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Finite difference approximation of state and control constrained optimal control problem with delay

by

IRENA LASIECKA

Polish Academy of Sciences
Institute for Organization, Management and Control Sciences
Warszawa, Poland

A finite difference approach to the problem of minimizing an integral cost functional subject to a differential equation with delay and state and control constraints is presented in the paper. The problem is viewed as a variational minimization problem subject to nonholomic constraints and is treated using Lagrange multipliers. Error estimate for the control is established under appropriate smoothness and boundedness conditions.

1. Introduction

The convergence properties of finite-difference approximation are examined for state and control constrained optimal control problem with delay. An approximation of constrained optimal control problem for system described by ordinary differential equations was considered in a number of papers.

In [1] a finite-difference approach applied to state and control constrained control problem gouverned by nonlinear ordinary differential equation was analysed and the convergence result was obtained.

In [2] the Ritz—Trefftz approximation for control problem with quadratic cost, linear dynamics and linear inequality state and control constraints was considered. The rate of convergence the solution of finite dimensional approximation to the solution of continuous optimal control problem was estimated. The both approaches presented in [1, 2] do not assure in general that the state of continuous system corresponding to discrete control determined by numerical algorithm satisfies constraints. So it seems that a problem to be important for treating state constrained optimal control problem is to find an approximation of the set of admissible states such that the above mentioned requirement is satisfied.

In [3] a finite-difference approximation for control constrained nonlinear optimal control problem with delay was anylysed and the error bound has been established. The main purpose of this paper is to generalize the results presented in [3] to problems of delay optimal control with constraints imposed both on state and control.

An approximation of state constraints set different to that given in [1, 2] is proposed. The method of approximation presented in the paper assures that the trajectory of continuous system corresponding to discret control belongs to the set of the admissible states. Moreover, the rate of convergence for the solution of discret problem to the solution of continuous problem is estimated. The general considerations are illustrated by an example.

2. Notations and conventions

$H^1[0, T; R^n]$ denotes the Sobolev space defined by

$$H^1[0, T; R^n]=\left\{\{x \in L^2[0, T; R^n]; \frac{dx}{dt} \in L^2[0, T; R^n]\right\}. \tag{2.1}$$

$C[0, T; R^n]$ denotes the space of all continuous on $[0, T]$ functions with the values in R^n. (2.2)

The norm in $C[0, T; R^n]$ is given by $|x|=\sup_{t\in[0,T]} |x(t)|$

$V[0, T; R^k]$ denotes the space of bounded variation functions defined on $[0, T]$ with the values in R^n induced by the norm $|x|_V=\operatorname{var}_{t\in[0,T]} x(t)$. (2.3)

$PC[0, T; R^n]$ — a space of piece-wise continuous functions (x: $[0, T]\to$ $\to R^n$ is called piece-wise continuous function if the number of discontinuouities is finite: $t_1, t_2, ..., t_p$ and for every $t \in [0, T]$ $x(t)=x(t-0)$). (2.4)

The norm in $PC[0, T; R^n]$ is given by

$$|x| \overset{\text{df}}{=} \sup_{t\in[0,T]} |x(t)|$$

$$\langle\cdot, \cdot\rangle \text{ — scalar product in } L^2[0, T; R^n] \tag{2.5}$$

$$\|\cdot\| \text{ — a norm in } L^2[0, T; R^n] \tag{2.6}$$

$\langle\cdot, \cdot\rangle_K$ — scalar product in $L^2[Kh; (K+1)h; R^n]$ where $h>0$:

$$m \overset{\text{df}}{=} \frac{T}{h} \text{ is assumed to be an integer.} \tag{2.7}$$

$$\|\cdot\|_K \text{ — a norm in } L^2[Kh; (K+1)h, R^n] \tag{2.8}$$

$$|\cdot| \text{ — a norm in } L^\infty[0, T; R^n] \tag{2.9}$$

$$(,) \text{ — scalar product in } R^n \tag{2.10}$$

$$\langle\langle , \rangle\rangle \text{ — general form of functional in } C[0, T; R^n] \text{ i.e.} \tag{2.11}$$

$$\langle\langle , \rangle\rangle = \int_0^T f(t)\, dg(t); f \in C[0, T; R^n]; g \in V[0, T; R^n]$$

$$P'(x^o) \text{ — Frechet derivative of operator } P \text{ at the point } x^o \tag{2.12}$$

$$f_x(x^o) \text{ — partial derivative of } f \text{ with respect to } x \text{ at the point } x^o \tag{2.13}$$

$$U(x^o) \text{ — denotes a neighbourghood of } x^o. \tag{2.14}$$

$$\text{For } x \in PC[0, T; R^n] \cap \bigcap_{i=0}^{p-1} H^1[t_i, t_{i+1}; R^n],\ t_0 = 0 \tag{2.15}$$

$$\beta_x \overset{\text{df}}{=} \sum_{i=1}^{p-1} \left(\left\| \frac{dx}{dt} \right\|_i + \delta_i \right)$$

where p is the number of discontinuity points of x; $\left\| \frac{dx}{dt} \right\|_i^2 = \int_{t_i}^{t_{i+1}} \left| \frac{dx(t)}{dt} \right|^2 dt$, t_i, $i = 1, \ldots, p$ a point of discontinuity of x, $\delta_i \overset{\text{df}}{=} x(t_i + 0) - x(t_i - 0)$.

3. Continuous optimal control problem

Continuous optimal control problem statement and some smoothness properties of optimal solution are given in this section. Since the optimal control problem can be viewed as a variational problem of minimizing a cost functional on the Hilbert space subject to some constraints, so using Lagrange functional another formulation of the original problem is presented.

The problem of existence of normal Lagrange's multipliers is discussed and some sufficient conditions of existence are formulated.

3.1. Problem description

The following optimal control problem is analysed in the paper: minimize $J(x, u) \overset{\text{df}}{=} \int_0^T \Phi(x(t), u(t), t)\, dt$ subject to constraints:

$$\frac{dx(t)}{dt} + f(x(t), x(t-h), u(t), t) = 0 \text{ a.e. } t \in [0, T] \tag{3.1.1}$$

$$x(Q) = \varphi(Q) \quad Q \in [-h, 0] \tag{3.1.2}$$

$$S(x(t), t) \in K \quad t \in [0, T] \tag{3.1.3}$$

$$u \in \Omega \tag{3.1.4}$$

where

$x \in H^1[0, T; R^n]$; $dx/dt \in PC[0, T; R^n]$

f: $R^n \times R^n \times R^m \times [0, T] \to R^n$

$\Phi: R^n \times R^n \times [0, T] \to R^1$
$S: R^n \times [0, T] \to R^k$
$\varphi \in H^1 [-h, 0; R^n]$
$\Omega \subset PC [0, T; R^m]$ a given convex, closed subset of all functions $u \in PC [0, T; R^m]$ such that u is absolutely continuous function on all intervals of continuity of u. i.e. $u \in H^1 [t_i, t_{i+1}; R^m]$ where t_i, t_{i+1}; $i=1, \ldots, p$, the points of discontinuity of u.
$K \subset R^k$ — a given closed convex cone with vertex at zero. Int $K \neq \emptyset$
Assume that the following hypothesis are satisfied:

H1. $f(\xi, \gamma, \eta, t)$, $\Phi(\xi, \eta, t)$ are continuously differentiables for every $\xi, \gamma \in R^n$, $\eta \in R^m$, $t \in [0, T]$.

H2. $J(x, u)$ is radially unbounded i.e.:

$$J(x(u), u) \underset{|u| \to \infty}{\longrightarrow} \infty$$

where $x(u)$ denotes the solution of (3.1.1) corresponding to u.

H3. (i) $S(\xi, t)$ is continuously differentiable with respect to ξ and t.

(ii) $S(\xi, t)$ is K-convex i.e.

for any $0<\alpha<1$, $\xi_1, \xi_2 \in R^n$, $t_1, t_2 \in [0, T]$,
$S(\alpha\xi_1 + (1-\alpha)\xi_2, \alpha t_1 + (1-\alpha) t_2) - \alpha S(\xi_1, t_1) - (1-\alpha) S(\xi_2, t_2) \in K$.

H4. f is a strongly monotonne function i.e.:

$$\exists \alpha > 0; \forall x_1, x_2, y \in PC [0, T; R^n]; u \in PC [0, T; R^m], 0<\tau<T$$

$$\int_0^\tau (f(x_1(t), y(t), u(t), t) - f(x_2(t), y(t), u(t), t), x_1(t) - x_2(t))\, dt \geqslant \int_0^\tau |x_1(t) - x_2(t)|^2\, dt.$$

Using the standard arguments simillar to that given in [4] it is easy to show that H1-H2 imply the existence of optimal solution. We shall refer to the above described problem as problem P_0 and the optimal solution of P_0 will be denoted by x^o, u^o.

Some property of optimal solution x^o, u^o is given by the following:

Lemma 3.1. If hypothesis H1 and H2 are satisfied then there exists $\delta < \infty$ such that

$$x^o, u^o \in G \subset C [0, T; R^n] \times PC [0, T; R^m]$$

where

$$G \stackrel{\mathrm{df}}{=} \{(x, u) \in C [0, T; R^n] \times PC [0, T; R^m];\ |x| < \delta;\ |u| < \delta\}.$$

Proof. By hypothesis H2 it can be shown (see [3]) that:

$$\exists \delta > 0, \ |u^o| < \delta.$$

Then after applying Theorem 1.3 given in [5] we arrive at the desired result.

3.2. Lagrange formalism — dual control problem

Optimal control problem with constraints on control and state can be transformed by duality (according to Rockefeller formalism [6]) into a dual control problem without constraints. Hence it is a natural idea to replace the primal (original) problem by the dual problem. In this section the duality formalism is introduced and the connections between the both problems are investigated.

Denote $A(x, y, u)(t) \overset{\text{df}}{=} f(x(t), y(t), u(t), t)$

$A: PC[0, T; R^n] \times PC[0, T; R^n] \times PC[0, T; R^m] \to L^2[0, T; R^n]$

$\tilde{S}(x)(t) \overset{\text{df}}{=} S(x(t), t)$

$\tilde{S}: PC[0, T; R^n] \to PC[0, T; R^k]$.

Let us introduce Lagrange functional

$$L: H^1[-hT; R^n] \times PC[0, T; R^n] \times L^2[0, T; R^n] \times V[0, T; R^k] \to R^1$$

defined by the following formula:

$$L(x, u, \lambda, \eta) = J(x, u) + \left\langle \lambda; \frac{dx}{dt} + A(x, y, u) \right\rangle + \langle\langle \tilde{S}(x), \eta \rangle\rangle. \tag{3.2.1}$$

Since $\tilde{S}: H^1[0, T; R^n] \to C[0, T; R^k]$ then the dual control problem consists in finding

$$\max_{\substack{\lambda \in L^2[0, T; R^n] \\ \eta \in -\tilde{K}^*}} \quad \min_{\substack{x \in H^1[-h, T; R^n] \\ x(Q) = \varphi(Q),\, Q \in [-h, 0] \\ u \in \Omega}} L(x, u, \lambda, \eta)$$

where

$$\begin{aligned} &\tilde{K} \overset{\text{df}}{=} \{y \in C[0, T; R^k];\ y(t) \in K,\ t \in [0, T]\} \\ &\tilde{K}^* \overset{\text{df}}{=} \{\eta \in V[0, T; R^k],\ \langle\langle x, \eta \rangle\rangle \geqslant 0\ \forall x \in \tilde{K}\} \end{aligned} \tag{3.2.2}$$

In order to establish the relations between the primal problem and dual problem we recall the following result given in [7].

Lemma 3.2 [7]. Let:

(i) E, E_1, E_2 be Banach spaces; and $M \subset E$,

(ii) $K_2 \subset E_2$ be a convex cone with vertex at zero,

(iii) $S_1: E \to E_1$; $S_2: E \to E_2$,

(iv) $J(x^o) = \min_{x \in Y} J(x)$, where $Y = \{x \in M;\ S_1(x) = 0;\ S_2(x) \in K_2\}$,

(v) $S_1'(x^o)$ is a surjection on E_1,

(vi) $\exists \bar{x} \in \operatorname{Ker} S_1'(x^o)$, $S_2(x^o) + S_2'(x^o)\bar{x} \in \operatorname{Int} K_2$

(vii) M possess a "good conical approximation" at x^o, i.e. the set

$$M_{x0} \overset{\text{df}}{=} \{x \in E;\ \exists \varepsilon_1 > 0\ \exists U(\bar{x})\ \forall 0 < \varepsilon < \varepsilon_1\ \forall \bar{\bar{x}} \in U(\bar{x})\ x^o + \varepsilon \bar{\bar{x}} \in M\}$$

is a convex cone,

(viii) $\exists \bar{x} \in \operatorname{Ker} S_1'(x^o) \cap M_{x^o}$; $S_2(x^o) + S_2'(x^o)\bar{x} \in K_2$,

(ix) $L(x, \lambda, \eta) \overset{\text{df}}{=} J(x) + (S_1(x), \lambda)_{E_1, E_1^*} + ((S_2(x), \eta))_{E_2, E_2^*}$

Then there exists $\lambda^o \in E_1^*$; $\eta^o \in -K_2^*$ such that

(i) $\delta_x L(x^o, \lambda^o, \eta^o)(x-x^o) \geq 0, \ \forall x \in M_{x^o}$,

(ii) $\delta_\lambda L(x^o, \lambda^o, \eta^o) = 0$,

(iii) $((S_2(x^o), \eta^o)) = 0$.

To apply the result of Lemma 3.2 to Problem P_0 with Lagrange's functional defined by (3.2.1) we put:

$$\begin{aligned} &E = H^1[0, T; R^n] \times PC[0, T; R^m] \\ &E_1 = L^2[0, T; R^n], \\ &E_2 = C[0, T; R^n], \\ &K_2 = \tilde{K} = \{y \in C[0, T; R^k];\ y(t) \in K, \forall t \in [0, T]\} \\ &M = H^1[0, T; R^n] \times \Omega \\ &S_1(x, u) = \frac{dx}{dt} - A(x, y, u) \text{ where } y(t) \stackrel{\mathrm{df}}{=} x(t-h) \\ &S_2(x, u) = \tilde{S}(x). \end{aligned} \tag{3.2.4}$$

So, the optimization problem P_0 is reduced to finding $\min J(x, u)$ over $Y \subset E$ where $Y = \{(x, u) \in M;\ S_1(x) = 0;\ S_2(x) \in K_2\}$ with M, S_1, S_2, K_2 defined by (3.2.4).

Now we are going to verify assumptions (v), (vii) of Lemma 3.2.

It is known (see [8]) that the Volterra equation

$$x(t) - \int_0^t [f_x(x^o(t), y^o(t), u^o(t))\, \bar{x}(t) + f_y(x^o(t), y^o(t), u^o(t)\, \bar{y}(t)]\, dt = a_1(t) \tag{3.2.5}$$

has a solution $x \in H^1[0, T; R^n]$ for any $a_1 \in H^1[0, T; R^n]$.

Moreover for any $b \in L^2[0, T; R^n]$ there exists $a \in H^1[0, T; R^n]$ such that:

$$b = \frac{da}{dt} \tag{3.2.6}$$

So, by (3.2.5) and (3.2.6) we obtain that: for any $b \in L^2[0, T; R^n]$ there exists $x \in H^1[0, T; R^n]$ such that:

$$\frac{dx}{dt} - A_x(x^o, y^o, u^o)\, x - A_y(x^o, y^o, u^o)\, y = b.$$

It means that for any $b \in L^2[0, T; R^n]$ there exists $x \in H^1[0, T; R^n]$ such that

$$S_1'(x^o, u^o)(x, 0) = b$$

what completes the proof of the fact that $S_1'(x^o, u^o)$ is onto E_1, so assumption (v) is satisfied.

(vii) By convexity of Ω it follows that

$$M_{x^o, u^o} = \bigcup_{\lambda > 0} \lambda(u - u^o) \times C[0, T; R^n],\ u \in \Omega \tag{3.2.7}$$

is a convex cone, so it is a "good conical approximation" of M and assumption (vii) is satisfied.

The above considerations yield:

Lemma 3.3 Let:

(i) x^o, u^o be an optimal solution of problem P_0,
(ii) $L(x, u, \lambda, \eta)$ be defined by (3.2.1),
(iii) there exists $(\bar{x}, \bar{u}) \in H^1[0, T; R^n] \times PC[0, T; R^m]$
such that:

$$\frac{d\bar{x}(t)}{dt} + f_x(x^o(t), y^o(t), u^o(t), t)\,\bar{x}(t) + f_y(x^o(t), y^o(t), u^o(t), t)\,\bar{y}(t) + \\ + f_u(x^o(t), y^o(t), u^o(t), t)\,\bar{u}(t) = 0 \quad (3.2.8)$$

$$\bar{x}(Q) = 0, \quad Q \in [-h, 0]$$

and

$$S(x^o(t), t) + S_x(x^o(t), t)\,\bar{x}(t) \in \operatorname{Int} K, \ \forall t \in [0, T].$$

(iv) There exist $(\bar{x}, \bar{u}) \in H^1[0, T; R^n] \times \Omega$ such that $(\bar{x}, \bar{u})$ satisfy (3.2.8) and

$$S(x^o(t), t) + S_x(x^o(t), t), \bar{x}(t) \in K, \ \forall t \in [0, T]$$

then there exist: $\lambda^o \in L^2[0, T; R^n]$, $\eta^o \in -\tilde{K}^* \subset V[0, T, R^K]$ (K^* given by (3.2.3)), such that

(i) $\langle \delta_x L(x^o, u^o, \lambda^o, \eta^o), x - x^o \rangle + \langle \delta_y L(x^o, u^o, \lambda^o, \eta^o), y - y^o \rangle = 0$
$\forall x \in H^1[-hT; R^n]$; $x(Q) = \varphi(Q)$ $Q \in [-h, 0]$,
(ii) $\delta_\lambda L(x^o, u^o, \lambda^o, \eta^o) = 0$,
(iii) $\langle \delta_u L(x^o, u^o, \lambda^o, \eta^o), u - u^o \rangle \geqslant 0 \ \forall u \in \Omega$,
(iv) $\langle\langle \tilde{S}(x^o), \eta^o \rangle\rangle = 0$.

The proof of the Lemma follows immediately from Lemma 3.2 if we note that our assumptions (iii) and (iv) respectively imply (vi) and (viii) in Lemma (3.2) and that for M_{x^o, u^o} given by (3.2.7) point (i) in the result of Lemma 3.2 reduces to (i) and (iii) in Lemma 3.3.

If Lagranglan $L(x, u, \lambda, \eta)$ is strictly convex then it enjoys saddle-point properties. Actually suppose that:

$$\text{H5} \left\langle \begin{bmatrix} L_{xx}(\tilde{a}) & L_{xy}(\tilde{a}) & L_{xu}(\tilde{a}) \\ L_{yx}(\tilde{a}) & L_{yy}(\tilde{a}) & L_{yu}(\tilde{a}) \\ L_{ux}(\tilde{a}) & L_{uy}(\tilde{a}) & L_{uu}(\tilde{a}) \end{bmatrix} \begin{bmatrix} x \\ y \\ u \end{bmatrix}, \begin{bmatrix} x \\ y \\ u \end{bmatrix} \right\rangle \geqslant \gamma \|u\|^2$$

where $\gamma > 0$; $\tilde{a} \overset{\mathrm{df}}{=} (\tilde{x}, \tilde{u}, \tilde{\lambda}, \tilde{\eta})$ is any point of some neighbourhood of the optimal point $(x^o, u^o, \lambda^o, \eta^o)$

Lemma 3.4. If all assumptions of Lemma 3.3 and hypothesis H5 are satisfied then: $L(x, u, \lambda, \eta)$ has a saddle point at $(x^o, u^o, \lambda^o, \eta^o)$ i.e.:

$$L(x^o, u^o, \lambda, \eta) \leqslant L(x^o, u^o, \lambda^o, \eta^o) \leqslant L(x, u, \lambda^o, \eta^o) \text{ for any } x \in H^1[-h, T; R^n];$$
$$x(Q) = \varphi(Q), \ Q \in [-h, 0],$$
$$\lambda \in L^2[0, T; R^n]$$
$$u \in \Omega$$
$$\eta \in -K^*.$$

Proof. Expanding $L(x, u, \lambda^o, \eta^o)$ into Taylor series about (x^o, u^o) we obtain:

$$L(x, u, \lambda^o, \eta^o) = L(x^o, u^o, \lambda^o, \eta^o) + \langle \delta_x L(x^o, u^o, \lambda^o, \eta^o), x - x^o \rangle +$$
$$+ \langle \delta_y L(x^o, u^o, \lambda^o, \eta^o), y - y^o \rangle + \langle \delta_u L(x^o, u^o, \lambda^o, \eta^o), u - u^o \rangle + H$$

where H is the second derivative of the operator L evaluated at $(\tilde{x}, \tilde{u}, \tilde{\lambda}, \tilde{\eta}) \in \in U(x^o, u^o, \lambda^o, \eta^o)$.

After applying Hypothesis H5 and Lemma 3.3 we arrive at:

$$L(x, u, \lambda^o, \eta^o) \geqslant L(x^o, u^o, \lambda^o, \eta^o) \ \forall u \in \Omega$$

what proves the right-hand side inequality. The left-hand side inequality results from the fact that:

$$\left\langle \lambda, \frac{dx^o}{dt} + A(x^o, y^o, u^o) \right\rangle = 0$$

and

$$\langle\langle \tilde{S}(x^o), \eta \rangle\rangle \leqslant 0 \ \ \forall \eta \in -K^*.$$

Namely,

$$L(x^o, u^o, \lambda^o, \eta^o) = J(x^o, u^o) \geqslant J(x^o, u^o) + \left\langle \lambda, \frac{dx^o}{dt} + A(x^o, y^o, u^o) \right\rangle +$$
$$+ \langle\langle \tilde{S}(x^o), \eta \rangle\rangle = L(x^o, u^o, \lambda, \eta) \ \forall \eta \in -K^*. \quad \text{Q.E.D.}$$

Observe that hypothesis H5 implies the uniqueness of optimal solution to problem P_0.

Summing up the results obtained in this section we conclude that the equivalence of primal and dual problem was established providing that hypothesis H5 and assumptions of Lemma 3.3 are satisfied.

4. Discrete optimal control problem

The main goal of this section is to approximate continuous optimal control problem by finite dimensional one. To do that the finite-difference method is used.

In order to define the discret problem an approximation of the set of admissible states is constructed and its properties are discussed. After formulating the finite-dimensional control problem the Lagrange's multipliers approach is applied to it.

4.1. An approximation of state and control spaces

In this section we are going to introduce some definitions essential in the sequel.

First let us define an approximation of $L^2[-h, T; R^n]$ and $H^1[-h, T; R^n]$. Let τ be a given parameter such that $\tau \to 0$. Denote: $M \stackrel{\text{df}}{=} \frac{T}{\tau}$; $N \stackrel{\text{df}}{=} \frac{h}{\tau}$. It is assumed that M and N are integers.

Define the space $E_\tau[-h, T+\tau; R^n]$ which is to be an approximation of $L^2[-h, T+\tau; R^n]$ and $H^1[-h, T+\tau; R^n]$ in the following way: $E_\tau[-h, T+\tau; R^n] = \{x_\tau; x_\tau(t) = \sum_{r=N}^{M} x_\tau(r\tau) W_r(t)\}$, where $x_\tau(r\tau) \in R^n$; $W_r(t)$ is a characteristic function of the interval $[r\tau; r+1\tau)$.

Let be given an operator $P_\tau: PC[-h, T; R^n] \to E_\tau[-h, T; R^n]$ defined by

$$P_\tau x(t) = \sum_{r=-N}^{M-1} x(t_r) W_r(t) \text{ where } t_r \in [r\tau; (r+1)\tau] \tag{4.1.1}$$

It is easy to verify that the following estimations take place:

$$\forall x \in H^1[-h, T; R^n] \; \|P_\tau x - x\| + \|P_\tau x - x\|_{-1} \leqslant \tau \left[\left\|\frac{dx}{dt}\right\| + \left\|\frac{dx}{dt}\right\|_{-1}\right], \tag{4.12.a}$$

$$\forall x \in H^2[-h, T; R^n] \left\|\nabla_\tau x - \frac{dx}{dt}\right\| \leqslant \tau\sqrt{2}\left\|\frac{d^2x}{dt^2}\right\|, \tag{4.1.2.b}$$

$$\forall x \in H^1[-h, T; R^n] \; |P_\tau x - x| + |P_\tau x - x|_{-1} \leqslant \tau^{\frac{1}{2}} \left[\left\|\frac{dx}{dt}\right\| + \left\|\frac{dx}{dt}\right\|_{-1}\right], \tag{4.1.3}$$

$$\forall x \in H^1[-h, T; R^n]; \text{ and such that } \frac{dx}{dt} \in L^\infty[-h, T; R^n], \tag{4.1.4}$$

$$|P_\tau x - x| + |P_\tau x - x|_{-1} \leqslant \tau \left[\left|\frac{dx}{dt}\right| + \left|\frac{dx}{dt}\right|_{-1}\right]$$

$$\forall x \in PC[-h, T; R^n] \cap \bigcap_{i=1}^{P} H^1[t_i, t_{i+1}, R^n]$$

$$\|P_\tau x - x\| + \|P_\tau x - x\|_{-1} \leqslant \tau^{\frac{1}{2}} \sum_{i=1}^{P} \left[\left\|\frac{dx}{dt}\right\|_i \tau^{\frac{1}{2}} + \delta_i\right]. \tag{4.1.5}$$

where: p is a number of points of discontinuously of x on $[-0, T]$,

$$\left\|\frac{dx}{dt}\right\|_i^2 = \int_{t_i}^{t_{i+1}} \left|\frac{dx(t)}{dt}\right|^2 dt;$$

t_i — the points of discontinuity of x; $\delta_i \overset{\mathrm{df}}{=} x(t_i^+) - x(t_i^-)$.

$\mathscr{P}_\tau \Omega \subset E_\tau[0, T; R^m]$ is called an approximation of Ω if and only if:

$$\mathscr{P}_\tau \Omega \text{ is a convex, closed set in } E_\tau[0, T; R^m] \tag{4.1.6}$$

$$\forall u \in \Omega \exists u_\tau \in \mathscr{P}_\tau \Omega, \; \|u - u_\tau\| \leqslant C\tau^{1/2} \text{ [1]} \tag{4.1.7}$$

$$\forall u_\tau \in \mathscr{P}_\tau \Omega \exists u \in \Omega, \; \|u - u_\tau\| \leqslant C\tau^{1/2}. \tag{4.1.8}$$

In finite-difference schemes the differential is replaced by finite-difference operator

$\nabla_\tau: E_\tau[0, T+\tau; R^n] \to E_\tau[0, T; R^n]$ defined by

$$\nabla_\tau x_\tau(t) = \frac{x_\tau(t+\tau) - x_\tau(t)}{\tau} \text{ for } t \in [0, T]. \tag{4.1.9}$$

[1] The constant C is used throughout the paper to designate a generic constant.

4.2. An approximation of the set of admissible states

A most natural approach to an approximation of the set of admissible states is to replace original continuous constraints (3.1.3) by discrete constraints of the type (see [1, 2]):

$$S(x_\tau(t), t_\tau) \in K, \tag{4.2.1}$$

where

$$t_\tau = \sum_{r=0}^{M-1} r\tau\, W_r(t),$$

$$x_\tau \in E_\tau [0, T; R^n].$$

In this case the finite-dimensional control problem to be an approximation of problem P_0 consist in finding $\min J(x, u)$ subject to:

$$\begin{aligned}
&\nabla x_\tau(t) + f\big(x_\tau(t), x_\tau(t-h), u_\tau(t), t_\tau\big) = 0,\\
&x_\tau(Q) = P_\tau\, \varphi(Q) \quad Q \in [-h, 0)\\
&x_\tau(0) = \varphi(0)\\
&S(x_\tau(t), t_\tau) \in K\\
&u_\tau \in \mathscr{P}_\tau\, \Omega
\end{aligned} \tag{4.2.2}$$

where $x_\tau \in E_\tau [0, T+\tau; R^n]$; $u_\tau \in E_\tau [0, T; R^m]$.

However in this case there arises the question whether the solution of continuous problem

$$\begin{aligned}
&\frac{dx(t)}{dt} + f\big(x(t), x(t-h), u_\tau(t), t\big) = 0, && t \in [0, T],\\
&x(Q) = \varphi(Q), && Q \in [-h, 0],
\end{aligned} \tag{4.2.3}$$

corresponding to discrete optimal control u_τ (u_τ determined by (4.2.2)) satisfy continuous constraints:

$$S\big(x(t), t\big) \in K, \quad t \in [0, T].$$

Obviously, we would like to obtain the solution of (4.2.3) $x(u_\tau)$ which satisfies $S(x(u_\tau)(t), t) \in K$.

However this requirement in general is not satisfied with the approximation of constraints given by (4.2.1).

Therefore in this section we are going to study the problem of finding such an approximation the set of admissible states that the above mentioned condition is satisfied.

In order to present some effective method of determining such approximation let us consider the following "parametric" family of optimization problems.

Problem P_{K_F}
$\min J(x_\tau, u_\tau)$ subject to

$$\begin{aligned}&\nabla x_\tau(t)+f\left(x_\tau(t), x_\tau(t-h), u_\tau(t), t_\tau\right)=0,\\ &x_\tau(Q)=P_\tau\,\varphi(Q), \qquad Q\in[-h,0)\\ &x_\tau(0)=\varphi(0),\\ &S\left(x_\tau(t), t_\tau\right)\in K_F\subset K\\ &u_\tau\in\mathscr{P}_\tau\,\Omega,\end{aligned} \tag{4.2.4}$$

where $K_F\in\mathscr{F}(K)$, $\mathscr{F}(K)$ is a family of convex closed cones belonging to K and there exists a convex closed cone $K_0\subset R^n$ such that: $\forall K_F\in\mathscr{F}(K)$, $K_0\subset K_F$.

The Lemma analogus to Lemma 3.1 concerning some properties of optimal solution to problem P_{K_F} is presented below.

Lemma 4.1. Assume that hypothesis H1 and H2 are satisfied. Then

$$\exists 0<\delta<\infty\exists\tau_0>0\ \forall K_F\in\mathscr{F}(K)\ \forall\tau<\tau_0\ |u_\tau^{K_F}|\leqslant\delta$$
$$|x_\tau^{K_F}|\leqslant\delta$$

where $u_\tau^{K_F}$ and $x_\tau^{R_F}$ are optimal solutions associated with Problem P_{K_F}.

The proof of the Lemma follows by using arguments similar to those given in the proof of Lemma 3.1 (see [3]) and employing the fact that $K_F\supset K_0$ $\forall K_F\in\mathscr{F}(K)$.

Define an operator $P_\tau^{-1}: E_\tau[0,T;R^n]\to H^1[0,T;R^n]$ by the following formula:

$$P_\tau^{-1}x_\tau(t)=x_\tau(r\tau)+\frac{[x_\tau(r+1)_\tau-x_\tau(r\tau)]\,[t-r\tau]}{\tau} \tag{4.2.5}$$

for $t\in[r\tau;(r+1)\tau]$, $r=0,1,\dots,M-1$.

Another words $P_\tau^{-1}x_\tau$ is piecewise linear function constructing with the help of x_τ such that

$$P_\tau^{-1}x_\tau(r\tau)=x_\tau(r\tau), \qquad r=0,1,\dots,M-1.$$

It is easy to verify that the following estimations take place:

$$\|x_\tau-P_\tau^{-1}x_\tau\|\leqslant\tau\,\|\nabla x_\tau\|, \tag{4.2.6}$$

$$|x_\tau-P_\tau^{-1}x_\tau|\leqslant\tau^{\frac{1}{2}}\,\|\nabla x_\tau\|, \tag{4.2.7}$$

$$|x_\tau-P_\tau^{-1}x_\tau|\leqslant\tau\,|\nabla x_\tau|. \tag{4.2.8}$$

Assume $x_\tau^{K_F}$, $u_\tau^{K_F}$ to be the solution of problem P_{K_F} associated with some cone $K_F\in\mathscr{F}(K)$. Denote by $\bar{x}\overset{\mathrm{df}}{=}\bar{x}(u_\tau^{K_F})$ the solution of equation (4.2.3) corresponding to control $u_\tau^{K_F}$. We are going to show that the difference (in the sense of L^∞ norm) between $P_\tau^{-1}(x_\tau^{K_F})$ and $\bar{x}(u_\tau^{K_F})$ can be a priori estimated independly of the choice of K_F. This result is formulated in the following:

Lemma 4.2. Assume that:

(i) $x_\tau^{K_F}$, $u_\tau^{K_F}$ — are solutions of problem P_{K_F}; $\bar{x}$ is the solution of state equation (4.2.3) corresponding to control $u_\tau^{K_F}$.

(ii) Hypothesis H1, H2, H4 are satisfied.

(iii) f satisfies Lipschitz condition with constant L_0 on the set $G(t)$

$$G(t) \overset{\text{df}}{=} \{(x(t), y(t), u(t), t) \in R^n \times R^n \times R^m \times [0, T];\ x, y, u \in G\}$$

(G defined in Lemma 3.1).

(iv) τ is chosen such that:

$$\tau < \frac{\alpha}{2L_0^2}$$

then:

$$|\bar{x}(u_\tau^{K_F}) - P_\tau^{-1} x_\tau^{K_F}| \leqslant a\tau$$

where a does not depend either on τ or on $K \in \mathscr{F}$ $(K_{\mathscr{F}})$.

Proof. Note that by (4.2.7)

$$|x - P_\tau^{-1} x_\tau^{K_F}| \leqslant |\bar{x} - x_\tau^{K_F}| + |x_\tau^{K_F} - P_\tau^{-1} x_\tau^{K_F}| \leqslant |\bar{x} - x_\tau^{K_F}| + \tau |\nabla x_\tau^{K_F}|. \tag{4.2.9}$$

So the term $|\bar{x} - x_\tau^{K_F}|$ must be estimated. In order to do that step method is used.

Denote $\tilde{x}_\tau = P_\tau \bar{x}$, $\bar{x}_\tau \overset{\text{df}}{=} x_\tau^{K_F}$, $\bar{u}_\tau \overset{\text{df}}{=} u_\tau^{K_F}$.

Define $\delta_\tau(t) \overset{\text{df}}{=} (\nabla(\tilde{x}_\tau - \bar{x}_\tau)(t), (\tilde{x}_\tau - \bar{x}_\tau)(t)) + (A(\bar{x}_\tau, \bar{y}_\tau, \bar{u}_\tau)(t_\tau) +$
$- A(\tilde{x}_\tau, \tilde{y}_\tau, \bar{u}_\tau)(t_\tau), (\tilde{x}_\tau - \bar{x}_\tau)(t))$.

Observe that:

$$(\nabla x_\tau(t), x_\tau(t)) = \frac{\nabla |x_\tau(t)|^2 - \tau |\nabla x_\tau(t)|^2}{2} \tag{4.2.10}$$

(it is easily deduced from the definition of ∇x_τ).

Denote $r_t = \text{entier } t/\tau$.

After integrating $\delta_\tau(t)$ from 0 to $r_t \tau$ (for $t < h$) and after applying (4.2.10) and hypothesis H4 we have:

$$\int_0^{r_t \tau} \delta_\tau(t)\, dt \geqslant \frac{1}{2} |\tilde{x}_\tau(t) - \bar{x}_\tau(t)|^2 - \frac{\tau}{2} \int_0^{r_t \tau} |\nabla(\tilde{x}_\tau(t) - \bar{x}_\tau(t))|^2\, dt +$$
$$+ \alpha \int_0^{r_t \tau} |\tilde{x}_\tau(t) - \bar{x}_\tau(t)|^2\, dt. \tag{4.2.11}$$

Taking advantage of the facts that $\bar{x}_\tau$ and $\bar{x}$ are the solution of state equations (4.2.4) and (4.2.3) respectively and that A satisfies Lipschitz condition we get:

$$\int_0^{r_t \tau} |\nabla(\tilde{x}_\tau - \bar{x}_\tau)(t)|^2\, dt = - \int_0^{r_t \tau} (A(\bar{x}_\tau, \bar{y}_\tau, \bar{u}_\tau)(t_\tau), \nabla(\tilde{x}_\tau - \bar{x}_\tau)(t))\, dt +$$
$$+ \int_0^{r_t \tau} \left(\frac{d}{dt} \bar{x}(t) - \nabla \tilde{x}_\tau(t), \nabla(\tilde{x}_\tau - \bar{x}_\tau)(t) \right) dt + \int_0^{r_t \tau} (A(\bar{x}, \bar{y}, \bar{u}_\tau)(t)),$$

$$\nabla(x_\tau-\tilde{x}_\tau)(t))\,dt\leqslant L_0\left[\int_0^{r_t\tau}\left(|(\bar{x}_\tau-\bar{x})(t)|^2+|t_\tau-t|^2+|(\bar{y}_\tau-\bar{y})(t)|^2\right)dt\right]^{1/2}\times$$
$$\times\left[\int_0^{r_t\tau^0}|\nabla(\bar{x}_\tau-\tilde{x}_\tau)(t)|^2\,dt\right]^{1/2}+$$
$$+\tau\sqrt{2}\left\|\frac{d^2\bar{x}}{dt^2}\right\|_*\left[\int_0^{r_t\tau}|\nabla(\bar{x}_\tau-\tilde{x}_\tau)(t)|^2\,dt\right]^{1/2}\,{}^{2)}$$

where

$$\left\|\frac{d^2\bar{x}}{dt^2}\right\|_*=\left[\sum_{i=0}^{N-1}\int_{r\tau}^{(r+1)\tau}\left|\frac{d^2\bar{x}(t)}{dt^2}\right|^2dt\right]^{1/2}$$

(in the last inequality we used the estimation (4.1.5)).

Hence by (4.1.2)

$$\left[\int_0^{r_t\tau}|\nabla(\bar{x}_\tau-\tilde{x}_\tau)(t)|^2\,dt\right]^{1/2}\leqslant\tau\sqrt{2}\left\|\frac{d^2\bar{x}}{dt^2}\right\|+$$
$$+L_0\left[2\int_0^{r_t\tau}|(\bar{x}_\tau-\tilde{x}_\tau)(t)|^2\,dt\right]^{1/2}+L_0\,\tau\left[\left\|\frac{d\varphi}{dt}\right\|_{-1}+2\left\|\frac{d\bar{x}}{dt}\right\|_0+T\right]\cdot \quad (4.2.12)$$

By substituting (4.2.12) into (4.2.11) we haxe:

$$\int_0^{r_t\tau}\delta_\tau(t)\,dt\geqslant\frac{1}{2}|(\bar{x}_\tau-\tilde{x}_\tau)(t)|^2+(\alpha-2\tau L_0^2)\int_0^{r_t\tau}|(\bar{x}_\tau-\tilde{x}_\tau)(t)|^2\,dt+$$
$$-L_0^2\,\tau^3\left[\left\|\frac{d\varphi}{dt}\right\|_{-1}^2+2\left\|\frac{d\bar{x}}{dt}\right\|_0^2+T^2\right]-\tau^2\left\|\frac{d^2x}{dt^2}\right\|_*^2. \quad (4.2.13)$$

On the other hand using inequality

$$ab\leqslant 2\varepsilon a^2+\frac{1}{4\varepsilon}b^2,\qquad \varepsilon>0 \quad (4.2.14)$$

and Lipschitz condition we obtain:

$$\int_0^{r_t\tau}\delta_\tau(t)\,dt=\int_0^{r_t\tau}\left(\frac{d}{dt}\bar{x}(t)-\nabla\tilde{x}_\tau(t),(\bar{x}_\tau-\tilde{x}_\tau)(t)\right)dt+$$
$$+\int_0^{r_t\tau}\left(A(\bar{x},\bar{y},\bar{u}_\tau)(t)-A(\bar{x}_\tau,\bar{y}_\tau,\bar{u}_\tau)(t_\tau),(\bar{x}_\tau-\tilde{x}_\tau)(t)\right)dt\leqslant$$
$$\leqslant\frac{1}{4\varepsilon}\left\{L_0^2\left[\|\bar{x}-\bar{x}_\tau\|+\|\bar{y}-\bar{y}_\tau\|_*^2+\|t-t_\tau\|_*^2\right]+\tau^2\left\|\frac{d^2\bar{x}}{dt^2}\right\|_1\right\}+$$
$$+2\varepsilon\int_0^{r_t\tau}|(\bar{x}_\tau-\tilde{x}_\tau)(t)|^2\,dt. \quad (4.2.15)$$

[2)] Observe that $\frac{d\bar{x}}{dt}$ is piecewise absolutely continuous function (since u_τ^0 is step function), so $\left\|\frac{d^2\bar{x}}{dt^2}\right\|_{r\tau\,(r+1)\,\tau}$ is defined, moreover $\left|\frac{d^2(x)}{dt^2}\right|<C$ for $t\in[r\tau,(r+1)\tau)$ where C does not depend on τ (it results from the fact that u_τ^0 is bounded independently on τ). Then $\left\|\frac{d^2\bar{x}}{dt^2}\right\|_*$ is bounded independently on τ.

After combining (4.2.15) and (4.2.13) we arrive at:

$$\frac{1}{2}|(\bar{x}_\tau-\tilde{x}_\tau)(t)|^2+(\alpha-2\tau L_0^2-2\varepsilon)\int_0^{r_t\tau}|(\bar{x}_\tau-\tilde{x}_\tau(t)|^2\,dt\leqslant$$

$$\leqslant L_0^2\tau^2\left(\frac{1}{4\varepsilon}+\tau\right)\left[2\left\|\frac{d\bar{x}}{dt}\right\|_0^2+\left\|\frac{d\varphi}{dt}\right\|_{-1}^2+T^2\right]+2\tau^2\left\|\frac{d^2x}{dt^2}\right\|_*\left(\frac{1}{4\varepsilon}+\tau\right). \quad (4.2.16)$$

By the analogous way we prove that for $K=1,\dots,m-1$ and $Kh\leqslant t\leqslant(K+1)h$

$$\int_{Kh}^{r_t\tau}\delta_\tau(t)\,dt\geqslant\frac{1}{2}|(\bar{x}_\tau-\tilde{x}_\tau)|^2-\frac{1}{2}|(\bar{x}_\tau-\tilde{x}_\tau)(Kh)|^2+$$

$$+(\alpha-2\tau L_0^2)\int_{Kh}^{r_t\tau}|(\bar{x}_\tau-\tilde{x}_\tau)(t)|^2\,dt-2\tau L_0^2\,\|\bar{x}_\tau-\tilde{x}_\tau\|_{K-1}^2+$$

$$-\tau\left[L_0^2\tau^2\left(\left\|\frac{d\bar{x}}{dt}\right\|_K^2+\left\|\frac{d\bar{x}}{dt}\right\|_{K-1}^2+T^2\right)+2\tau^2\left\|\frac{d^2\bar{x}}{dt^2}\right\|_*^2\right. \quad (4.2.17)$$

On the other hand

$$\int_{Kh}^{r_t\tau}\delta_\tau(t)\,dt\leqslant\frac{1}{4\varepsilon}\left\{2L_0^2\,\|\bar{x}_\tau-\tilde{x}_\tau\|_{K-1}^2+2\tau^2L_0^2\left(\left\|\frac{d\bar{x}}{dt}\right\|_K^2+\right.\right.$$

$$\left.\left.+\left\|\frac{d\bar{x}}{dt}\right\|_{K-1}^2+T^2\right)+2\tau^2\left\|\frac{d^2\bar{x}}{dt^2}\right\|_*^2\right\}+2\varepsilon\int_{Kh}^{r_t\tau}|(\bar{x}_\tau-\tilde{x}_\tau)(t)|^2\,dt. \quad (4.2.18)$$

Combining (4.2.16) and (4.2.17) and (4.2.18) we obtain:

$$(\alpha-2\tau L_0^2-2\varepsilon)\int_{Kh}^{r_t\tau}|(\bar{x}_\tau-\tilde{x}_\tau)(t)|^2\,dt+\frac{1}{2}|(\bar{x}_\tau-\tilde{x}_\tau)(t)|^2\leqslant$$

$$\leqslant 2L_0^2\left(\tau+\frac{1}{2\varepsilon}\right)\|\bar{x}_\tau-\tilde{x}_\tau\|_{K-1}^2+\tau^2L_0^2\left(\tau+\frac{1}{2\varepsilon}\right)\left(\left\|\frac{d\bar{x}}{dt}\right\|_{K-1}+\right.$$

$$\left.+2\left\|\frac{d\bar{x}}{dt}\right\|_K+T\right)^2+\tau^2\left\|\frac{d^2\bar{x}}{dt^2}\right\|_*^2\left(\frac{1}{2\varepsilon}+\tau\right)+\frac{1}{2}|(\bar{x}_\tau-\tilde{x}_\tau)(Kh)|^2. \quad (4.2.19)$$

Let ε be chosen such that $\alpha-2\tau L_0^2-2\varepsilon>0$ (it is possible due to the assumption $\alpha-2\tau L_0^2>0$).

Denote

$$\alpha_0=\alpha-2L_0^2-2\varepsilon,$$

$$\alpha_1=L_0^2\left(\tau+\frac{1}{2\varepsilon}\right),$$

$$a_0=\tau^2\alpha_1\left(2\left\|\frac{d\bar{x}}{dt}\right\|_0^2+\left\|\frac{d\varphi}{dt}\right\|_{-1}^2+T^2\right)+\tau^2\left\|\frac{d^2\bar{x}}{dt^2}\right\|_*^2\left(\frac{1}{4\varepsilon}+\tau\right),$$

$$a_K=\tau^2\alpha_1\left(2\left\|\frac{d\bar{x}}{dt}\right\|_K^2+\left\|\frac{d\bar{x}}{dt}\right\|_{K-1}^2+T^2\right)+\tau^2\left\|\frac{d^2\bar{x}}{dt^2}\right\|_*^2\left(\frac{1}{2\varepsilon}+\tau\right),$$

for $K=1,\dots,M-1$.

Thus (4.2.16) and (4.2.19) we can rewrite in the following form:

$$\alpha_0 \int_0^{r_t \tau} |(\bar{x}_\tau - \tilde{x}_\tau)(t)|^2 \, dt + \frac{1}{2} |(\bar{x}_\tau - \tilde{x}_\tau)(t)|^2 \leqslant a_0 \quad \text{for } t \leqslant h, \tag{4.2.20}$$

$$\alpha_0 \int_{Kh}^{r_t \tau} |(\bar{x}_\tau - \tilde{x}_\tau)(t)|^2 \, dt + \frac{1}{2} |(\bar{x}_\tau - \tilde{x}_\tau)(t)|^2 \leqslant a_K + 2\alpha_1 \|\bar{x}_\tau - \tilde{x}_\tau\|^2_{K-1} +$$
$$+ \frac{1}{2} |\bar{x}_\tau(Kh) - \tilde{x}_\tau(Kh)|^2 \quad \text{for} \quad Kh \leqslant t \leqslant (K+1)h. \tag{4.2.21}$$

From (4.2.20) and (4.2.21) it follows that

$$\frac{1}{2} |(\bar{x}_\tau - \tilde{x}_\tau)(t)|^2 + \alpha_0 \int_{Kh}^{r_t \tau} |(\bar{x}_\tau - \tilde{x}_\tau)(t)|^2 \, dt \leqslant a_0 \left(\frac{2\alpha_1}{\alpha_0} + 1 \right)^K +$$
$$+ a_1 \left(\frac{2\alpha_1}{\alpha_0} + 1 \right)^{K-1} + \dots a_K .$$

Hence for $t \in [Kh, (K+1)h]$ we have:

$$\frac{1}{2} |(\bar{x}_\tau - \tilde{x}_\tau)(t)|^2 \leqslant \left(\frac{2\alpha_1}{\alpha_0} + 1 \right)^K \sum_{i=0}^{K} a_i .$$

Recall that $|\bar{x}_\tau(t) - \tilde{x}_\tau(t)| \leqslant \tau \left| \frac{d\bar{x}}{dt} \right|$ (see (4.1.3)), so

$$\sup_{t \in [0,T]} |\bar{x}_\tau(t) - \tilde{x}_\tau(t)| \leqslant 2m \left(\frac{x\alpha_1}{\alpha_0} + 1 \right)^{\frac{m-1}{2}} \left[2\tau^2 \alpha_1 \left(\left\| \frac{d\bar{x}}{dt} \right\|^2 + \right. \right.$$
$$\left. \left. + \left\| \frac{d\varphi}{dt} \right\|^2_{-1} + T^2 \right) + \tau^2 \left\| \frac{d^2 \bar{x}}{dt^2} \right\|^2_* \left(\frac{1}{2\varepsilon} + \tau \right) \right]^{\frac{1}{2}} +$$
$$+ \tau \left| \frac{d\bar{x}}{dt} \right| \leqslant \tau \left[2m \left(\frac{2\alpha_1}{\alpha_0} + 1 \right)^{\frac{m-1}{2}} \left[\alpha_1^{\frac{1}{2}} \left(\left\| \frac{d\bar{x}}{dt} \right\| + \right. \right. \right.$$
$$\left. \left. \left. + \left\| \frac{d\varphi}{dt} \right\|_{-1} + T \right) + \left\| \frac{d^2 \bar{x}}{dt^2} \right\|_* \left(\frac{1}{2\varepsilon} + \tau \right)^{\frac{1}{2}} \right] + \left| \frac{dx}{dt} \right| \right]. \tag{4.2.22}$$

After combining (4.2.22) and (4.2.9) we get

$$|\bar{x} - P_\tau^{-1} x_\tau^{K_F}| \leqslant a\tau$$

where

$$a = |\nabla x_\tau^{K_F}| + 2m \left(\frac{2\alpha_1}{\alpha_0} + 1 \right)^{\frac{m-1}{2}} \left[\alpha_1^{\frac{1}{2}} \left(\left\| \frac{d\bar{x}}{dt} \right\| + \left\| \frac{d\varphi}{dt} \right\|_{-1} + T \right) + \right.$$
$$\left. + \left\| \frac{d^2 \bar{x}}{dt^2} \right\| \left(\frac{1}{2\varepsilon} + \tau \right)^{\frac{1}{2}} + \left| \frac{d\bar{x}}{dt} \right| \right.$$

Since $\nabla x_\tau^{K_F}(t_\tau) = -A(x_\tau^{K_F}, y_\tau^{K_F}, u_\tau^{K_F})(t_\tau)$ and $\frac{d\bar{x}}{dt} = -A(\bar{x}, \bar{y}, u_\tau^{K_F})$ then after employing the results of Lemma 3.1., equation (4.1) and differentiability of f (hypothesis H1) we find that $|\nabla x_\tau^{K_F}|$, $\left\|\frac{d\bar{x}}{dt}\right\|$ and $\left|\frac{d\bar{x}}{dt}\right|$, $\left\|\frac{d^2\bar{x}}{dt^2}\right\|_*$ are bounded by constants independent of τ and K_F. So we prove that a does not depend on τ, K_F what completes the proof of the Lemma.

Now let us define

$$D_\tau^* = \{x \in H^1 [0, T; R^n];\ S(x, t), t) \in K + p_\tau\ t \in [0, T]\} \tag{4.2.23}$$

where $p_\tau \in K$ is determined by the following condition:

$$d(p_\tau, 0) = \min_{\hat{p}_\tau \in P} d(\hat{p}_\tau, 0);\ P \stackrel{\text{df}}{=} \{\hat{p}_\tau \in K\ d(\hat{p}_\tau, \delta K) = La\,\tau\}^{3)\,4)}$$

see Fig. (1).

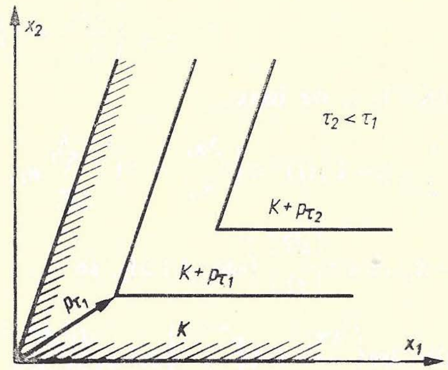


Fig. 1

It is easy to verify that p_τ satisfies the following conditions

$$\begin{aligned} &\text{(i)}\quad K + p_{\tau_1} \subset K + p_{\tau_2} \subset K \quad \tau_1 > \tau_2, \\ &\text{(ii)}\quad \exists b > 0\ La\tau \leqslant |p_\tau|_{R^K} \leqslant b\tau, \\ &\text{(iii)}\quad \forall k_1 \in K + p_\tau\ \forall k_2 : |k_1 - k_2|_{R^K} \leqslant La\tau \Rightarrow k_2 \in K. \end{aligned} \tag{4.2.24}$$

If we denote

$$D \stackrel{\text{df}}{=} \{x \in H^1 [0, T; R^n]; S(x(t), t) \in K, \quad t \in [0, T]\}$$

then from (4.2.24 (i)) immediately follows that:

$$D_{\tau_1}^* \subset D_{\tau_2}^* \subset D_{\tau_K}^* \ldots \subset D,\ \tau_1 > \tau_2 > \tau_K.$$

---

3) $d(a, B) = \inf_{z \in B} |a - z|$; δK — boundary of K.

4) L is the Lipschitz constant for S i.e.

$$|S(x_1(t), t) - S((x_2)(t), t)| \leqslant L\,|x_1(t) - x_2(t)|$$
$$\forall x_1, x_2 \in G\quad \forall t \in [0, T].$$

It is easy to observe that the sets D_τ^* are convex closed and moreover

$$D=\overline{\bigcup_{\tau>0} D_\tau^*}.$$

Hence

$$\forall \varepsilon>0\ \exists \tau\ \forall x \in D\ \exists x_\tau^* \subset D_\tau^*\ |x-x_\tau^*| \leqslant \varepsilon\,|x|. \tag{4.2.25}$$

In order to construct approximation of the set of admissible states we are going to prove the following Lemma to be essential in the sequel:

Lemma 4.3. Assume that $S(\xi, t)$ satisfies Lipschitz condition with constant L. Then the following implication takes place:

$$\forall x_\tau^* \in D_\tau^*\ \forall x \in H^1[0, T; R^n];\ \ |x_\tau^*-x| \leqslant a\tau \Rightarrow x \in D.$$

Proof. By Lipschitz condition for any $t \in [0, T]$ we have

$$|S(x_\tau^*(t), t)-S(x(t), t)|_{R^K} \leqslant L\,|x_\tau^*(t)-x(t)|_{R^n} \leqslant La\tau.$$

Since $S(x_\tau^*(t), t) \in K+p_\tau$ then from (4.2.24 (iii)) we obtain that $t \in [0, T]\ S(x(t), t) \in K$ and thus $x \in D$ Q.E.D.

It follows from the results of Lemmas 4.2, 4.3 that $\bar{x}$ (solution of (4.2.3)) belongs to the set of admissible states if the cone K_F is chosen in such a way that $P_\tau^{-1}\, x_\tau^{K_F} \in D_\tau^*$ ($x_\tau^{K_F}$ solution of problem P_{K_F}). So, the problem to be posed now is that of finding K_F such that $P_\tau^{-1}\, x_\tau^{K_F} \in D_\tau^*$. It is easy to prove that $K_F \overset{\text{df}}{=} K+p_\tau$ satisfies this requirement. Actually K_F belongs to the family $\mathscr{F}(K)$ since $K+p_\tau$ is a convex closed cone, $K+p_\tau \subset K$ and $K_0 \subset K+p_\tau$ with $K_0 \overset{\text{df}}{=} K+p_{\tau_0}$ (τ_0 given in Lemma 4.11).

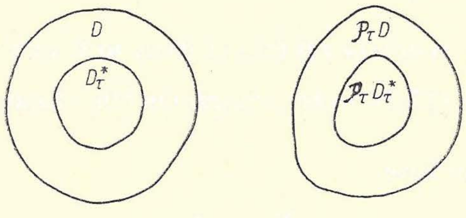
Furthermore the validity of the relation $P_\tau^{-1}\, x_\tau^{K_F} \in D_\tau^*$ for $K_F=K+p_\tau$ results from the following:

Lemma 4.4. Assume that hypothesis H3 is satisfied.

Let us put

$$\mathscr{P}_\tau D_\tau^* \overset{\text{df}}{=} \{x_\tau \in E_\tau[0, T; R^n];\ S(x_\tau(t), t_\tau) \in K+p_\tau.\}. \tag{4.2.26}$$

If $x_\tau \in \mathscr{P}_\tau D_\tau^*$ then $P_\tau^{-1}\, x_\tau \in D_\tau^*$ (see Fig. 2).



$\mathscr{P}_\tau D \overset{\text{df}}{=} \{x_\tau \in E_\tau[0T; R^n];\ S(x_\tau(t), t_\tau) \in K\}$

Fig. 2

Proof. Note that $t=\frac{t-r\tau}{\tau}(r+1)^{\tau}+\left(1-\frac{t-r\tau}{\tau}\right)r\tau$.

For $t\in[r\tau;(r+1)\tau)$

$$P_{\tau}^{-1}x_{\tau}(t)=\frac{t-r\tau}{\tau}x\left((r+1)\tau\right)+\left(1-\frac{t+r\tau}{\tau}\right)x(r\tau).$$

Denote $\alpha=\frac{t-r\tau}{\tau}$. Then for $t\in[r\tau;(r+1)\tau)$ by K-convexity of S we have

$$S\left(P_{\tau}^{-1}x_{\tau}(t),t\right)=S\left(\alpha x\left((r+1)\tau\right)+(1-\alpha)x(r\tau);\alpha(r+1)\tau+\right.$$
$$\left.(1-\alpha)r\tau\right)\in\alpha S\left(x(r\tau),r\tau\right)+(1-\alpha)S\left(x\left((r+1)\tau\right);(r+1)\tau\right)+$$
$$+K\subset\alpha(K+p_{\tau})+(1-\alpha)(K+p_{\tau})+K\subset K+p_{\tau}$$

what completes the proof of the fact that $P_{\tau}^{-1}x_{\tau}\in D_{\tau}^{*}$.

Now we are going to show that $\mathscr{P}_{\tau}D_{\tau}^{*}$ given by (4.2.26) is "a good approximation" of D in the following sense:

$$\forall x\in D\ \exists x_{\tau}\in\mathscr{P}_{\tau}D_{\tau}^{*},\ |x-x_{\tau}|\leqslant f_1(\tau), \tag{4.2.27}$$

$$\forall x_{\tau}\in\mathscr{P}_{\tau}D_{\tau}^{*}\ \exists x\in D,\ |x-x_{\tau}|\leqslant f_2(\tau), \tag{4.2.28}$$

where f_i: $[0,\tau_0]\to R^1$ are the continuous functions such that $f_i(\tau)\underset{\tau\to 0}{\longrightarrow}0$, $i=1,2$.

Actually, by (4.2.25)

$$\forall x\in D\exists x_{\tau}^{*}\in D_{\tau}^{*}\ |x-x_{\tau}^{*}|\leqslant\varepsilon(\tau)\,|x|.$$

Observe that for any $x_{\tau}^{*}\in D_{\tau}^{*}$, $t\in[r\tau;(r+1)\tau)$ and P_{τ} defined by (4.1.1) with $t_r=r\tau$ we obtain:

$$S\left(P_{\tau}x_{\tau}^{*}(t),t_{\tau}\right)=S\left(x_{\tau}^{*}(r\tau),r\tau\right)\in K+p_{\tau}\left(\text{since } S\left(x_{\tau}^{*}(t),t\right)\in K+p_{\tau}\ \forall t\in 0,T\right).$$

Hence $P_{\tau}x_{\tau}^{*}\in\mathscr{P}_{\tau}D_{\tau}^{*}$.

Moreover by (4.1.4)

$$|P_{\tau}x_{\tau}^{*}-x_{\tau}^{*}|\leqslant\tau\,|x_{\tau}^{*}|\leqslant\tau[|x_{\tau}^{*}-x|+|x|]\leqslant\tau\,[|x|+|x|\,\varepsilon(\tau)]\leqslant\tau\,(\varepsilon(\tau)+1)\,|x|$$

Then

$$\forall x\in D\exists\,P_{\tau}x_{\tau}^{*}\in\mathscr{P}_{\tau}D_{\tau}^{*}\ |x-P_{\tau}x_{\tau}^{*}|\leqslant f_1(\tau)$$

where

$$|x|\cdot f_1(\tau)=\varepsilon(\tau)+\tau\left(\varepsilon(\tau)+1\right)\ \text{tends to 0 with } \tau\to 0$$

what assures condition (4.2.27) to be satisfied. On other hand by Lemma 4.4 $\forall x_{\tau}\in\mathscr{P}_{\tau}D_{\tau}^{*}$; $P_{\tau}^{-1}x_{\tau}\in D_{\tau}^{*}$.

Recalling (4.2.7) we have

$$|x_{\tau}-P_{\tau}^{-1}x_{\tau}|\leqslant\tau\,|x_{\tau}|.$$

Furthermore, since $D_{\tau}^{*}\subset D$ then $\forall\,x_{\tau}\in\mathscr{P}_{\tau}D_{\tau}^{*}$ we have $P_{\tau}^{-1}x_{\tau}\in D$ and $|x_{\tau}-P_{\tau}^{-1}x_{\tau}|\leqslant\tau\,|x_{\tau}|$. So condition (4.2.28) is fulfilled with $f_2(\tau)=\tau\,|x_{\tau}|$.

Summing up the results obtained in this section we recall that the approximation of the set of admissible states defined by (4.2.26) is a "good approximation" in the sense given by (4.2.27), (4.2.28) and moreover $\bar{x}(u_\tau^{K_F})$ (where $u_\tau^{K_F}$ optimal control corresponding problem P_{K_F} with $K_F = K + p_\tau$) belongs to the set of admissible states (i.e. $S(\bar{x}(t), t) \in K$).

4.3. Discret optimal control problem — problem statement

We are going to formulate the finite-difference approximation of initial problem P_0

Problem P_τ

Find $(x_\tau^o, u_\tau^o) \in E_\tau[0, T+\tau; R^n] \times E_\tau[0, T; R^m]$ minimizing $J(x_\tau, u_\tau) = \int_0^T \Phi\big(x_\tau(t), u_\tau(t), t\big)\, dt$ subject to constraints:

$$\nabla x_\tau(t) + f\big(x_\tau(t).\ x_\tau(t-h), u_\tau(t), t_\tau\big) = 0 \tag{4.3.1}$$

$$x_\tau(Q) = P_\tau\, \varphi(Q),\ Q \in [-h, 0] \tag{4.3.2}$$

$$x_\tau(0) = \varphi(0) \tag{4.3.3}$$

$$x_\tau \in \mathcal{P}_\tau D_\tau^* \tag{4.3.4}$$

$$u_\tau \in \mathcal{P}_\tau \Omega\,. \tag{4.3.5}$$

It can be shown by using the standard arguments (see [4]) that hypothesis H1, H2 imply the existence of solution to problem P_τ.

A problem to be studied in the next section is the estimation of the error (in the sense of L^2-norm) between the solutions of problems (P_0) and (P_τ). In order to find this estimation the Lagrange's multipliers approach to finite-dimensional problem will be applied in the same way as it was done in Section 3.2. for continuous problem. At first under some additional convexity assumption the equivalence between the primal and dual discret problem will be established.

Define: $L_\tau\colon E_\tau[-h, T+\tau; R^n] \to E_\tau[0, T; R^m] \times E_\tau[0, T; R^n] \times E_\tau[0, T; R^n] \to R^1$ by the following formula:

$$L_\tau(x_\tau, u_\tau, \lambda_\tau, \eta_\tau) = J(x_\tau, u_\tau) + \langle \lambda_\tau, \nabla x_\tau + A_\tau(x_\tau, y_\tau, u_\tau)\rangle + \langle\!\langle \tilde{S}_\tau(x_\tau) - p_\tau, \eta_\tau \rangle\!\rangle \tag{4.3.6}$$

where

$$A_\tau(x_\tau, y_\tau, u_\tau)(t) \overset{\mathrm{df}}{=} A(x_\tau, y_\tau, u_\tau)(t_\tau);\ \tilde{S}_\tau(x_\tau)(t) \overset{\mathrm{df}}{=} \tilde{S}(x_\tau)(t_\tau);\ p_\tau(t) \overset{\mathrm{df}}{=} p_\tau$$
$$\text{for } t \in [0, T].$$

After applying the result of Lemma 3.2 in this case we can state that: $\exists \lambda_\tau^o \in E_\tau[0, T; R^n];\ \eta_\tau^o \in E_\tau[0, T; R^K]$ such that

$$\langle \delta_x L_\tau(x_\tau^o, u_\tau^o, \lambda_\tau^o, \eta_\tau^o), x_\tau - x_\tau^o\rangle + \langle \delta_y L_\tau(x_\tau^o, u_\tau^o, \lambda_\tau^o, \eta_\tau^o), y_\tau - y_\tau^o\rangle = 0$$

$$\forall x_\tau \in E_\tau[-h, T+\tau, R^n];\ x_\tau(Q) = P_\tau\, \varphi(Q), Q \in [-h, 0);\ x_\tau(0) = \varphi(0), \tag{4.3.7}$$

$$\delta_{\lambda_\tau} L_\tau(x_\tau^o, u_\tau^o, \lambda_\tau^o, \eta_\tau^o) = 0, \tag{4.3.8}$$

$$\langle \delta_u L_\tau(x_\tau^o, u_\tau^o, \lambda_\tau^o, \eta_\tau^o), u_\tau - u_\tau^o \rangle \geq 0 \ \forall u_\tau \in \mathscr{P}_\tau \Omega, \tag{4.3.9}$$

$$\langle\!\langle \tilde{S}(x_\tau^o) - p_\tau, \eta_\tau^0 \rangle\!\rangle = 0. \tag{4.3.10}$$

Assume the following hypothesis to be satisfied

$$\text{H5}' \left\langle \begin{bmatrix} L_{xx}(\tilde{a}_\tau) & L_{yx}(\tilde{a}_\tau) & L_{ux}(\tilde{a}_\tau) \\ L_{xy}(\tilde{a}_\tau) & L_{yy}(\tilde{a}_\tau) & L_{uy}(\tilde{a}_\tau) \\ L_{xu}(\tilde{a}_\tau) & L_{yu}(\tilde{a}_\tau) & L_{uu}(\tilde{a}_\tau) \end{bmatrix} \begin{bmatrix} x_\tau \\ y_\tau \\ u_\tau \end{bmatrix}, \begin{bmatrix} x_\tau \\ y_\tau \\ u_\tau \end{bmatrix} \right\rangle \geq \gamma \|u_\tau\|^2$$

where

$$\tilde{a}_\tau = (\tilde{x}_\tau, \tilde{u}_\tau, \tilde{\lambda}_\tau, \tilde{\eta}_\tau) \in U(x_\tau^o, u_\tau^o, \lambda_\tau^o, \eta_\tau^o).$$

Observe that hypothesis H5′ implies the uniqueness of optimal solution (x_τ^o, u_τ^o).

Lemma 4.5. If hypothesis H5′ is satisfied then Lagrangian $L_\tau(x_\tau, u_\tau, \lambda_\tau, \eta_\tau)$ posses a saddle point at $(x_\tau^o, u_\tau^o, \lambda_\tau^o, \eta_\tau^o)$ i.e.

$$L_\tau(x_\tau^o, u_\tau^o, \lambda_\tau, \eta_\tau) \leq L_\tau(x_\tau^o, u_\tau^o, \lambda_\tau^o, \eta_\tau^o) \leq L_\tau(x_\tau, u_\tau, \lambda_\tau^o, \eta_\tau^o)$$

$$\forall x_\tau \in E_\tau[-h, T; R^n]; \ x_\tau(Q) = P_\tau \varphi(Q), Q \in [-h, 0); \ x_\tau(0) = \varphi(0)$$

$$\forall u_\tau \in \mathscr{P}_\tau \Omega; \ \forall \lambda_\tau \in E_\tau[0, T+\tau; R^n]$$

$$\forall \eta_\tau \in -K_\tau^* \quad \text{where } K_\tau = \{x_\tau \in E_\tau[0, T; R^n]; \ x_\tau(t) \in K, \ t \in [0, T]\}$$

$$K_\tau^* = \{\eta_\tau \in E_\tau[0, T; R^n]; \ \int_0^T x_\tau(t)\, d\eta_\tau(t) \geq 0 \ \forall x_\tau \in K_\tau\}.$$

The proof of the Lemma is analogous to that of Lemma 3.4.

5. Error estimation for optimal and discrete solutions

The problem considered below is that of finding the rate of convergence for both the optimal state and the optimal control of finite-dimensional problem P_0 to the optimal state and the optimal control of continuous problem P_τ.

The main result of the paper is given in the following.

Theorem 5.1. Assume

(i) (x^o, u^o), (x_τ^0, u_τ^0) are optimal solutions corresponding to P_0 and P_τ problems respectively.

(ii) Hypothesis H1–H5 and assumption of Lemma 3.3 are satisfied.

(iii) f, S, Φ satisfy Lipschitz condition on the set $G(t)$ with constants L_0, L, L_1 respectively.

Then

$$\|u^o - u^o_\tau\| \leqslant C\tau^{\frac{1}{4}}$$

where C does not depend on τ

Proof. Denote:

$$\tilde{x}^o_\tau = P_\tau x^o$$

$$\tilde{\lambda}^o_\tau = P_\tau \lambda^o$$

$\tilde{u}^o_\tau \in \mathscr{P}_\tau \Omega$ an approximation of u^o obtained on the basis of (4.1.7) condition (recall that $\|u^o - \tilde{u}^o_\tau\| \leqslant C\tau^{\frac{1}{2}}$).

Let us choose $\tilde{\eta}^o_\tau \in -K^*_\tau$ such that:

$$\langle\!\langle x_\tau, \eta^o - \tilde{\eta}^o_\tau \rangle\!\rangle \leqslant 0 \ \forall x_\tau \in K_\tau . \tag{5.1}$$

We are going to show that such $\tilde{\eta}^o_\tau$ exists. Actually, observe that by properties of Stiljtes integral [8] we have

$$\langle\!\langle x_\tau, \eta^o \rangle\!\rangle = \sum_{r=0}^{M-1} x_\tau(r\tau) \int_{r\tau}^{(r+1)\tau} d\eta^o(t),$$

$$\langle\!\langle x_\tau, \tilde{\eta}^o_\tau \rangle\!\rangle = \sum_{r=0}^{M-1} x_\tau(r\tau)\, \delta_r$$

where $\delta_r \overset{\mathrm{df}}{=} \tilde{\eta}_\tau((r+1)\tau) - \tilde{\eta}_\tau(r\tau)$.

Thus, in order that (5.1) be satisfied the following conditions have to hold

$$\sum_{r=0}^{M-1} x_\tau(r\tau) \int_{r\tau}^{(r+1)\tau} d\eta^o(t) \leqslant \sum_{r=0}^{M-1} x_\tau(r\tau)\, \delta_r \ \forall x_\tau \in K_\tau \tag{5.2}$$

$$\sum_{r=0}^{M-1} x_\tau(r\tau)\, \delta_r \leqslant 0 \ \forall x_\tau \in K_\tau . \tag{5.3}$$

If we put

$$\delta_r \overset{\mathrm{df}}{=} \int_{r\tau}^{(r+1)\tau} d\eta^o(t)$$

than (5.2) is obviously fulfilled.

Taking into account the fact that $\eta^o \in -\tilde{K}^*$ and that $x_\tau(t) \overset{\mathrm{df}}{=} x_\tau(r\tau)\ \forall t \in [0, T]$ belongs to $\tilde{K}$ we have:

$$\forall r = 0, 1, \ldots, M-1, \ x_\tau(r\tau) \int_0^T d\eta^o(t) = \int_0^T x_r(t)\, d\eta^o(t) \leqslant 0 .$$

Thus

$$\sum_{r=0}^{M-1} x_\tau(r\tau)\, \delta_r = \sum_{r=0}^{M-1} x_\tau(r\tau) \int_{r\tau}^{(r+1)\tau} d\eta^o(t) \leqslant 0$$

what proves the validity of (5.3).

Further denote:

$$\hat{x} \stackrel{\mathrm{df}}{=} P_\tau^{-1} x_\tau^o .$$

Let $\hat{u}$ be an element of Ω corresponding to u^o according to condition (4.1.8) Hence

$$\|\hat{u} - u_\tau^o\| \leqslant C\tau^{1/2}.$$

Recalling the result of Lemma 4.5 and using the above notations we have:

$$L_\tau (x_\tau^o, u_\tau^o, \lambda_\tau^o, \eta_\tau^o) \leqslant L_\tau (\tilde{x}_\tau^o, \tilde{u}_\tau^o, \lambda_\tau^o, \eta_\tau^o). \tag{5.4}$$

On the other hand expending $L(\hat{x}, \hat{u}, \lambda^o, \eta^o)$ into Taylor series about $(x^o, u^o, \lambda^o, \eta^o)$, employing hypothesis H5 and the results of Lemma 3.3 we get:

$$\begin{aligned} L(\hat{x}, \hat{u}, \lambda^o, \eta^o) \geqslant L(x^o, u^o, \lambda^o, \eta^o) + \langle \delta_x L(x^o, u^o, \lambda^o, \eta^o), \hat{x} - x^o \rangle + \\ + \langle \delta_y L(x^o, u^o, \lambda^o, \eta^o), \hat{y} - y^o \rangle + \langle \delta_u L(x^o, u^o, \lambda^o, \eta^o), \hat{u} - u^o \rangle + \\ + \gamma \|\hat{u} - u^o\| \geqslant L(x^o, u^o, \lambda^o, \eta^o) + \gamma \|\hat{u} - u^o\|^2. \end{aligned} \tag{5.5}$$

Since $\tilde{\eta}_\tau^o \in -K_\tau^*$ Lemma 4.5 implies:

$$L_\tau (x_\tau^o, u_\tau^o, \lambda_\tau^o, \eta_\tau^o) \geqslant L_\tau (x_\tau^o, u_\tau^o, \tilde{\lambda}_\tau^o, \tilde{\eta}_\tau^o). \tag{5.6}$$

Let us add and subtract from the right-hand side of this inequality the term $L(\hat{x}, \hat{u}, \tilde{\lambda}_\tau^o, \tilde{\eta}_\tau^o)$. Taking advantage of (5.5) we obtain from (5.6)

$$\begin{aligned} L_\tau (x_\tau^o, u_\tau^o, \lambda_\tau^o, \eta_\tau^o) \geqslant L_\tau (x_\tau^o, u_\tau^o, \tilde{\lambda}_\tau^o, \tilde{\eta}_\tau^o) - L(\hat{x}, \hat{u}, \tilde{\lambda}_\tau^o, \tilde{\eta}_\tau^o) + \\ + L(\hat{x}, \hat{u}, \tilde{\lambda}_\tau^o, \tilde{\eta}_\tau^o) \geqslant L_\tau (x_\tau^o, u_\tau^o, \tilde{\lambda}_\tau^o, \tilde{\eta}_\tau^o) - L(\hat{x}, \hat{u}, \tilde{\lambda}_\tau^o, \tilde{\eta}_\tau^o) + \\ + L(x^o, u^o, \lambda^o, \eta^o) + \gamma \|\hat{u} - u^o\|^2. \end{aligned} \tag{5.7}$$

Combining (5.4) and (5.7) we arrive at:

$$\begin{aligned} L_\tau (x_\tau^o, u_\tau^o, \tilde{\lambda}_\tau^o, \tilde{\eta}_\tau^o) - L(\hat{x}, \hat{u}, \lambda^o, \eta^o) + L(x^o, u^o, \lambda^o, \eta^o) + \\ + \gamma \|\hat{u} - u^o\|^2 \leqslant L_\tau (\tilde{x}_\tau^o, \tilde{u}_\tau^o, \lambda_\tau^o, \eta_\tau^o). \end{aligned}$$

Hence

$$\begin{aligned} \gamma \|\hat{u} - u^o\|^2 \leqslant L_\tau (\tilde{x}_\tau^o, \tilde{u}_\tau^o, \lambda_\tau^o, \eta_\tau^o) - L(x^o, u^o, \lambda^o, \eta^o) + \\ + L(\hat{x}, \hat{u}, \lambda^o, \eta^o) - L_\tau (x_\tau^o, u_\tau^o, \tilde{\lambda}_\tau^o, \tilde{\eta}_\tau^o). \end{aligned} \tag{5.8}$$

In order to obtain the estimation of $\|\hat{u} - u^o\|^2$ we have to compute the differences between the discret Lagrangeans evaluated at $(\tilde{x}_\tau^o, \tilde{u}_\tau^o, \lambda_\tau^o, \eta_\tau^o)$ and $(x_\tau^o, u_\tau^o, \tilde{\lambda}_\tau^o, \tilde{\eta}_\tau^o)$ and continuous Lagrangians at the points $(x^o, u^o, \lambda^o, \eta^o)$ and $(\hat{x}, \hat{u}, \lambda^o, \eta^o)$ respectively.

Using definitions of L_τ and L and taking into account the fact that (x^o, u^o) satisfy the state equation (3.1.1) and state constraints we get:

$$\begin{aligned} L_\tau (\tilde{x}_\tau^o, \tilde{u}_\tau^o, \lambda_\tau^o, \eta_\tau^o) - L(x^o, u^o, \lambda^o, \eta^o) = J(\tilde{x}_\tau^o, \tilde{u}_\tau^o) - J(x^o, u^o) + \\ + \langle \nabla \tilde{x}_\tau^o + A_\tau (\tilde{x}_\tau^o, \tilde{y}_\tau^o, \tilde{u}_\tau^o), \lambda_\tau^o \rangle + \langle\!\langle \tilde{S}_\tau (\tilde{x}_\tau^o) - \tilde{p}_\tau, \eta_\tau^o \rangle\!\rangle \leqslant J(\tilde{x}_\tau^o, \tilde{u}_\tau^o) + \\ - J(x^o, u^o) + \left\langle \nabla \tilde{x}_\tau^o - \frac{d}{dt} x^o, \lambda_\tau^o \right\rangle + \langle \lambda_\tau^o, A_\tau (\tilde{x}_\tau^o, \tilde{y}_\tau^o, \tilde{u}_\tau^o) - \\ A(x^o, y^o, u^o) \rangle + \langle\!\langle \tilde{S}_\tau (\tilde{x}_\tau^o) - p_\tau - \tilde{S}(x^o), \eta_\tau^o \rangle\!\rangle . \end{aligned}$$

To the right hand side inequality (5.9) we added the term $-\langle\!\langle \tilde{S}(x^o), \eta_\tau^o \rangle\!\rangle$ which will be shown to be positive. So we are going to prove that: $\eta_\tau^o \in -\tilde{K}^*$, then $\langle\!\langle S(x^o), \eta_\tau^o \rangle\!\rangle \leqslant 0$.

Actually suppose that $x \in \tilde{K}$; then

$$\langle\!\langle x, \eta_\tau^o \rangle\!\rangle = \sum_{r=0}^{M-1} x(t_r)\,\delta_r \quad \text{where } \delta_r = \eta_\tau((r+1)\tau) - \eta_\tau(r\tau),\ t_r \in [r\tau; (r+1)\tau).$$

Let us denote $x_\tau(r\tau) = x(t_r)$. Obviously $x_\tau(t) \in K$ (since $x(t) \in K$), so $x_\tau \in K_\tau \subset E_\tau[0, T; R^n]$.

Hence

$$\langle\!\langle x, \eta_\tau^o \rangle\!\rangle = \sum_{r=0}^{M-1} x_\tau(r\tau)\,\delta_r = \langle\!\langle x_\tau, \eta_\tau^o \rangle\!\rangle \leqslant 0 \quad (\text{since } \eta_\tau^o \in -K^*)$$

what completes the proof of the fact that $\eta_\tau^o \in -\tilde{K}^*$.

After applying Schwartz inequality in (5.9) and employing the part that Φ, J, S satisfy Lipschitz conditions we get:

$$L_\tau(\tilde{x}_\tau^o, \tilde{u}_\tau^o, \lambda_\tau^o, \eta_\tau^o) - L(x^o, u^o, \lambda^o, \eta^o) \leqslant L_1 \left[\|\tilde{x}_\tau^o - x^o\| + \|\tilde{u}_\tau^o - u^o\|\right] +$$
$$+ \|\lambda_\tau^o\| \left[\left\|\nabla \tilde{x}_\tau^o - \frac{d}{dt} x^o\right\| + L_0 \left(\|x^o - \tilde{x}_\tau^o\| + \|y^o - \tilde{y}_\tau^o\| + \|u^o - \tilde{u}_\tau^o\| + \|t - t_\tau\|\right)\right] +$$
$$+ |\eta_\tau^o|_V \left[L\,|\tilde{x}_\tau^o - x^o| + |t - t_\tau| + |\tilde{p}_\tau|\right] \leqslant$$
$$\leqslant L_1 \left[\tau \left\|\frac{dx^o}{dt}\right\| + \tau^{\frac{1}{2}} C\right] + \|\lambda_\tau^o\| \left[\tau^{\frac{1}{2}} \beta_{\frac{dx^o}{dt}} + \right.$$
$$\left. L_0 \left(\tau \left(2 \left\|\frac{dx^o}{dt}\right\| + \left\|\frac{d\varphi}{dt}\right\|_{-1}\right) + \tau^{\frac{1}{2}} C + \tau T\right)\right] + |\eta_\tau^o|_V \left[L\tau \left|\frac{dx^o}{dt}\right| + (b+1)\tau\right].$$

In the last inequality the estimations (4.1.2), (4.1.4), (4.1.5), (4.1.7), (4.1.8) were employed. Hence

$$L_\tau(\tilde{x}_\tau^o, \tilde{u}_\tau^o, \lambda_\tau^o, \eta_\tau^o) - L(x^o, u^o, \lambda^o, \eta^o) \leqslant \tau^{\frac{1}{2}} C_0 \tag{5.10}$$

where

$$C_0 \overset{\text{df}}{=} \left\|\frac{dx^o}{dt}\right\| \left(L_1 \tau^{\frac{1}{2}} + 2\|\lambda_\tau^o\| L_0 \tau^{\frac{1}{2}}\right) + \left|\frac{dx^o}{dt}\right| L\tau^{\frac{1}{2}} |\eta_\tau^o|_V +$$
$$+ \|\lambda_\tau^o\| \left\|\frac{d\varphi}{dt}\right\|_{-1} L_0 \tau^{\frac{1}{2}} + L_1 C + \|\lambda_\tau^o\| \beta_{\frac{dx^o}{dt}} + \|\lambda_\tau^o\| L_0 C +$$
$$+ |\eta_\tau^o|_V (b+1)\tau^{\frac{1}{2}} + \|\lambda_\tau^o\| T\tau^{\frac{1}{2}}. \tag{5.11}$$

On the other hand using definitions of L and L_τ as well as the fact that (x_τ^o, u_τ^o) satisfy (4.3.1) we obtain

$$L(\hat{x}, \hat{u}, \lambda^o, \eta^o) - L_\tau(x_\tau^o, u_\tau^o, \tilde{\lambda}_\tau^o, \tilde{\eta}_\tau^o) = J(\hat{x}, \hat{u}) - J(x_\tau^o, u_\tau^o) +$$
$$+ \left\langle \frac{d\hat{x}}{dt} + A(\hat{x}, \hat{y}, \hat{u}), \lambda^o \right\rangle + \langle\!\langle \tilde{S}(\hat{x}), \eta^o \rangle\!\rangle - \langle\!\langle \tilde{S}(x_\tau^o) - p_\tau, \tilde{\eta}_\tau^o \rangle\!\rangle =$$

$$=J(x, u)-J(x_\tau^o, u_\tau^o)+\left\langle \frac{d\hat{x}}{dt}-\nabla x_\tau^o, \lambda^o \right\rangle+\langle A(\hat{x}, \hat{y}, \hat{u})+$$
$$-A(x^o, y^o, u^o), \lambda^o\rangle+\langle\!\langle \tilde{S}(\hat{x})-\tilde{S}_\tau(x_\tau^o)+\tilde{p}_\tau, \eta^o\rangle\!\rangle+$$
$$+\langle\!\langle \tilde{S}_\tau(x_\tau^o)-\tilde{p}_\tau, \eta^o-\tilde{\eta}_\tau^o\rangle\!\rangle .$$

After applying Schwartz inequality, using Lipschitz conditions and (5.1) we get:

$$L(\hat{x}, \hat{u}, \lambda^o, \eta^o)-L_\tau(x_\tau^o, u_\tau^o, \tilde{\lambda}_\tau^o, \tilde{\eta}_\tau^o)\leqslant L_1\,[\|\hat{x}-x_\tau^o\|+\|\hat{u}-u_\tau^o\|]+$$
$$+\|\lambda^o\|\left[\left\|\frac{d\hat{x}}{dt}-\nabla x_\tau^o\right\|+L_0\,(\|\hat{x}-x_\tau^o\|+\|\hat{y}-y_\tau^o\|+\|\hat{u}-u_\tau^o\|+\|t-t_\tau\|)+\right.$$
$$+|\eta^o|_V\,[L\,(|\hat{x}-x_\tau^o|+|t-t_\tau|+b\tau]\leqslant L_1\,[\tau\,\|\nabla \hat{x}_\tau^o\|+\tau^{\frac{1}{2}}\,C]+$$
$$\left.+\|\lambda^o\|\,[\tau^{\frac{1}{2}}\,\beta_{\frac{d\hat{x}}{dt}}+L_0\left(2\tau\,\|\nabla x_\tau^o\|+\tau\left\|\frac{d\varphi}{dt}\right\|_{-1}+\tau^{\frac{1}{2}}\,C+\tau T\right)\right]+$$
$$+\,|\eta^o|_V\,(L\tau\,|\nabla x_\tau^o|+\tau+b\tau).$$

Hence

$$L(\hat{x}, \hat{u}, \lambda^o, \eta^o)-L_\tau(x_\tau^o, u_\tau^o, \tilde{\lambda}_\tau^o, \tilde{\eta}_\tau^o)\leqslant C_1\,\tau^{\frac{1}{2}} \tag{5.12}$$

where

$$C_1 \overset{\mathrm{df}}{=} \|\nabla x_\tau^o\|\,[L_1\,\tau^{\frac{1}{2}}+2L_0\,\tau^{\frac{1}{2}}]+|\nabla x_\tau^o|\;|\eta^o|_V\,L\tau^{\frac{1}{2}}+2\left\|\frac{d\varphi}{dt}\right\|_{-1}L_0\,\tau^{\frac{1}{2}}+$$
$$\|\lambda^o\|\,(\beta_{\frac{d\hat{x}}{dt}}+L_0\,C)+L_1\,C+|\eta^o|_V+(b+1)\,\tau^{\frac{1}{2}}+L_0\,T\tau^{\frac{1}{2}}\,\|\lambda^o\|. \tag{5.13}$$

Substituting (5.10) and (5.12) into (5.8) we arrive at:

$$\gamma\,\|\hat{u}-u^o\|^2\leqslant C_2\,\tau^{\frac{1}{2}} \tag{5.14}$$

where $C_2 \overset{\mathrm{df}}{=} C_1+C_0$, C_1, C_0 are given respectively by (5.11), (5.13).

Hence by (4.1.8) and (5.14)

$$\|u_\tau^o-u^o\|\leqslant\|u_\tau^o-\hat{u}\|+\|\hat{u}-u^o\|\leqslant C\tau^{\frac{1}{2}}+\frac{C_2}{\gamma}\,\tau^{\frac{1}{4}}\leqslant C\tau^{\frac{1}{4}}$$

what completes the proof of Theorem 5.1.

Corollary 5.1. If the assumptions of Theorem 5.1 are satisfied and moreover $\tau<\dfrac{\alpha}{2L_0^2}$ then:

$$\|x^o-x_\tau^o\|\leqslant C\tau^{\frac{1}{4}}$$

and

$$J(x^o, u^o)-J(x_\tau^o, u_\tau^o)\leqslant C\tau^{\frac{1}{4}}.$$

The proof of Corollary 5.1 immediately follows from Theorem 5.1. and from the inequality

$$\|x^o-x_\tau^o\|\leqslant C\,\|u^o-u_\tau^o\|$$

which was proved in [3].

If we assume that the set of admissible controls belongs to H^1 [0, T; R^m] then by (4.1.2) the following estimations take place:

$$\|u^o - P_\tau u^o\| \leqslant \tau \left\| \frac{du^o}{dt} \right\|, \tag{5.15}$$

$$\left\| \nabla x^o_\tau - \frac{d}{dt} x^o \right\| \leqslant \tau \left\| \frac{d^2 x^o}{dt^2} \right\|. \tag{5.16}$$

Employing the same arguments to that given in the proof of Theorem 5.1 and taking into account (5.15) and (5.16) we obtain

Theorem 5.2. Assume taht

(i) (x^o, u^o), (x^o_τ, u^o_τ) are the solutions of problems P_0 and P_τ and $u^o \in \Omega \subset \subset H^1$ [0, T; R^m], where Ω is a convex closed set in H^1 [0, T; R^m].

(ii) Assumptions of Theorem 5.1. are satisfied then

$$\|u^o - u^o_\tau\| \leqslant C\tau^{\frac{1}{2}}.$$

Theorem 5.2 implies the following

Corollary 5.2. If the assumptions of Theorem 5.2 are satisfied and moreover $\tau < \dfrac{\alpha}{2L_0^2}$, then

$$\|x^o - x^o_\tau\| \leqslant C\tau^{\frac{1}{2}}$$

and

$$J(x^o, u^o) - J(x^o_\tau, u^o_\tau) \leqslant C\tau^{\frac{1}{2}}.$$

Remark. If $u \in PC$ [0, T; R^m[is assumed to satisfy Hölder condition with constant $0 < \alpha < 1$ in all intervals of continuity of $u(t)$ then as it was shown in [9],

$$\|u^o - P_\tau u^o\| \leqslant C\tau^\alpha, \quad 0 < \alpha < 1$$

then in this case Theorem 5.1 is valid with

$$\|u^o - u^o_\tau\| \leqslant C\tau^{\frac{\alpha}{2}} \text{ where } 0 < \alpha < 1.$$

6. Example

As a special case of constraints given by (3.13) the following form of state constraints is considered

$$g(x(t), t) \leqslant 0, \quad t \in [0, T] \tag{6.1}$$

where $g: R^n x [0, T] \to R^1$, is assumed to be twice continuously differentiable function, convex and to satisfy the following condition:

$$\text{(a)} \ \left(A_u^T (x^o, y^o, u^o)(t) \cdot g_x (x^o(t), t), I_m\right) \neq 0 \quad t \in [0, T] \tag{6.2}$$

where $I_m = (1, ..., 1) \in R^m$;

$$\text{(b)} \ g(x^o(0), 0) < 0, \quad x^o \in R^n. \tag{6.2}$$

It is easy to verify that hypothesis H3 is satisfied with

$$S(x(t), t) \stackrel{\text{df}}{=} g(x(t), t) \text{ and}$$

$$K \stackrel{\text{df}}{=} \{\xi \in R^1; \xi \leqslant 0\}. \tag{6.3}$$

Observe, that $\tilde{K}$ and $\tilde{K}^*$ take the following forms in this case:

$$\tilde{K} = \{y \in C[0, T]; \ y(t) \leqslant 0, \ t \in [0, T]\} \tag{6.4}$$

$$K^* = \{\eta \in V[0, T, R^1], \eta(t) \text{ nonincreasing, } d\eta(t) \geqslant 0, \ t \in [0, T]\} \tag{6.5}$$

We are going to show that under some observability condition we can employ the result of Lemma 3.3. This fact is given by the following

Lemma 6.1. Assume that:

(i) (x^o, u^o) is a solution of problem P_0 with state constraint of the form (6.1);

(ii) $L(x, u, \lambda, \eta) \stackrel{\text{df}}{=} J(x, u) + \left\langle \dfrac{dx}{dt} + A(x, y, u), \lambda \right\rangle + \langle\!\langle \tilde{g}(x), \eta \rangle\!\rangle$

where $\tilde{g}(x)(t) \stackrel{\text{df}}{=} g(x(t), t)$;

(iii) $A_u^T (x^o, y^o, u^o)(t) \, g_x (x^o(t), t) \neq 0, \quad 0 \in R^m$;

(iv) g satisfies: $g(x^o(0), 0) < 0$;

(v) $0 \in \Omega$

then there exists $\lambda_0 \in L^2 [0, T; R^n]$, $\eta^o \in -\tilde{K}^*$ (given by (6.5)) such that:

(i) $\langle \delta_x L(x^o, u^o, \lambda^o, \eta^o), x - x^o \rangle + \langle \delta_y L(x^o, u^o, \lambda^o, \eta^o), y - y^o \rangle = 0$,
$\forall x \in H^1 [-h, T; R^n]$, $x(Q) = \varphi(Q)$, $Q \in [-h, 0]$,

(ii) $\delta_\lambda L(x^o, u^o, \lambda^o, \eta^o) = 0$,

(iii) $\langle \delta_u L(x^o, u^o, \lambda^o, \eta^o), u - u^o \rangle \geqslant 0, \ \forall u \in \Omega$,

(iv) $\langle\!\langle \tilde{g}(x^o), \eta^o \rangle\!\rangle = 0$.

Proof. The Lemma is a special case of the Lemma 3.3. Therefore to prove it we have to verify the fulfillment of all assumptions of Lemma 3.3. It is easy to see that it is enough to check (ii) and (iv) observe that in our case (iii) takes on the form: $\exists \bar{x} \in H^1 [0, T; R^n]$, $\exists \bar{u} \in PC[0, T; R^m]$ such that

$$\frac{d\bar{x}(t)}{dt} + A_x (x^o, y^o, u^o)(t), \bar{x}(t) + A_y (x^o, y^o, u^o)(t) \cdot \bar{y}(t) +$$
$$+ A_u (x^o, y^o, u^o)(t) \cdot \bar{u}(t) = 0, \ \bar{x}(Q) = 0, \ Q \in [-h, 0] \tag{6.6}$$

and

$$g(x^o(t), t)+(g_x(x^o(t), t), \bar{x}(t))_{R^n}<0 \quad \forall t \in [0, T] \tag{6.7}$$

Denote $R \stackrel{\text{df}}{=} \{t \in [0, T];\ g(x^o(t), t)=0\}$.

Since $g(x^o(0), 0)<0$ (by assumption (6.2b) then for all $t \in [0, \text{min} t, t \in R]$ $\bar{u}(t) \stackrel{\text{df}}{=} 0$; $\bar{x}(t) \stackrel{\text{df}}{=} 0$ satisfy (6.6) and (6.7). Therefore in order to (6.7) be satisfied in all interval $[0, T]$ it is sufficient to assure the first derivative at (6.7) with respect to t i.e. $\frac{d}{dt}[(g\, x^o(t), t)+(g_x(x^o(t), t), \bar{x}(t))]$ to be negatif.

So, after computing the first derivative of (6.7) we arrive at

$$\begin{aligned}(g_x(x^o(t), t), \dot{x}(t))_{R^n}+g_t(x^o(t), t)+(g_{xx}(x^o(t), t), \dot{x}^o(t)+\\ +g_{xt}(x^o(t), t)+A_x^T(x^o, y^o, u^o)(t)\, g_x(x^o(t), t), \bar{x}(t))_{R^n}+\\ +(A_y^T(x^o, y^o, u^o)(t)\, g_x(x^o(t), t), \bar{y}(t))_{R^n}+\\ +(A_u^T(x^o, y^o, u^o)(t)\, g_x(x^o(t), t), \bar{u}(t))_{R^m}<0.\end{aligned} \tag{6.8}$$

If we denote

$$\begin{aligned}&d_0(t) \stackrel{\text{df}}{=} (g_x(x^o(t), t)\, \dot{x}^o(t))_{R^n}+g_t(x^o(t).\, t)\\ &d_1(t) \stackrel{\text{df}}{=} g_{xx}(x^o(t), t)\, \dot{x}^o(t)+g_{xt}(x^o(t), t)+A_x^T(x^o, y^o, u^o)(t)\, g_x(x^o(t), t)\\ &d_2(t)=A_y^T(x^o, y^o, u^o)(t)\, g_x(x^o(t), t)\\ &d_3(t)=A_u^T(x^o, y^o, u^o)(t)\, g_x(x^o(t), t)\end{aligned}$$

then (6.8) can be rewritten in the form:

$$d_0(t)+(d_1(t), \bar{x}(t))_{R^n}+(d_2(t), \bar{y}(t))_{R^n}+(d_3(t), \bar{u}(t))_{R^m}<0 \tag{6.8a}$$

From the smoothness properties of g it results that:

$$\begin{aligned}d_0(t) &\in C[0, T; R^1]\\ d_1(t) &\in C[0, T; R^n]\\ d_2(t) &\in C[0, T; R^n]\\ d_3(t) &\in C[0, T; R^m]\end{aligned}$$

Since $\bar{x}(t)$ and $\bar{u}(t)$ have to satisfy (6.6) then $\bar{x}(t)$ can be expressed in the form (see [10])

$$\begin{aligned}\bar{x}(t)=X(t, t_0)\, x(t_0)+\int_{t_0-h}^{t_0} X(t, \sigma+h)\, A_y(x^o, y^o, u^o)(\sigma+h)\, \bar{x}(\sigma)\, d\sigma)+\\ +\int_{t_0}^{t} X(t, s)\, A_u(x^o, y^o, u^o)(s)\, \bar{u}(s)\, ds \quad \text{for } t>t_0\end{aligned}$$

where

$$\frac{\partial}{\partial t} X(t, \sigma)=A_x(x^o, y^o, u^o)(t)\, X(t, \sigma)+A_y(x^o, y^o, u^o)(t)\, X(t-h, \sigma)$$

$$X(t, \sigma)=\begin{cases}0, & t<\sigma.\\ 1, & t=\sigma.\end{cases}$$

Hence (6.8a) is equivalent to

$$\left(d_1(t), \int_{t_0}^{t} X(t,s)\, A_u(x^o, y^o, u^o)(s)\, \bar{u}(s)\, ds\right)_{R^n} + (d_3(t), \bar{u}(t))_{R^m} <$$
$$< -d_0(t) - (d_2(t), \bar{y}(t))_{R^n} - (d_1(t), X(t, t_0)\, x(t^o))_{R^n} +$$
$$-\left(d_1(t), \int_{t_0-h}^{t_0} X(t, \sigma+h), A_y(x^o, y^o, u^o)(\sigma+h)\, \bar{x}(\sigma)\, d\sigma\right)_{R^n}. \tag{6.8b}$$

Observe that for $t_1 < t < t_1 + h$ inequality (6.8b) takes a form:

$$\left(d_1(t), \int_{t_1}^{t} X(t,s)\, A_u(x^o, y^o, u^o)(s)\, \bar{u}(s)\, ds\right)_{R^n} + (d_3(t), \bar{u}(t))_{R^m} <$$
$$< -d_0(t) \quad (\text{since } \bar{x}(t) = 0 \text{ for } t \in [-h, t_1]).$$

So, if $d_3(t) \neq 0$ then the existence of $\bar{u}(t)$, $t \in (t_1, t_1+h)$, realizing the above inequality it follows from the fact that Volterra equation

$$\bar{u}_k(t) - (d_1(t), \int_{t_1}^{t} X(t,s)\, A_u(x^o, y^o, u^o)(s)\, \bar{u}_k(s)\, ds)_{R^n} = -|d_0(t)| - 1$$

($u_k(t)$ denotes a k-coordinate of $\bar{u}(t)$ corresponding to $d_k(t)$ which is different from zero), has a solution for any $d_0(t)$ (in this point we employed the continuity of d_0, d_1).

Furthermore for $t_1 + lh < t < t_1 + (l+1)h$, $l = 1, \ldots, E\left(\frac{T-t_1}{h}\right) + 1$, $\bar{u}(t)$ can be determined by the following equation:

$$\bar{u}_k(t) - \left(d_1(t), \int_{t_1-lh}^{t} X(t,s)\, A_u(x^o, y^o, u^o)\, \bar{u}_k(s)\, ds\right)_{R^n} = -d_0(t) +$$
$$+ (d_1(t), X(t, t_1+lh)\, \bar{x}(t_1+lh))_{R^n} + (d_2(t), \bar{y}(t))_{R^n} + (d_1(t),$$
$$\int_{t_1(l-1)h}^{t_1+lh} X(t, \sigma+h)\, A_y(x^o, y^o, u^o)(\sigma+h)\, \bar{x}(\sigma)\, d\sigma)_{R^n}. \tag{6.8c}$$

Actually observe that all terms on the right hand side of equation (6.8c) are known (since $\bar{x}(t)$ for $t \in [t_1 + (l-1)h, t_1 + lh]$ was determined on the preceding step). Then due to the fact that the function on the right hand side of (6.8c) is continuous and that Volterra equation has a solution for any continuous function on the right hand side of equation there exists a solution of (6.8c). Therefore if even one coordinate of $d_3(t) = A_u^T(x^o, y^o, u^o)(t)\, g_x(x^g(t), t)$ is different from zero then we can to choose $\bar{u}(t)$ and $\bar{x}(t)$ in such a way that (6.8) is satisfied what establish the validity of (iii) in Lemma 3.3.

The assumption (iv) of Lemma 3.3 takes on the form

$$\exists \bar{x} \in H^1[0, T; R^n];\; u \in \Omega \text{ such that}$$

$$\frac{d\bar{x}}{dt} + A_x(x^o, y^o, u^o)\, \bar{x} + A_y(x^o, y^o, u^o)\, \bar{y} + A_u(x^o, y^o, u^o)\, \bar{u} = 0 \tag{6.9}$$
$$x(Q) = 0,\; Q \in [-h, 0]$$

$$g(x^o(t), t) + g_x(x^o(t), t)\bar{x}(t) \leq 0, \quad t \in [0, T]. \tag{6.10}$$

If we take $\bar{x} \stackrel{\mathrm{df}}{=} 0$ and $\bar{u} \stackrel{\mathrm{df}}{=} 0$, then (6.9) and (6.10) are obviously satisfied. By assumption (v) we known that $\bar{u}=0 \in \Omega$ what completes the proof of (v) in Lemma 3.3. Q.E.D.

It is easy to observe that in the case of cone K given by (6.3) p satisfying (4.2.24) is equal to:

$$p_\tau = -La\tau. \tag{6.11}$$

So, the discret constraints in $\mathscr{P}_\tau$ problem take the form

$$g\left(x_\tau(t), t_\tau\right) + La\tau \leqslant 0 \tag{6.12}$$

summing up the results obtained in this section we conclude that in our case the estimations given by theorems (5.1) (5.2) will be valid with $b \stackrel{\mathrm{df}}{=} La$ if the assumptions of Lemma 6.1 are fulfilled (since Lemma 6.1 is equivalent to Lemma 3.3 for considered types of constraints).

7. Conclusions and remarks

1. The error bound for continuous and discret solutions of delayed optimization problem with state and control constraint was established. For optimal control of the class $PC\,[0, T; R^m]$ the rate of convergence equal to $0\,(\tau^{\frac{1}{4}})$ was derived (see Theorem 5.1). In the case where optimal control belongs to Sobolev space $H^1\,[0, T; R^m]$ the rate of convergence can be estimated by $0\,(\tau^{\frac{1}{2}})$ (Theorem 5.2). The estimations presented in Sec. 5 provide that the problem of the speed of convergence is strictly closed with the smoothness properties of primal and dual variables. It is easily seen that in order to achieve the rate of convergence better than $0\,(\tau^{1/2})$ the appropriate regularity of Lagrange multipliers is required. Namely additionally assuming that the rows of the matrix

$$f_u^T\left(x^o(t), x^o(t-h), u^o(t), t\right) S_x\left(x^o(t), t\right)$$

are linearly independent it can be shown (based on the method proposed in [2]) that u^o, λ^o, η^o, are absolutely continuous. In this case the rate of convergence equal to $0\,(\tau)$ can be established.

2. For nonlinear optimization problem for systems with delay but without constraints the rate of convergence of order $0\,(\tau)$ was proved in [3].

3. Observe that the expressions determining constraints C_0, C_1 (see formulas (5.11), (5.13)) depend on $|\eta^o|_v$, $\|\lambda^o\|$ which are unknown in general case (the other quantities in (5.11), (5.13) — $\left\|\frac{dx^o}{dt}\right\|$, $\left|\frac{dx^o}{dt}\right|$ can be estimated a priori (see [3])). The adjoint equation (Lemma 3.3 (i)) establishes relation between η^o and λ^o but it seems to be impossible to estimate η^o, λ^o, independly — in terms of initial data of original problem. Therefore the inequalities given by Theorems (5.1), (5.2) rather can be understood as the estimations of the speed of convergence than the error bounds.

4. The results given in the paper can be generalized to the case of systems with any finite numbers of delays.

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Aproksymacja różnicowa zadania sterowania optymalnego dla układu z opóźnieniem przy ograniczeniach na stan i sterowanie

Artykuł poświęcony jest aproksymacji różnicowej problemu sterowania optymalnego z ograniczeniem stanu i sterowania dla układu opisanego nieliniowymi równaniami różniczkowymi z opóźnieniem. Problem optymalizacji rozważa się stosując teorię mnożników Lagrange'a. W pracy podane są oszacowania różnicy normy (w sensie przestrzeni L^2) sterowania i stanu optymalnego dla problemu dokładnego i aproksymowanego.

Разностная аппроксимация задачи оптимального управления для системы с запаздыванием при ограничениях на состояние и управление

Рассмотрена разностная аппроксимация задачи оптимального управления с ограничением состояния и управления для системы описываемой нелинейными дифференциальными уравнениями с запаздыванием. Для этой цели используется теория множителей Лагранжа. Дана оценка разницы нормы (в смысле пространства L^2) оптимального управления и состояния для точной и аппроксимируемой задачи.

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3. Wzory i symbole powinny być wpisane na maszynie bardzo starannie.

Szczególną uwagę należy zwrócić na wyraźne zróżnicowanie małych i dużych liter. Litery greckie powinny być objaśnione na marginesie. Szczególnie dokładnie powinny być pisane indeksy, (wskaźniki) i oznaczenia potęgowe. Należy stosować nawiasy okrągłe.

4. Spis literatury powinien być podany na końcu artykułu. Numery pozycji literatury w tekście zaopatruje się w nawiasy kwadratowe. Pozycje literatury powinny zawierać nazwisko autora (autorów) i pierwsze litery imion oraz dokładny tytuł pracy (w języku oryginału), a ponadto:

a) przy wydawnictwach zwartych (książki) — miejsce i rok wydania oraz wydawcę;

b) przy artykułach z czasopism: nazwę czasopisma, numer tomu, rok wydania i numer bieżący.

Pozycję literatury radzieckiej należy pisać alfabetem oryginalnym, czyli tzw. grażdanką.