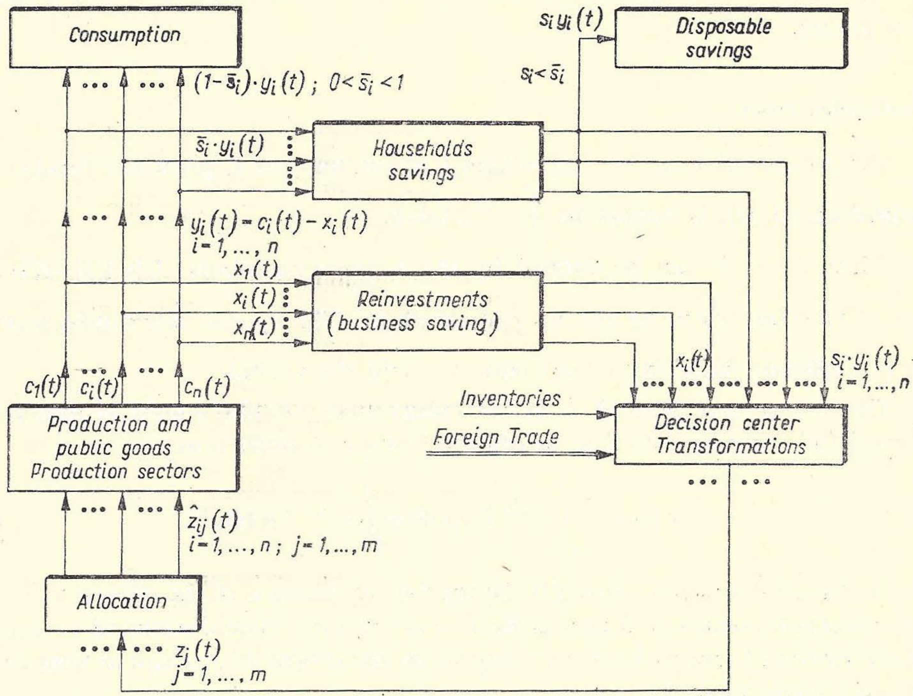

Fig. 3. A closed model of economic growth

Thus, we can write

$$\psi(L, C_1, \ldots, C_m)=\varphi(L, K)$$

and for the purposes of production any patterns of inputs $C_1, \ldots, C_m$ are equivalent so long as they yield the same value of the index K.

It should be amphasized that the index-function Φ and the collapsed function φ have the characteristics we usually associate with production functions.

The marginal rate of substitution which does not involve labour L can be obtained for the general class of production functions with "means" $\psi=f^{-1}[f(L)+ +f(C_1)+\ldots+f(C_m)]$, usually restricted to be homogeneous of first degree with the functions φ and Φ having all the desired properties of homogeneity and convexity. In the case of C.E.S. function (1) the marginal rate of substitution of, for instance, C_l for C_j i.e. z_{l+1} for z_{j+1} is $\frac{\delta_{l+1}}{\delta_{j+1}} \cdot \left(\frac{z_{j+1}}{z_{l+1}}\right)^{-\frac{r}{v}}$. In the model, investment is assumed to be carried out by a production and business sector out of funds coming from two sources: depreciation allowances and household savings[6]. Depreciation allowances are determined in accordance with a specific depreciation policy d, which is a function defined on $(0, \infty)$ and indicates the amount of resources $\lambda d(\tau)\, d\tau$ set aside during the interval $(t_0+\tau,\ t_0+\tau+d\tau]$ for purposes of reinvestment committed for this purpose when an investment of λ units was made at time t_0[7]. These set aside resources will be referred to as business and production saving.

Depreciation Policy

The *present value* of the time stream k at interest rate $\bar{r}$, given any function k defined on $(0, \infty)$, is defined by $\int_0^T e^{-\bar{r}t}\, k(t)\, dt$.

When $T \to \infty$ it can be defined by the Laplace transform $L[k(t)] = K(\bar{r}) = \int_0^\infty e^{-\bar{r}t}\, k(t)\, dt$, whenever the integral converges. The interest rate can be treated as a coefficient indicating a cost rate of using the capital.

The *current value* (worth) of an investment project $\tilde{k}$ after t units of time have elapsed following its initiation, at discound rate $\bar{r}$, is defined as

$$\tilde{w}(t) = e^{\bar{r}t} \int_t^T e^{-\bar{r}\tau}\, \tilde{k}(\tau)\, d\tau = \int_t^T e^{\bar{r}(t-\tau)}\, \tilde{k}(\tau)\, d\tau. \tag{3}$$

Let the *rate of depreciation* $d(t)$ be the rate of decrease of the current value of the investment, which is in turn defined as the present value discounted to time t, of the stream of returns $\tilde{k}(\tau)$, $\tau > t$, due to an investment of one unit at time zero, at some interest rate $\bar{r}$.

Consider the depreciation policy of sector i defined by

$$d_i(t) = \tilde{k}_i(t) - \bar{r}\left(v_i - \int_0^t d_i(\tau)\, d\tau\right) \tag{4}$$

for some $\bar{r} > 0$, $t > 0$ and some $v_i > 0$, where $v_i = \tilde{w}_i(0)$, $i = 1, \ldots, n$, is the initial book value of the capital investment of one unit, the term in the parentheses represents the book value at time t of the original investment. Multiplying this by $\bar{r}$, which

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[6] In the centralized economy the rate of business and production saving (funds for allocation available from production sectors) and the depreciation rate are subject to the Decision Center policy.

[7] The word "reinvestment" has been used since it is assumed that the investment which determines a level of further production follows an initial investment, given exogeneously.

can be interpreted as an accounting interest rate, gives the accounting cost rate at time t of the use of the capital, equivalent to the value of the original investment.

The rate of depreciation at time t is chosen to equalize this cost rate and the rate of net yield of the investment, $\tilde{k}_i(t)-d_i(t)$.

The depreciation policy d_i associated with the investment project $\tilde{k}_i(t)$ is assumed to satisfy the condition

$$\left|\int_0^\infty d_i(t)\,dt\right|<\infty, \quad i=1,\ldots,n.$$

From the definitions (4) and (3) and the assumption that $\tilde{w}_i(t)=v_i-\int_0^t d_i(\tau)\,d\tau$ follows that the *declining value depreciation policy* associated with $\tilde{k}_i$ at discount rate $\bar{r}$ is given by

$$d_i(t)=-\tilde{w}_i'(t)$$

and may also be expressed as

$$d_i(t)=\tilde{k}_i(t)-\bar{r}\tilde{w}_i(t).$$

Thus, there exists the explicit solution for $d_i(t)$ of equation (4), given $\tilde{k}_i(t)$.

Allocation Model

The *aggregate reinvestment* in sector i, determined by the depreciation expense $x_i(t)$, is defined by

$$x_i(t)=\int_0^t d_i(\tau)\,\psi_i\big(z^i(\tau)\big)\,d\tau+x_{0i}(t), \qquad i=1,\ldots,n \tag{5}$$

where $x_{i0}(t)$ is an exogenous term denoting the rate of business and production saving resulting from commitments already made prior to time 0 (it includes depreciation policies initiated before that date).

The *net product* of sector i at time t, $y_i(t)$, is the difference between the gross product $c_i(t)$ (the total rate of return from past investments) and the total rate of business and production reinvestment $x_i(t)$, (depreciation expense), i.e.

$$y_i(t)=c_i(t)-x_i(t)=\int_0^t k_i(\tau)\,\psi_i\big(z^i(\tau)\big)\,d\tau+y_{0i}(t), \tag{6}$$

where $k_i(\tau)=\tilde{k}_i(\tau)-d_i(\tau)$, $i=1,\ldots,n$, is the cost rate of using the capital equal to the value of the original investment at time τ and $y_{0i}(t)=c_{0i}(t)-x_{0i}(t)$. The net product is assumed to be paid out to households, which in turn save a constant fraction $\bar{s}_i$, $0<\bar{s}_i<1$, of their net incomes. This constant fraction is reinvested in selected sectors of the economy.

The experience of last years has shown that a part of the households saving can be (and has been) used as credits in agriculture and private housing. It is well known that the dynamic development of agricultural production can stimulate the development of an economy. In Poland, for instance, over 80% of cultivated land is privately owned. The money saved, over a long period of time, by individuals can be partly used for the development of agriculture and selected areas of economy which are crucial to the overall development of a society. They can be also used in those areas in which the supply, in spite of large investment and effort, still does not satisfies demands. Encouraging individuals to use their saving for building houses can allow to better satisfy the demand for housing facilities. A proper utilization of households saving can accelerate the development of the whole economy. However, one should not use individual saving for productive investments.

The amount of credits coming from individual saving should be evaluated very carefully and result from thorough investigations.

Assume that a fraction s_i, $s_i < \bar{s}_i$ of net households income is used for credits to develope agriculture and private housing. This means that at any instant of time there exists a disposable, over a short period of time, e.g. one year, saving that can be paid back to individuals. Then, the system can be closed by stipulating the equality of gross saving and investment where the gross saving includes this part of household saving which over a long time period has been used for financing credits in agriculture and housing.

The equality can be written in the following form

$$\sum_{j=1}^{m} w_j(t)\, z_j(t) = \sum_{i=1}^{n} [\bar{x}_i(t) + s_i\, \bar{y}_i(t)] \tag{7}$$

where $\sum_{i=1}^{n} z_{ji}(t) = z_j(t)$, $\bar{y}_i(t) = \bar{c}_i(t) - \bar{x}_i(t)$,

$$\bar{x}_i(t) = \int_0^t d_i(\tau)\, \psi_i\big(w_j(t)\, z_{ij}(\tau)\big)\, d\tau + x_{0i}(t),$$

$$\bar{c}_i(t) = \int_0^t \tilde{k}_i(\tau)\, \psi_i\big(w_j(t)\, z_{ij}(\tau)\big)\, d\tau + c_{0i}(t),$$

and $w_j(t)$, $j = 1, \ldots, m$, denotes prices of labour and capital services of government for productive investments and public consumption sectors.

A structural constraint must be adopted in the model to assure that the production of a given sector i in natural units, is sufficient to satisfy the demand of all sectors for the i-th aggregated sector good treated as an input to production and public goods production sectors

$$\sum_{j=1}^{m} z_{ij}(t) \leqslant \frac{\bar{c}_i(t)}{p_i(t)} \tag{8}$$

where $p_i(t)$, $i = 1, \ldots, n$, is the aggregated sector price.

It is assumed that both sector and, labour and capital prices are exogenous in the model. The aggregated sector prices are viewed as equilibrium prices. In general they must depend on the quantity of output, the price for labour and for government capital expenditures and the consumption structure.

The discounted cumulative net product per capita from n production sectors (*the net national product*) per capita over time interval $[0, T]$ is

$$\pi(T)=\sum_{i=1}^{n}\int_{0}^{T}\frac{e^{-\bar{r}t}}{z_1(t)}\,\bar{y}_i(t)\,dt, \tag{9}$$

where $z_1(t)$ denotes labour force at time t and $e^{-\bar{r}t}$ is the discounting function[8].

For the model described by equations (1)–(9) one could think of formulating two distinct optimization problems. In both of them the same objective can be used, i.e. maximization over interval $[0, T]$, of the discounted net national product $\pi(T)$ which is equivalent to maximization of the per capita consumption in the system since

$$\text{Consumption}=\sum_{i=1}^{n}(1-\bar{s}_i)\,\bar{y}_i,\quad 0<s_i<\bar{s}_i<1,\quad i=1, \dots, n.$$

The above closed system has only theoretical and illustrative meaning since it could be applied only in the case when the net balance of foreign trade is zero and the inventories are kept constant over time at their initial value. Therefore, the problem will be formulated to optimize the consumption per capita in the open system with foreign trade balance and inventories included in the disposable national income.

III. Solution

Solution is given to only one optimization problem formulated for the model in which the structural equation is checked after the problem has been solved.

Optimization Problem I

Assume the following values to be given:

a) the discount rate $\bar{r}>0$;

b) the time interval $[0, T]$, $T>0$;

c) the continuous, positive functions, depreciation $d_i(t)$ and investment return $\tilde{k}_i(t)$, defined over $(0, \infty)$ for all $i=1, \dots, n$;

d) the parameters of the C.E.S. production function (1), i.e. positive numbers r, δ_j, $-v$, where $\sum_{j=1}^{m}\delta_j=1$ and $v\in(-1, 0]$;

e) the sector prices $p_i(t)$, $i=1, \dots, n$, and the prices for labour $w_1(t)$, productive investments $w_2(t)$ and government expenditures $w_3(t), \dots, w_m(t)$.

Then, we can look for the optimal allocation strategy, i.e. the optimum values $z_{ij}(t)=\hat{z}_{ij}(t)$, $i=1, \dots, n$; $j=1, \dots, m$, such that the global net product per capita

[8] For the discussion of discounting functions in the investment optimization problem see [1] p. 41–45: Strotz phenomenon.

$\pi(t)$, given by (9), is optimum, provided that the funds (for allocation) Z_j in each class of government evxpenditures j are given exogenously and are defined by

$$Z_j(t)=\sum_{i=1}^{n}\int_0^t \bar{z}_{ij}(\tau)\,d\tau, \quad j=1,\ldots,m, \tag{10}$$

where $\bar{z}_{ij}(t)=w_j(t)\,z_{ij}(t)$, $t\in[0,T]$.

The values $Z_j(t)$ depend on inventories, net balance of foreign trade, prices and the national product per capita generated over time $[0,t]$. These functional dependencies are briefly discussed at the end of this section. Substituting $\bar{z}_{ij}(\tau)$ in $\bar{c}_i(t)$, $\bar{x}_i(t)$ and using (1), (2), (7) and (8) the problem of maximization of the net national product per capita can be written

$$\max_{z_{ij}(\tau)\in\Omega}\left\{\pi(T)=\sum_{i=1}^{n}\int_0^T \frac{e^{-\bar{r}t}}{z_1(t)}\,\bar{y}_i(t)\,dt=\right.$$

$$\left.=\sum_{i=1}^{n}\int_0^T \frac{e^{-\bar{r}t}}{z_1(t)}\left[\int_0^t k_i(\tau)\left(\sum_{i=1}^{m}\delta_j\,[\bar{z}_{ij}(\tau)]^{-\nu}\right)^{-\frac{r}{\nu}}d\tau+\bar{y}_{0i}(t)\right]dt\right\}, \tag{11}$$

where

$$\Omega=\left\{z_{ij}(\tau):\sum_{i=1}^{n}\int_0^T \bar{z}_{ij}(\tau)\,d\tau\leqslant Z_j,\ z_{ij}(\tau)>0,\ \tau\in[0,T],\ i=1,\ldots,n:\ j=1,\ldots,m\right\}, \tag{12}$$

Theorem 1. Let n production operators $c_i(t)$ be given by (2) and the assumptions a—d be satisfied. Then, there exists the unique, optimum allocation strategy $z_{ij}(\tau)=\hat{z}_{ij}(\tau)$ for $\tau\in[0,T]$,

$$\hat{z}_{ij}(\tau)=\frac{Z_j}{F}\,\frac{f_i(\tau)}{w_j(\tau)},\ i=1,\ldots,n;\ j=1,\ldots,m \tag{13}$$

which yields the global net product per capita $\pi(T)$ over time interval $[0,T]$ (with $\bar{y}_{0i}(t)$ given)

$$\hat{\pi}(T)=\max_{z_{ij}(\tau)\in\Omega}\sum_{i=1}^{n}\int_0^T \frac{e^{-\bar{r}t}}{z_1(t)}\int_0^t k_i(\tau)\left(\sum_{j=1}^{m}\delta_j\,[w_j(\tau)\,z_{ij}(\tau)]^{-\nu}\right)^{-\frac{r}{\nu}}d\tau\,dt=$$

$$=F^q\left[\sum_{j=1}^{m}\delta_j\,Z_j^{-\nu}\right]^{-\frac{r}{\nu}} \tag{14}$$

where

$$F=\sum_{i=1}^{n}\int_0^T f_i(\tau)\,d\tau, \tag{15}$$

$$f_i(\tau)=\left\{k_i(\tau)\int_\tau^T \frac{e^{-\bar{r}t}}{z_1(t)}\,dt\right\}^{1/q},\ q=1-r. \tag{16}$$

Theorem 1 has been proved in Appendix A.

It is assumed that the sum of "investment resources" over time is given

$$\sum_{j=1}^{m}Z_j=Z \tag{17}$$

where Z is exogenous. However, in planning practice Z is a disposable part of the net national income generated over the previous planning interval to be allocated to several categories of resources for labour (individual consumption) productive investments and other government expenditures including public consumption and services. These resources are then assigned to the n production and public goods production sectors.

Assuming $t=T_0$ to be a base year (beginning of a planning interval) and $t=0$ to be the beginning of the previous planning interval

$$Z=\sum_{i=1}^{n}\int_{0}^{T_0}[\bar{x}_i(t)+s_i\bar{y}_i(t)]\,dt+Z_{in}(0)-Z_{in}(T_0),$$

where Z_{in} denotes inventories with a net balance of the foreign trade incorporated into it.

Thus, it is necessary to find an allocation strategy $\hat{Z}_j$, $j=1, \dots, m$, which maximizes the function (11).

The problem can be formulated as follows:

$$\max_{Z_j\in\Omega_1}\left\{\sum_{j=1}^{m}\alpha_j Z_j^{-v}=[\hat{\pi}(T)]^{-v/r}\right\} \tag{18}$$

where

$$\Omega_1=\left\{Z_j:\ \sum_{j=1}^{m}Z_j\leqslant Z, Z_j\geqslant 0,\ j=1, \dots, m\right\},$$

and

$$\alpha_j=\delta_j F^{-\frac{v}{r}(1-r)}$$

The optimum allocation strategy[9]

$$\hat{Z}_j=\frac{\alpha_j^{\frac{1}{1+v}}}{\sum_{j=1}^{m}(\alpha_j)^{\frac{1}{1+v}}}Z,\ j=1, \dots, m. \tag{19}$$

and

$$[\hat{\pi}(T)]^{-v/r}=\left(\sum_{j=1}^{m}\alpha_j^{\frac{1}{1+v}}\right)^{1+v}Z^{-v}.$$

Thus, the optimum net product per capita

$$\hat{\pi}(T)=\left(\sum_{j=1}^{m}\alpha_j^{\frac{1}{1+v}}\right)^{-\frac{1+v}{v}r}Z^r. \tag{20}$$

One may compute now the marginal cost of a change in Z_j, $\frac{\partial\pi(T)}{\partial\hat{Z}_j}$ which depends on the cost of using the invested capital, labour growth, the discounting function and the parameters of the C.E.S. function.

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9) See Appendix B.

IV. Extension of the Model

Within presented framework another optimization problem can be formulated.

Optimization Problem II

Assuming conditions a to d to hold and given "savings" Z_j, $j=1, \ldots, m$, and demands $z_{ij}(t)$, $i=1, \ldots, n$; $j=1, \ldots, m$, find the sector prices $p_i(t)$ and prices $w_j(t)$ for labour, capital and capital expenditures, which yield the maximum per capita consumption in the model or equivalently the maximum net product per capita given by (11).

The above problem will not be pursued further in this paper.

One of the most difficult problems in the socialist economy seems to be the model of prices. Prices should provide market equilibrium and the maximum of a social utility. The resulting, optimum consumption structure should stimulate the incentives of producers, compensate the impact of personal saving on the market and provide for inexpensive basic consumption goods.

It should be emphasized that the aggregated sector prices p_i, the same for all commodities produced by sector i, are by far not a perfect approach.

However, even their impact on the structural relation (8) and on the optimal allocation strategy is very difficult to investigate.

The optimization problem has been formulated for an open system in which the resources available for allocation are assumed to be given exogeneously and the optimum allocation strategy is obtained under assumptions that the "investments" made prior to time zero yield given returns. This seems to be no drawback since in economic planning of centrally governed countries one has to know or assume given the amount of resources at time t to be allocated after that time. These given numbers can be checked for consistency with projections based on estimates of resources in previous years which are in turn based on historical data. For instance $Z_j(t) = \varepsilon_j'(t)\, Z(t)$, where $\sum_{j=1}^{m} \varepsilon_j = 1$ can be selected as a solution of an optimization problem yielding optimum of a utility function subject to budget constraints[10]. The values $\varepsilon_j(t)$ change over time due to changes in the GNP per capita and prices. They can be estimated based on "ex post" specification of the GNP per capita, prices and their elasticities. Another possible extension of the paper could be the investigation of the invariance of the system with respect to the personal saving yielding its best utilization.

Also, the optimal solution in the closed model with zero balance of foreign trade and constant inventories would give more insight into allocation mechanism.

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[10] The utility function can assume either Cobb-Douglas or C.S.E. form — see e.g. Appendix B.

Appendix A

Proof of Theorem 1

The global net product (11), is

$$\pi(T)=\sum_{i=1}^{n}\int_{0}^{T}\frac{e^{-\bar{r}t}}{z_1(t)}\int_{0}^{t}k_i(\tau)\left(\sum_{j=1}^{m}(\delta_j[\bar{z}_{ij}(\tau)]^{-\nu}\right)^{-\frac{r}{\nu}}d\tau\,dt), \tag{21}$$

where $\bar{z}_{ij}(\tau)=w_j(\tau)\,z_{ij}(\tau)$ and $\bar{y}_{0i}(t)=0$ [11].

Changing the integration order we have

$$\pi(T)=\sum_{i=1}^{n}\int_{0}^{T}\left\{\left(\sum_{j=1}^{m}\delta_j[\bar{z}_{ij}(\tau)]^{-\nu}\right)^{-\frac{r}{\nu}}\int_{\tau}^{T}\frac{e^{-\bar{r}t}}{z_1(t)}k_i(\tau)\,dt\right\}d\tau. \tag{22}$$

Denoting by

$$Y_{ij}(\tau)=\delta_j\left\{\int_{\tau}^{T}\frac{e^{-\bar{r}t}}{z_1(t)}k_i(\tau)\,dt\right\}^{-\frac{\nu}{r}}[z_{ij}(\tau)]^{-\nu} \tag{23}$$

and substituting $Y_{ij}(\tau)$ into (22) yields

$$\pi(T)=\sum_{i=1}^{n}\int_{0}^{T}\left(\sum_{j=1}^{m}Y_{ij}(\tau)\right)^{l}d\tau, \tag{24}$$

where $l=-\dfrac{r}{\nu}$.

The Minkowski inequality for integrals yields

$$\sum_{i=1}^{n}\int_{0}^{T}\left[\sum_{j-1}^{m}Y_{ij}(\tau)\right]^{l}d\tau\leqslant\left\{\sum_{j=1}^{m}\left[\int_{0}^{T}\sum_{i=1}^{n}Y_{ij}^{l}(\tau)\,d\tau\right]^{\frac{1}{l}}\right\}^{l}. \tag{25}$$

The equality in (23) holds iff

$$Y_{ij}(\tau)=c_j^1\,Y_{i,j+1}(\tau),\ i=1,\dots,n;\ j=1,\dots,m \tag{26}$$

where c_j^1 is a positive constant.

Consider the expression

$$\int_{0}^{T}\sum_{i=1}^{n}[Y_{ij}(\tau)]^{l}\,d\tau=\sum_{i=1}^{n}\int_{0}^{T}[Y_{ij}(\tau)]^{l}\,d\tau=\sum_{i=1}^{n}\int_{0}^{T}\delta_j^l\int_{\tau}^{T}\frac{e^{-\bar{r}t}}{z_1(t)}k_i(\tau)\,dt\,[\bar{z}_{ij}(\tau)]^{r}\,d\tau$$

and denote

$$f_i(\tau)=\left\{\int_{\tau}^{T}\frac{e^{-\bar{r}t}}{z_1(t)}k_i(\tau)\,dt\right\}^{\frac{1}{q}},\qquad q=1-r. \tag{27}$$

Thus,

$$\int_{0}^{T}[Y_{ij}(\tau)]^{l}\,d\tau=\delta_j^l\int_{0}^{T}f_i^q(\tau)\,[\bar{z}_{ij}(\tau)]^{r}\,d\tau. \tag{28}$$

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[11] This assumption does not affect the optimal solution.

Applying Hölder inequality

$$\delta_j^l \int_0^T f_i^q(\tau)\,[\bar{z}_{ij}(\tau)]^r\,d\tau \leqslant \delta_j^l \left\{\int_0^T f_i(\tau)\,d\tau\right\}^q \left\{\int_0^T \bar{z}_{ij}(\tau)\,d\tau\right\}^r. \tag{29}$$

The equality in (29) holds iff

$$\bar{z}_{ij}(\tau) = c_j^2 f_i(\tau), \qquad i=1,\dots,n;\ j=1,\dots,m; \tag{30}$$

where c_j^2 is a positive constant.

The optimum strategy $\bar{z}_{ij}(\tau)$ yields the equality in constraints (12).

$$\sum_{i=1}^{n} \int_0^T \bar{z}_{ij}(\tau)\,d\tau = Z_j, \qquad j=1,\dots,m. \tag{31}$$

Substituting (30) into (31) yields

$$c_j^2 = \frac{Z_j}{F}, \qquad j=1,\dots,m, \tag{32}$$

where $F=\sum_{i=1}^{n}\int_0^T f_i(t)\,dt$. Thus, using (30) and (32)

$$\hat{\bar{z}}_{ij}(\tau) = \frac{Z_j}{F}\cdot f_i(\tau) \quad \text{and} \quad \hat{z}_{ij}(\tau) = \frac{Z_j}{F}\cdot\frac{f_i(\tau)}{w_j(t)}. \tag{33}$$

The optimal value of the global profit per capita $\hat{\pi}(T)$, using (22), (23), (25), (28), (32) and (33) yields

$$\hat{\pi}(T) = \left\{\sum_{j=1}^{m}\left[\int_0^T \sum_{i=1}^{n} Y_{ij}^l(\tau)\,d\tau\right]^{\frac{1}{l}}\right\}^l =$$

$$= \left\{\sum_{j=1}^{m}\left[\sum_{i=1}^{n} \delta_j^l \left\{\int_0^T f_i(\tau)\,d\tau\right\}^q \left\{\int_0^T \hat{\bar{z}}_{ij}(\tau)\,d\tau\right\}^r\right]^{\frac{1}{l}}\right\}^l =$$

$$= \left\{\sum_{j=1}^{m}\left[\sum_{i=1}^{n} \delta_j^l \left(\frac{Z_j}{F}\right)^r \left\{\int_0^T f_i(\tau)\,d\tau\right\}^q \left\{\int_0^T f_i(\tau)\,d\tau\right\}^r\right]^{\frac{1}{l}}\right\}^l =$$

$$= \frac{\sum_{i=1}^{n}\int_0^T f_i(\tau)\,d\tau}{(F)^r}\left\{\sum_{j=1}^{m} \delta_j Z_j^{\frac{r}{l}}\right\}^l = F^{1-r}\left[\sum_{j=1}^{m} \delta_j Z_j^{-\nu}\right]^{-\frac{r}{\nu}}.$$

Since $q=1-r$ and

$$c_j^1 = \frac{Y_{ij}(\tau)}{Y_{i,j+1}(\tau)} = \frac{\delta_j Z_j}{\delta_{j+1} Z_{j+1}} = \text{const} > 0.$$

the equation (26) is satisfied.

Thus, we have proved Theorem 1 and found the optimal solution to the investment allocation problem. Q.E.D.

Appendix B

I. Let us consider the following problem:

$$\max_x \left\{\Phi(x)=\sum_{i=1}^{m} c_i x_i^{\alpha}\right\} \tag{34}$$

subject to

$$g(x)=\sum_{i=1}^{m} x_i - X \leqslant 0,$$

where $x_i \geqslant 0$, $0<\alpha<1$, $c_i>0$, X — given positive number and $x=(x_1, ..., x_m)$.

It is implicit that $\Phi(x)$ and $g(x)$ are differentiable at $\bar{x}=(\bar{x}_1, ..., \bar{x}_m)$, where $\bar{x}$ is the optimal solution to (34). Thus, we can apply Kuhn—Tucker conditions.

$$\nabla_x L(\bar{x}, \bar{\lambda})=0, \quad \nabla_\lambda L(\bar{x}, \bar{\lambda})\leqslant 0, \quad \lambda \nabla_\lambda L(\bar{x}, \bar{\lambda})=0, \tag{35}$$

$$\lambda \geqslant 0, \quad \text{where} \quad L(x, \lambda)=\Phi(x)+\lambda g(x).$$

Conditions (35) for the optimization problem (34) can be written

$$\alpha c_i x_i^{\alpha-1}-\lambda=0, \qquad \sum_{i=1}^{n} x_i - X \leqslant 0,$$

$$\lambda\left(\sum_{i=1}^{n} x_i - X\right)=0$$

$$\lambda \geqslant 0$$

yielding

$$\hat{x}_i=\frac{c_i^{\frac{1}{1-\alpha}}}{\sum_{j=1}^{n}(c_j)^{\frac{1}{1-\alpha}}} X, \quad i=1, ..., m \tag{36}$$

and

$$\hat{\Phi}=\Phi(\hat{x})=\left(\sum_{i=1}^{n} c_i^{\frac{1}{1-\alpha}}\right)^{1-\alpha} X^{\alpha}. \tag{37}$$

II. Let $\Phi(y)$ assume the form of the Contant Elasticity of Substitution (C.E.S.) function

$$\Phi(y)=\left(\sum_{i=1}^{m} \delta_i y_i^{-v}\right)^{-p/v}, \quad \text{where} \quad v \in (-1, 0],\ p>0,\ \delta_i \geqslant 0,\ \sum_{i=1}^{n} \delta_i=1.$$

Thus, the optimization problem is

$$\max_y \left\{\Phi(y)=\left(\sum_{i=1}^{m} \delta_i y_i^{-v}\right)^{-p/v}\right\} \tag{38}$$

subject to

$$g(y)=\sum_{i=1}^{m} y_i - Y \leqslant 0$$

$$y_i \geqslant 0, \quad \text{where} \quad y=(y_1, \dots, y_m), \quad Y \text{ is a given number}.$$

The above optimization problem is equivalent to

$$\max_y \left\{ [\Phi(y)]^{-v/p} = \sum_{i=1}^{m} \delta_i y_i^{-v} \right\} \tag{39}$$

subject to

$$g(y)=\sum_{i=1}^{m} y_i - Y \leqslant 0, \quad y_i \geqslant 0.$$

Using (36)

$$\hat{y}_i = \frac{(\delta_i)^{\frac{1}{1+v}}}{\sum_{j=1}^{m} \delta_j^{\frac{1}{1+v}}} Y, \quad i=1, \dots, m. \tag{40}$$

Thus, for given p and v by (37)

$$\hat{\Phi} = \Phi(\hat{y}) = \left(\sum_{j=1}^{m} \delta_j^{\frac{1}{1+v}} \right)^{\frac{-p(1+v)}{v}} Y^p. \tag{41}$$

References

1. Chakravarty S.: Capital and development planning. M.I.T. Press, Cambridge, Mass. 1969.
2. Cichocki K.: Optimization of investment processes. Center for Control Sciences, University of Minnesota, Jan. 1975 (unpublished).
3. Cichocki K.: Modelling and optimization of economic system in socialist countries. *ASME J. Dynamic Systems, Measurement and Control* 98G3 (1976).
4. Chipman J. S.; A reneval model of economic growth. The continuous case. Department of Economics, Dec. 1974 (unpublished).
5. Kulikowski R.: Optimization of complex investment and production processes. *Bull. Acad. Pol. Sci. Ser. Sci. Tech.* **22**, 2 (1974).
6. Kulikowski R.: Modelling of production, utility structure, prices and technological change. *Contr. a. Cybernet.* (Warszawa) **4**, 2 (1975).

Received, February 1976.

Optymalna alokacja inwestycji w modelu wzrostu gospodarczego

W pracy podano model wzrostu gospodarczego, który umożliwia optymalny rozdział zasobów. Zasoby powstają w wyniku akumulacji produkcji ("oszczędzania") w okresie [0, T].

Alokacja odbywa się w Centrum Decyzyjnym i polega na podziale zasobów dla potrzeb konsumpcji indywidualnej, inwestycji produkcyjnych i wydatków rządowych na konsumpcję zbiorową i usługi. Zasoby te są następnie przydzielane sektorom produkcyjnym i sektorom produkującym dobra konsumpcji publicznej.

Zasoby, stanowiące część zakumulowanej produkcji, są wynikiem polityki deprecjacyjnej sektorów produkcyjnych oraz oszczędności gospodarstw indywidualnych.

Matematycznie rozpatruje się nieliniowy system dynamiczny, w którym jako kryterium wybrano maksymalizację globalnej produkcji netto na głowę w czasie [0, T]. Sformułowane zadanie posiada jednoznaczne globalne rozwiązanie optymalne zależne od wielkości egzogenicznych. Zaproponowany model może być również wykorzystany do określania optymalnych cen systemu.

Оптимальное распределение капиталовложений в модели экономического роста

В работе дана модель экономического роста, которая позволяет произвести оптимальное распределение ресурсов. Эти ресурсы возникают в результате аккумуляции производства („экономии") за период [0, T].

Распределение происходит в Центре Принятия Решений и состоит в разделении ресурсов на нужды индивидуального потребления, производственных капиталовложений и правительственных затрат на общественное потребление и услуги. Эти ресурсы затем передаются производственным секторам и секторам производящим ценности общественного потребления.

Ресурсы, являющиеся частью аккумулированного производства, появляются в результате политики обеспенивания проводимой производственными секторами и экономии индивидуальных хозяйств.

Математически рассматривается нелинейная динамическая система, в которой в качестве критерия выбрана максимизация глобального производства нетто на душу за время [0, T]. Сформулированная задача имеет однозначное глобальное оптимальное решение зависящее от згзогенных величин. Предлагаемая модель может быть также использована для определения оптимальных цен системы.