

**Synthesis of linear multidimensional control
providing maximal dumping of Lyapunov function
of the system**

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A new performance index for the parametric synthesis of multivariable control systems has been proposed. This performance index is of a form of a ratio of the value of the system's Lyapunov function at a chosen time $T > 0$ to the value of this function at the initial time. Relationship between this performance index and square mean indexes in case of linear systems has been shown. Choice of initial conditions for which the synthesis is performed has been discussed.

1. Introduction

There are many control systems in which, the controller's goal is to bring the controlled parameters of the system to the vicinity of the working point, in the shortest possible time after appearance of a big disturbance. For example, a controller that maintains tension and frequency of a power generating unit should restore the state close to nominal after a short circuit or other rapid changes of load in some section of the system.

The basic problem of designing such types of control systems is to obtain systems' parameters that make acceptable a compromise between overshoots of particular variables and the rate of response of the system to the control error. In general, minimization of mean square type indexes yields systems with too big overshoots. A few years ago modal synthesis technique was popular [1, 4], however in many multivariable systems it is rather difficult to give modal values that yield desired characteristics of the system [6].

In the present paper a performance criterion is proposed which gives fast and at the same time sufficiently damped behaviour of the system.

2. Model of the system and definition of the performance index

2.1. General case

Consider the following system

$$\dot{x} = f(x, u), \quad x(0) = x_0, \quad (1)$$

$$y = g(x), \quad (2)$$

where

$x = x(t) \in R^n$ is a state vector at time t ,

$y = y(t) \in R^q$ is a vector of outputs at time t ,

$u = u(t) \in R^p$ is a vector controls at time t ,

$$f: R^n \times R^p \rightarrow R^n, \quad (3)$$

$$g: R^n \rightarrow R^q.$$

Assume, that the control u is a function of parameters F , which must be determined, and of outputs y

$$u = h(F, y), \quad F \in \mathcal{F} \subset R^m. \quad (4)$$

The set \mathcal{F} is for the moment only assumed to contain for sure only those values of parameters F for which the closed-loop system (1), (2), (4) is stable. Moreover, there may be other conditions imposed on the systems in the specific cases.

The performance index is assumed to be of the form

$$J_T(x_0, F) = \frac{V[x(T), F]}{V(x_0, F)} \quad (5)$$

where $V(\cdot)$ is a Lyapunov function of system (1), (2), (4), $x(T)$ is the state vector at time $t=T$.

The performance index (5) expresses a ratio of a value of Lyapunov function V at a chosen time instant $T > 0$ to a value of the same Lyapunov function at the initial time instant $t=0$. Other words, the index J_T determines damping in the time interval $(0, T]$ of a function, which being a Lyapunov function, characterises stability of the control system.

When interpreting the Lyapunov function in term of energy, the performance index (5) shows that part of the initial energy has been retained in the system at the time T .

The parametric synthesis consists in determining such values of parameters F , for which the performance index (5) assumes its minimum value.

For nonlinear systems, building a Lyapunov function in itself creates serious problems [8, 9]. Even for linear systems construction of the performance index of the form (5) is in general troublesome. However, for linear systems, the performance index can be represented in a different, intuitively clearer form.

Before proceeding further with the discussion, devoted exclusively to linear systems, let us consider the dependence of the synthesis problem formulation on the assumptions on initial condition.

In general, minimization of $J_T(x_0, F)$ over $F \in \mathcal{F}$ leads to a controller whose parameters depend on initial conditions x_0 . Such a controller can be realized as an adaptive controller. If we wish to remain in the class of conventional controller we have the following alternatives:

— minimization of $J_T(x_0, F)$ for given initial conditions

$$\min_{F \in \mathcal{F}} J_T(x_0^*, F) = J_T(x_0^*, F^o) \quad (6)$$

where: x_0^* — chosen initial conditions, F^o — optimum values of controller parameters;

— minimization of $E\{J_T(x_0, F)\}$, which requires a priori knowledge of multivariate probability distribution $p(x_0)$

$$\min_{F \in \mathcal{F}} E_{x_0} \{J_T(x_0, F)\} = E_{x_0} \{J_T(x_0, F^o)\}; \quad (7)$$

— minimization of $J_T(x_0, F)$ for the least favourable initial conditions

$$\min_{F \in \mathcal{F}} \max_{x_0} J_T(x_0, F) = J_T(\hat{x}_0, F^o) \quad (8)$$

where \hat{x}_0 — the least favourable initial conditions.

2.2. Linear case

For a linear time-invariant case the system's equations as

$$\dot{x} = Ax + Bu; \quad x(0) = x_0; \quad y = Cx; \quad u = Fy, \quad (9)$$

where A, B, C and F are matrices of constant elements of appropriate dimensions.

As a Lyapunov function for the system (9) a quadratic form of the state vector may be assumed [7]

$$V = x^T(T) Q x(T); \quad Q > 0. \quad (10)$$

In such a case the performance index (5) is

$$J_T(x_0, F) = \frac{x^T(T) Q x(T)}{x_0^T Q x_0}. \quad (11)$$

The effect of F on J_T is due to the fact that the elements of Q are functions of the elements of F . This dependence will be discussed later.

Necessary and sufficient condition for the quadratic form (10) to be a Lyapunov function of system (9) is that the following equation should be satisfied [7]

$$x^T(t) Q x(t) = \int_t^\infty x^T(\tau) R x(\tau) d\tau \quad (12)$$

where R is a positive semidefinite matrix ($R \geq 0$).

Equation (12) is satisfied if and only if there exists certain matrix $Q > 0$ which satisfies the following matrix equation

$$-R = (A + BFC)^T Q + Q(A + BFC). \quad (13)$$

Using (11) and (12) the performance index may be expressed in the form

$$J_T(x_0, F) = \frac{\int_T^\infty x^T(\tau) R x(\tau) d\tau}{\int_0^\infty x^T(\tau) R x(\tau) d\tau}. \quad (14)$$

On the basis of (14) the following interpretation of the performance index $J_T(x_0, F)$ may be given. If the scalar quantity $x^T(\tau) R x(\tau)$ is viewed as a squared generalized control error at time τ then the performance index (14) expresses ratio of the generalized integral error on the time interval $[T, \infty)$ to the generalized integral error on the time interval $[0, \infty)$.

Thus minimization of $J_T(x_0, F)$ is equivalent to elimination of maximum possible part of the generalized integral error on the time interval $(0, T)$ for (14) can be written in the form

$$J_T(x_0, F) = 1 - \frac{\int_0^T x^T(\tau) R x(\tau) d\tau}{\int_0^\infty x^T(\tau) R x(\tau) d\tau}. \quad (15)$$

Minimization of (14) is equivalent to maximization of the second term of the right hand side of (15), which is equal to the ratio of the integral error on the time interval $(0, T)$ to the integral error on the time interval $(0, \infty)$ (Fig. 1).

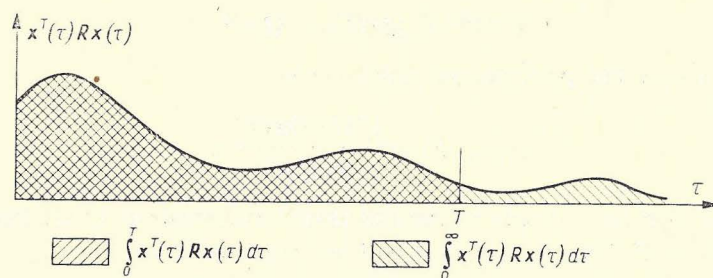


Fig. 1. Example of a time profile of a generalized control error

Formulation of the performance index in the form (15), which is reduced in practice to choosing the elements of R , most frequently diagonal, should not be a difficult task for the system designer.

3. Synthesis of a controller in a linear system under various hypothesis on initial conditions

Optimal parameter values of a controller are usually determined by means of an iterative procedure. This is complicated by the existence of local minima of the hypersurface $J_T(x_0, F)$ in the space of parameter F . Particular computational algorithm to be used depends on the assumptions on initial conditions.

In the simplest case of synthesis, for chosen initial conditions and chosen T and R the sequence of steps may be the following:

1. Assume values for the elements of F .
2. From matrix equation (13) determine Q .
3. For given $x_0 = x_0^*$ determine $x(T)$ from the equation

$$x(T) = \exp[(A + BFC)^T] x_0^*.$$

4. Determine $J_T(x_0^*, F)$ from equation (11).
5. Change F in the direction of minimizing $J_T(x_0^*, F)$.

Let us note that from (11)

$$J_T(x_0^*, F) = J_T(\alpha x_0^*, F) \text{ for } \alpha \in R' \quad (17)$$

so that minimization of (11) for chosen x_0^* is equivalent to its minimization for all x_0 being on the straight line passing through the origin and point $x = x_0^*$ (Fig. 2).

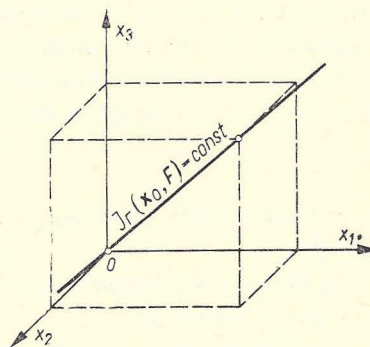


Fig. 2. Geometrical interpretation of relation (17)

When the probability distribution $p(x_0)$ of initial conditions is known, the synthesis is much more complicated as for chosen F expectation of $J_T(x_0, F)$ must be calculated

$$E\{J_T(x_0, F)\} = \int_{\Omega(x_0)} J_T(x_0, F) p(x_0) d\Omega(x_0) \quad (18)$$

where $\Omega(x_0)$ is the set of all possible values of x_0 .

The case of controller synthesis for the least favourable initial condition will be now discussed in more detail.

It should be emphasized that determining the least favourable initial conditions within the bounded region of their possible values is computationally very difficult.

Note, that on the basis of (17) the problem of determining the least favourable initial conditions in any bounded region Ω which has a nonempty interior in n -dimensional space, containing the origin is equivalent to the problem of determining such initial conditions in the whole space, as

$$\forall x_0 \in R^n, \exists \alpha \in R^1: \alpha x_0 = x_0^* \in \Omega \subset R^n. \quad (19)$$

However, if the region of admissible initial conditions has no interior in n -dimensional space then determining the least favourable initial conditions in the whole space we take into account also those initial conditions that can not take place.

In the synthesis of a controller for the least favourable initial conditions in the whole space extremum characteristics of quadratic forms' ratio may be used [3].

Substitution of (16) in (11) yields

$$J_T(x_0, F) = \frac{x_0^T \tilde{Q} x_0}{x_0^T Q x_0} \quad (20)$$

where

$$\tilde{Q} = \exp^T [(A + BFC) T] Q \exp [(A + BFC) T]. \quad (21)$$

Let $\hat{J}_T(F)$ denote maximum value of $J_T(x_0, F)$ for given F . Then

$$\hat{J}_T(F) = \max_{x_0} J_T(x_0, F) = \frac{\hat{x}_0^T \tilde{Q} \hat{x}_0}{\hat{x}_0^T Q \hat{x}_0} \quad (22)$$

where \hat{x}_0 denotes the initial conditions for which the right hand side term of (20) assumes its maximum value.

On the basis of the theorem on extremum characteristics of a quadratic forms' ratio [3] $\hat{J}_T(F)$ is equal to the largest root of the equation

$$\det(\tilde{Q} - \mu Q) = 0. \quad (23)$$

It follows from (23) that $\hat{J}_T(F)$ is equal to the largest eigenvalue of matrix $Q^{-1} \tilde{Q}$.

In the case of controller synthesis based on minimization of $\hat{J}_T(F)$ the sequence of steps is following:

1. Assume values for elements of F .
2. Calculate matrix Q from matrix equation (13).
3. Calculate \tilde{Q} from (21).
4. Calculate the largest eigenvalue of matrix $Q^{-1} \tilde{Q}$.
5. Change F in the direction of minimization of $\hat{J}_T(F)$.

4. Choice of starting values for the elements of F in the optimization procedure

The most serious computational problem of determining the optimal value of F is created by the necessity of determining global extremum of $\hat{J}_T(F)$ over F . To facilitate this problem it is suggested to minimize first a geometrical mean of eigen-

values of matrix $Q^{-1} \tilde{Q}$ what can be done rather easily. Justification of this approach follows from the fact that values of $J_T(x_0, F)$ change from the smallest to the largest eigenvalue of $Q^{-1} \tilde{Q}$ when x_0 changes over an unbounded region.

The matrix F obtained as a result of minimization of the geometrical mean can be used as a starting value for the optimization procedure $\tilde{J}_T(F)$.

To compute the geometrical mean of eigenvalues of a matrix a formula known from matrix analysis [3] may be used

$$\det(Q^{-1} \tilde{Q}) = \prod_{i=1}^n \mu_i \quad (24)$$

where μ_i are the eigenvalues of $Q^{-1} \tilde{Q}$ (note that this is a symmetric positive definite matrix).

If the geometrical mean of the eigenvalues of $Q^{-1} \tilde{Q}$ is denoted by $\tilde{J}_T(F)$, then from (24)

$$\tilde{J}_T(F) = [\det(Q^{-1} \tilde{Q})]^{1/n}. \quad (25)$$

Substituting (21) to (25) we get

$$\tilde{J}_T(F) = \{\det \exp[(A + BFC)T]\}^{2/n}. \quad (26)$$

Then from

$$\det \exp[(A + BFC)T] = \exp\{\text{Tr}[(A + BFC)T]\} \quad (27)$$

follows

$$\tilde{J}_T(F) = \{\exp\{\text{Tr}[(A + BFC)T]\}\}^{2/n} \quad (28)$$

or

$$\ln \tilde{J}_T(F) = \frac{2T}{n} \text{Tr}(A + BFC). \quad (28')$$

Formula (28) or (28') may be used in the following way:

- set a sufficiently small value of $\tilde{J}_T(F)$;
- find out whether there exists F for which equation (28') is satisfied for the assumed value of $\tilde{J}_T(F)$, if not — increase $\tilde{J}_T(F)$ until required F is found;
- use the matrix F thus obtained as a starting value for the optimizing procedure of $\tilde{J}_T(F)$.

5. Example

Consider a second order system

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= a_1 x_1 + a_2 x_2 + u. \end{aligned} \quad (29)$$

The controller equation is of the form

$$u = f_1 x_1 + f_2 x_2. \quad (30)$$

Choose

$$R = \text{diag}(r_1, r_2). \quad (31)$$

State equations of the closed-loop system corresponding to (29) and (30) are

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \tilde{a}_1 x_1 + \tilde{a}_2 x_2 \end{aligned} \quad (32)$$

where

$$\begin{aligned} \tilde{a}_1 &= a_1 + f_1, \\ \tilde{a}_2 &= a_2 + f_2. \end{aligned} \quad (33)$$

Parameters \tilde{a}_1 and \tilde{a}_2 instead of f_1 and f_2 will be viewed as variables for convenience.

Assume that the set \mathcal{F} is given by stability conditions

$$\tilde{a}_1 < 0; \tilde{a}_2 < 0 \quad (34)$$

or

$$f_1 < -a_1; f_2 < -a_2.$$

From matrix equation (13) we have

$$\begin{aligned} q_{11} &= \frac{1}{2} \left[\left(\frac{\tilde{a}_2}{\tilde{a}_1} + \frac{1}{\tilde{a}_2} \right) r_1 + \frac{\tilde{a}_1}{\tilde{a}_2} \right] r_2, \\ q_{12} &= q_{21} = r_1 / 2\tilde{a}_1 \\ q_{22} &= \frac{1}{2\tilde{a}_2} \left(\frac{r_1}{\tilde{a}_1} + r_2 \right). \end{aligned} \quad (35)$$

Values of $\hat{J}_T(F)$ were determined numerically from equations (13) (21) and (23). For $\tilde{a}_1 + \frac{\tilde{a}_2^2}{4} < 0$ we have

$$\begin{aligned} \exp[(A+BFC)T] &= \begin{bmatrix} \cos \omega T + \frac{\xi}{\omega} \sin \omega T & \frac{\sin \omega T}{\omega} \\ -\left(\frac{\xi^2}{\omega} + \omega\right) \sin \omega T & \cos \omega T - \frac{\xi}{\omega} \sin \omega T \end{bmatrix} \exp(-\xi T) \quad (36) \\ \omega &= \sqrt{-\tilde{a}_1 - \frac{\tilde{a}_2^2}{4}} \quad \xi = -\frac{\tilde{a}_2}{2}. \end{aligned}$$

Results for $T=4$, $r_1=r_2=1$ will be discussed.

Starting values for optimization are obtained, according to (28), from the equation

$$\ln \hat{J}_T(\tilde{a}_2) = T\tilde{a}_2. \quad (37)$$

Setting $\hat{J}_T(\tilde{a}_2)=0.02$ we get from (37) $\tilde{a}_2=-1$.

Figure 3 presents changes of parameters in the optimization procedure for three different starting points \tilde{a}_2 . It can be seen that the procedure has been stopping at several points on the $(\tilde{a}_1, \tilde{a}_2)$ plane. It has not been discovered whether this resulted from a flat ridge of $\tilde{J}_T(\tilde{a}_2)$ in the vicinity of an extremum or from existence of local extrema.

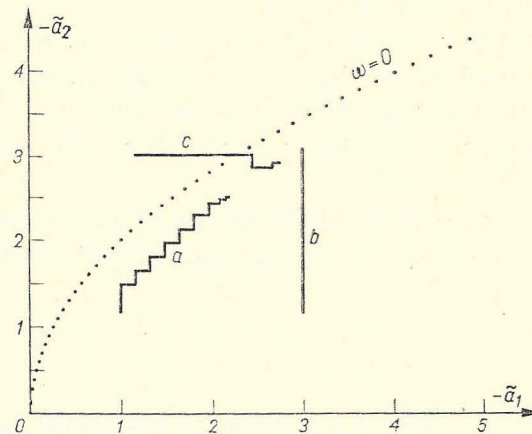


Fig. 3. Changes of parameters \tilde{a}_1 and \tilde{a}_2 in the optimization procedure for various initial values of the parameters

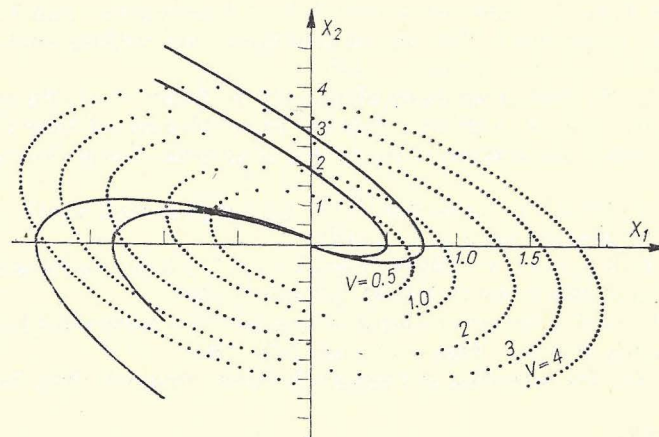


Fig. 4. Trajectories of system (32) for $\tilde{a}_1 = -2.165$, $\tilde{a}_2 = -2.5$ against a background of isoquants of V

The optimal values of the parameters give a system whose states change in time in an oscillatory manner with considerable damping, which testifies usefulness of the chosen performance index. This is illustrated by the curve $\omega=0$ on Fig. 3 which determines a boundary between oscillatory and aperiodic behaviour of the system's state.

Figure 4 presents trajectories of the system on (x_1, x_2) plane and isoquants of Lyapunov function corresponding to the values $\tilde{a}_1 = -2.165$ and $\tilde{a}_2 = -2.5$ obtained from the optimization procedure (curve a on Fig. 3)

$$V = 1.21 x_1^2 + 2 \times 0.23 x_1 x_2 + 0.29 x_2^2. \quad (38)$$

6. Conclusions

A new performance index for the parametric synthesis of multivariable control systems has been proposed. The performance index considered is of a form of a ratio of the values of Lyapunov function of the system at time $T > 0$ to the value of this function at the initial time.

Physical interpretation of the performance index shows reasonability of its application to the synthesis of systems requiring fast damping of large control errors with moderate overshoots while small deviations from the working point have no greater significance. The above example of controller synthesis for a second order system supports this view.

Choice of initial conditions used in the synthesis was discussed, with the emphasis on the synthesis in case of the least favourable initial conditions.

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Synteza liniowych wielowymiarowych układów regulacji na podstawie wskaźnika jakości charakteryzującego stopień tłumienia przebiegów

Zaproponowano nowy wskaźnik jakości do syntezy parametrycznej wielowymiarowych układów regulacji. Wskaźnikiem tym jest stosunek wartości funkcji Lapunowa układu w wybranej chwili $T > 0$ do wartości tej funkcji w chwili początkowej. Pokazano związek tego wskaźnika z wskaźnikami średniokwadratowymi w przypadku układów liniowych. Omówiono wybór warunków początkowych, dla których dokonuje się syntezy.

Podano przykład ilustrujący celowość zastosowania zaproponowanego wskaźnika do syntezy układów wymagających szybkiego tłumienia uchybu regulacji przy umiarkowanych przeregulowaniach.

**Синтез линейных многомерных систем регулирования
на основе показателя качества определяющего степень
демпфирования процессов**

Предлагается новый показатель качества для параметрического синтеза многомерных систем регулирования. Таким показателем является отношение значения функции Ляпунова системы в выбранный момент $T > 0$ к значению этой функции в начальный момент времени. Показана связь этого показателя и среднеквадратных показателей в случае линейных систем. Рассмотрен выбор начальных условий для которых проводится синтез. Дан пример иллюстрирующий целесообразность применения предлагаемого показателя к синтезу систем требующих быстрого погашения отклонения регулирования в случае среднего перерегулирования.

