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Control of Retarded Systems with Function Space Constraints Part II. Approximate Controllability

by

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The problem of approximate controllability for general linear functional-differential systems of retarded type is considered. It is proved that so called multipoint or weak multipoint controllability is a general necessary condition in order to the set of reachable final states be dense in a function space satisfying some natural axioms. For systems with discrete delays algebraic criteria for (weak) multipoint controllability are given. The dual space characterization of approximate controllability is established for general system while the state space is one of the commonly used C, W_1^r and $M_{\alpha\beta}^r$ as a generalization of $L^r = M_{00}^r$. These conditions are then developed into the form of certain observability problems for dual system constructible from a given one by simple matrix transposition. Hence easy checkable either sufficient or necessary conditions follow for general case. Complete algebraic testable characterization is derived for the case of one or finitely many commensurable discrete delays. In one delay case the spaces C, W_1^r, M_{11}^r, $1 \leq r < \infty$, are shown to be equivalent with respect to approximate controllability. The general system is never L^∞-approximately controllable. The numerical examples are given illustrating practical applicability of the obtained criteria. On the basis of them some conclusions on relations between approximate controllability and stabilizability are drawn.

0. Introduction and notation

0.1. Introduction

Although the controllability and observability problems have been extensively examined in general setting it still lacks results expressed directly in terms of system parameters.

For linear hereditary systems with a trajectory evolving in R^n one can distinguish many types of controllability and observability concepts which are, in general, unequivalent. This is due to the fact that both reaching of trajectory value $x(t)$ and the study of the behaviour of a full state of the system are interesting for applications. Testable algebraic conditions for reachability of trajectory value are well known. Appropriate references are given in Section 2. However conditions for state exact or approximate reachability were derived for special cases only (some authors use the term function space controllability).

In Section 1 we define R^n-, (weak) multipoint -, $\mathcal{F}$-, $\mathcal{F}$-approximate and $\mathcal{F}$-approximate null controllability of general linear autonomous system of retarded type

$$\dot{x}(t)=\int_{-h}^{0} d_s A(s)\, x(t+s)+Bu(t). \tag{0.1}$$

In this definitions $\mathcal{F}$ stands for arbitrary normed function space of states for (0.1). It is also motivated that $\mathcal{F}$-approximate controllability is the most important for applications (e.g. stabilization problems).

Besides indicating references for R^n-controllability we report in section 2 the previous results obtained for $\mathcal{F}$- and $\mathcal{F}$-approximate controllability, most of them concerning the system with one delay, a special case of (0.1)

$$\dot{x}(t)=A_0\, x(t)+A_1\, x(t-h)+Bu(t). \tag{0.2}$$

In Section 3 we prove that weak multipoint or multipoint controllability is a general necessary condition for $\mathcal{F}$-approximate controllability of (0.1). $\mathcal{F}$ is supposed to satisfy some natural axioms.

Finite, matrix rank conditions for multipoint controllability of systems with lumped delays, especially for (0.2), are presented in Section 4.

Then, in Section 5, concrete function spaces C, W_1^r, $M_{\alpha\beta}^r$ are considered in place of $\mathcal{F}$ (These spaces are characterized in paragraph 0.3 below). Approximate controllability is characterized by some identities involving fundamental matrix solution to (0.1) and functionals from dual spaces C^*, $(W_1^r)^*$, $(M_{\alpha\beta}^r)^*$. General implications between the properties of approximate controllability in the spaces mentioned above are established. It is proved that system (0.1) is never L^∞- (in general $M_{\alpha\beta}^\infty$-) approximately controllable and that for the case of W_1^∞ the necessary and sufficient condition is rank $B=n$.

In Section 6 for the same concrete spaces equivalent dual observability problems are posed for a dual observed system which is simply of type (0.1) but with transposed kernel $A'(s)$, free-motion and with output $y(t)=B'\,x(t)$. From this a conclusion is drawn that rank $B=n$ implies approximate controllability of system (0.1) in any of spaces of Section 5. Also an easily checkable necessary condition, common for all these spaces, is derived. If specialized to system (0.2) it has the form rank $[A_1; B]=n$. An important, from the point of view of applications, result is that approximate controllability in any of the spaces $W_1^r, C, M_{11}^r, M_{01}^r$ implies pole assignability (and hence stabilizability) of the system (0.1) with the aid of linear state feedback.

A complete set of algebraic, numerically checkable, criteria for approximate controllability of system (0.2) in the spaces C, W_1^r, $M_{\alpha\beta}^r$ is given. The properties of a maximal controlled invariant corresponding to some nondelayed linear system equivalent to system (0.2) are the most essential for approximate controllability of system (0.2). For system (0.2) it is proved that the spaces C, W_1^r and M_{11}^r are equivalent with respect to approximate controllability.

For some simple cases of system (0.2) such that $A_0=0$, the control interval $[0, 2h]$, etc. the algebraic criteria are simplified (Section 8).

Numerical examples are given in Section 9. They illustrate how the theory developed works in practical applications and also serve as counterexamples for that some implications do not hold in general. It is shown, for instance, that L^r- do not imply M_{01}^r-approximate controllability, it do not imply stabilizability either.

In concluding section the results of the paper are evaluated and applications to optimal control and stabilization problems are indicated.

0.2. Notation

We deal with linear spaces of R^n-valued functions defined on a closed interval $[a, b]\subset R^1$. The general notation for such space will be $\mathscr{F}(a, b; R^n)$. If a, b, n are understood we abbreviate to $\mathscr{F}$. When considering special spaces of e.g. continuous or square integrable functions we replace symbol $\mathscr{F}$ by common C or L^2 respectively.

For a vector (or matrix) q the transposed vector is denoted by q' and Euclidean norm by $|q|$. im and ker stand for image and kernel of an operator (matrix). I is the identity matrix. For $n\times n$ and $n\times m$ matrices A and B respectively and a subspace $X\subset R^n$ we denote by $\|A\|$ an operator norm, $[A; B]$ the augmented $n\times(n+m)$ matrix, A^+ the Moore—Penrose pseudoinverse, $\{A|X\}=X+AX+\ldots+A^{n-1}X$ the controllable subspace, $\{A|B\}=\{A|\text{im } B\}$, $A^{-1}X=A^+(X\cap \text{im } A)+\ker A$ the preimage of X under A. Y^* is the topological adjoint of space Y and for a set $Z\subset Y$ we denote $Z^\perp=\{y^*\in Y^*: y^*(Z)=0\}$ the annihilator of Z.

0.3. Special function spaces and their topological adjoints

Recall basic topological properties of following special function spaces which seem to be most important for applications and therefore they are extensively used in the paper. The space $C(a, b; R^n)$ of continuous R^n-valued functions defined on $[a, b]\subset R^1$ is known as Banach space when endowed with norm

$$\|x\|=\sup_{t\in[a,b]}|x(t)|,\quad x\in C. \tag{0.3}$$

A linear bounded functional $f^*\in C^*$ can be characterized by Riesz reprezentation theorem as follows.

$$f^*(x)=\int_a^b x'(t)\,d\bar{f}(t),\ x\in C \tag{0.4}$$

where $\bar{f}$ is a function of bounded variation and is normalized such that it is left-continuous on (a, b) and $\bar{f}(a)=0$. Since $\bar{f}$ has countably many discontinuities, it follows that (0.4) can be rewritten equivalently as

$$f^*(x)=\sum_{i=1}^{\infty} q_i' x(t_i)+\int_a^b x'(t)\,df(t),\qquad \forall x\in C \tag{0.5}$$

where now

$$t_i \in [a, b],\ q_i \in R^n,\ \sum_{i=1}^{\infty} |q_i| < \infty \tag{0.6}$$

and f is continuous of bounded variation, $f(a)=0$.

The space $M^r_{\alpha\beta}(a, b; R^n)$ is a generalization of common $L^r(a, b; R^n)$, $1 \leqslant r \leqslant \infty$. Consider the following functional

$$\|x\|^2 = \alpha^2 |x(a)|^2 + \beta^2 |x(b)|^2 + \left(\int_a^b (|x(t)|^r\, dt \right)^{2/r}. \tag{0.7}$$

$M^r_{\alpha\beta}(a, b; R^n)$, where $\alpha, \beta \geqslant 0$. $1 \leqslant r < \infty$, is the quotient space of all measurable functions x: $[a, b] \to R^n$ for which (0.7) exists and is finite by the linear subspace of elements x for which (0.7) becomes zero.

For $r = \infty$ the similar definition is assumed with (0.7) replaced by

$$\|x\|^2 = \alpha^2 |x(a)|^2 + \beta^2 |x(b)|^2 + (\operatorname*{ess\,sup}_{t \in [a,b]} |x(t)|)^2 \tag{0.8}$$

Clearly $M^r_{00} = L^r$. If $\alpha, \beta > 0$ the space $M^r_{\alpha\beta}(a, b; R^n)$ is topologically isomorphic to $R^n \times R^n \times L^r(a, b; R^n)$. Similarly for $\alpha = 0$, $\beta > 0$ or $\alpha > 0$, $\beta = 0$ the isomorphic space is $R^n \times L^r(a, b; R^n)$. For $r < \infty$ a functional $f^* \in (M^r_{\alpha\beta})^*$ is represented by

$$f^*(x) = \alpha q_1' x(a) + \beta q_2' x(b) + \int_a^b f'(t)\, x(t)\, dt,\ \forall x \in M^r_{\alpha\beta} \tag{0.9}$$

where $q_1, q_2 \in R^n$, $f \in L^{\bar{r}}(a, b; R^n)$, $1/\bar{r} + 1/r = 1$.

Finally we shall use the space $W^r_1(a, b; R^n)$ of absolutely continuous functions x: $[a, b] \to R^n$ with derivative $\dot{x} \in L^r(a, b; R^n)$ and the norm

$$\|x\|^2 = |x(a)|^2 + \|\dot{x}\|^2_{L^r} \tag{0.10}$$

or the topologically equivalent

$$\|x\|^2 = |x(b)|^2 + \|\dot{x}\|^2_{L^2}. \tag{0.11}$$

A functional $f^* \in (W^r_1)^*$, $r < \infty$ has the form

$$f^*(x) = q_1' x(a) + \int_a^b f'(t)\, x(t)\, dt = q_2' x(b) + \int_a^b f_2'(t)\, \dot{x}(t)\, dt \tag{0.12}$$

for all $x \in W^r_1$. Here $q_1 = q_2 \in R^n$, $f_1(t) = f_2(t) + q_2$ is of class $L^{\bar{r}}$, $1/\bar{r} + 1/r = 1$.

The space M^2_{01} was used extensively by Delfour and Mitter [18], Vinter [28] and other authors in various problems of control theory for hereditary systems. The space W^r_1 was used for examining controllability and closedness property of the attainable set for such systems by Banks at al. [3] ($r = 2$) and by Kurcyusz and Olbrot [29].

Finally the notation for the space of n-vector valued functions of bounded variation on $[a, b]$ will be $BV(a, b; R^n)$ or briefly BV.

1. Problem statement and motivation

1.1. Problem statement

The system under consideration is of the form

$$\dot{x}(t)=\int_{-h}^{0}[d_s A(s)]\,x(t+s)+Bu(t)=Lx_t+Bu(t) \text{ for a.a.t.}\geqslant 0 \tag{1.1}$$

where $x(t)\in R^n$, $u(t)\in R^m$, $L{:}C(-h,0;R^n)\to R^n$ is a linear bounded operator acting on the state x_t,

$$x_t=\{x(t+s): s\in[-h,0]\} \tag{1.2}$$

of the system (1.1) and B is $n\times m$ real matrix.

Assumptions on the kernel follow from Riesz representation theorem applied to operator L, i.e. elements of the matrix $A(s)$ are functions of bounded variation. Control u is taken from the class $L^p_{loc}(0,\infty;R^m)$, $p\geqslant 1$. For any continuous initial state x_0 there exists a unique absolutely continuous solution to (1.1) of the form [1], [13], [14],

$$x(t)=K(t)\,x_0+\int_0^t X(t-s)\,Bu(s)\,ds, \tag{1.3}$$

where the fundamental matrix solution is, by definition, the unique solution to

$$\dot{X}(t)=\int_{-h}^{0}[d_s A(s)]\,X(t+s), \text{ a.e. in } t\geqslant 0,\ X(0)=I,\ X(t)=0 \text{ for } t<0 \tag{1.4}$$

and the operator $K(t)$ is of integral form.

REMARK 1.1. One remark is needed of rather technical nature concerning the understanding of differential equation (1.4). For $t\geqslant h$ the integral on right-hand side exists in Riemann—Stieltjes (RS) sense [31] and one may consider the equation satisfied for all $t\geqslant h$. For $t<h$ this integral does not exists if e.g. $A(-t)-A(-t-0)\neq 0$ but since A is of bounded variation the number of jump points of A is countable on $[-h,0]$ so that the absolutely continuous solution to (1.4) can be obtained uniquely [30]. The situation can be made more regular if dividing A into two parts both of bounded variation

$$A(s)=\sum_{i=0}^{\infty}A_i(s+h_i)+\bar{A}(s),\ 0\leqslant h_i\leqslant h, \tag{1.5}$$

where the sum represents jump part and $\bar{A}$ is continuous in s. Then equation (1.4) is equivalent to

$$\dot{X}(t)=\sum_{i=0}^{\infty}A_i\,X(t-h_i)+\int_{-h}^{0}[d_s\bar{A}(s)]\,X(t+s) \text{ for all } t\geqslant 0. \tag{1.6}$$

Similar decomposition may be applied to system equation (1.1). This allows to consider the initial conditions for (1.1) as functions of bounded variation and to understand the remaining integral with continuous kernel in (RS) sense. All these difficulties are solved automatically when introducing Lebesgue—Stieltjes (LS) integral [32], [33] but for our purposes this somewhat more abstract notion is not necessary. The two following attainable sets of system (1.1) are important in control theory. The R^n-reachable set at $T>0$

$$\mathscr{R}(T)=\{x\in R^n\colon x=x(T) \text{ for some } u\in L^p \text{ and } x_0=0\} \tag{1.7}$$

and the set of reachable states on [0, T]

$$\mathscr{A}(T)=\{x(\cdot)\in W_1^p(-h,0;R^n)\colon x(\cdot)=x_T \text{ for some } u\in L^p \text{ and } x_0=0\}, \tag{1.8}$$

the latter being defined for $T\geqslant h$.

Basic definitions of controllability of system (1.1) are given below.

Definition 1.1. System (1.1) is R^n-controllable on $[0,T]$ iff $\mathscr{R}(T)=R^n$. System (1.1) is null function controllable on $[0,T]$ iff for each initial state x_0 there exists $u\in L^p(0,T;R^m)$ such that $x_T=0$.

Introducing a topological space $\mathscr{F}$ of R^n-valued functions defined on $[T-h,T]$ we define naturally $\mathscr{F}$-, $\mathscr{F}$-approximate and $\mathscr{F}$-approximate null controllability.

Definition 1.2. System (1.1) is $\mathscr{F}$-controllable on $[0,T]$ iff the reachable states cover the whole space $\mathscr{F}$ i.e. $\mathscr{A}(T)\supset\mathscr{F}$. System (1.1) is $\mathscr{F}$-approximately controllable iff the closure of the set $\mathscr{A}(T)\cap\mathscr{F}$ equals $\mathscr{F}$ i.e. $\mathscr{A}(T)$ is dense in $\mathscr{F}$. System (1.1) is $\mathscr{F}$-approximately null controllable on $[0,T]$ iff for any neighbourhood $U_{\mathscr{F}}$ of $0\in\mathscr{F}$ the inclusion im $\mathscr{K}(T)\subset\mathscr{A}(T)+U_{\mathscr{F}}$ holds where $(\mathscr{K}(T)x_0)(s)=K(T+s)x_0$ for any x_0 from a given space of initial states.

If the time interval is omitted in the definitions of this section it is understood that there exists a time T such that appropriate controllability property holds on $[0,T]$. Our aim is to examine $\mathscr{F}$-approximate controllability for various $\mathscr{F}$, especially to find checkable algebraic criteria characterizing approximate controllability in terms of system parameters.

We shall not examine conditions for approximate null controllability restricting ourselves to one useful observation following directly from Definition 1.2.

Corollary 1.1. Suppose that $\mathscr{F}$ is such that im $\mathscr{K}(T)\subset\mathscr{F}$. Then for system (1.1) $\mathscr{F}$-approximate controllability on $[0,T]$ implies $\mathscr{F}$-approximate null controllability on $[0,T]$.

We shall show that a necessary condition for approximate controllability is so called (weak) multipoint controllability defined as follows.

Definition 1.3. System (1.1) is (weakly) r-point controllable on $[0,T]$ iff for any points $x^1,\dots,x^r$ from R^n and any instants $t^1,\dots,t^r$ (from $(T-h,T]$) there exist an element x_T of $\mathscr{A}(T)$ such that $x(t^i)=x_T(-T+t^i)=x^i$, $i=1,\dots,r$. System (1.1) is (weakly) multipoint controllable iff it is (weakly) r-point controllable for each integer r.

1.2. Motivation

The following motivation indicates that approximate controllability is a necessary factor when considering regulator design problem for systems with delays. Suppose one considers stabilization problem for system (1.1), that is, starting from nonzero initial state x_0 one has to control the system in order to attain $x_T=0$ or of much smaller norm than $\|x_0\|$ at relatively short time T. However the exact null function controllability conditions are too difficult to use and in fact, too strong [2]. One may demand, of course, $\mathscr{F}$-controllability, where $\mathscr{F}=W_1^p$, but this leads to condition rank $B=n$ (see [3]), which means that the number of controls is equal to the number of state variables which is not the case in most real situations. So the only practical controllability assumption is $\mathscr{F}$-approximate controllability, where the choice of $\mathscr{F}$ may depend on technical requirements imposed on the system. We shall also show that approximate controllability property is generic, i.e. it is satisfied by all systems in the space of parameters with the exception of a set of measure zero.

2. Summary of previous results

The problem of R^n-controllability is well examined (see [4], [5], [6], [7] for results and extensive bibliography) and for stationary case rank conditions of classical Kalman type were obtained. These are especially clear in case of multiple state delays

$$\dot{x}(t)=\sum_{i=0}^{l} A_i\, x(t-h_i)+Bu(t), \qquad t\geqslant 0, \tag{2.1}$$

$$0=h_0<h_1<\ldots<h_l=h.$$

For this type of systems also null function controllability criteria in algebraic testable form were obtained (Olbrot [2]).

Let us report in more details results concerning function space controllability. First notice that if $\mathscr{F}$-controllability is considered then the space $\mathscr{F}$ must be a subspace of $W_1^p(T-h, T; R^n)$ since every state in $\mathscr{A}(T)$ is of this class. As was mentioned above the assumption $\mathscr{F}=W_1^p$ leads to simple but very strong condition as characterization for W_1^p-controllability, namely rank $B=n$ (Banks at al. [3]). There were several attempts to work with spaces different from W_1^p. Korytowski [8] takes $\mathscr{F}=W_i^\infty$ and defines the system (2.1) with $l=1$ function space controllable if there exist a time $T\geqslant h$ and an integer i such that for each sufficiently small $\varepsilon>0$ the set of all attainable restrictions $x(\cdot)|_{[T-h+\varepsilon,\, T-\varepsilon]}$ equals W_i^∞. The results were given in terms of some Laplace transforms and therefore are not suitable for numerical computation. It was proved however that controllability of the pair (A_1, B) is sufficient for function space controllability. The latter coincides with rank $B=n$ if $A_1=0$ (no delays). Similar Laplace transform type results for one delay case and shortened by ε final interval were obtained by Popov [9] who took $\mathscr{F}=C^{(p)}$ and Choudhury

[10] who examined reachability of null function on $[T-h+\varepsilon, T]$. L^2-approximate controllability was considered by Zmood [11]. However no checkable criteria were given except for the case $l=1$, $A_0=0$. This is a rank condition

$$\operatorname{rank}[B, A_1 B, \ldots, A_1^{k-1} B]=n, \tag{2.2}$$

where $k=\text{Entier}\,[T/h]$. (2.2) holds if and only if the system (2.1) with $l=1$, $A_0=0$ has its attainable set $\mathscr{A}(T)$ dense in $L^2(T-h, T; R^n)$.
Pandolfi [12] obtained sufficient algebraic conditions for W_1^2-approximate controllability of system (2.1) on $[0, T]$ with additional requirement $u(t)=0$ on $[T-h, T]$. Delfour and Mitter [18] showed equivalence between M_{01}^2-approximate controllability and positive definiteness of some symetric operator constructed on the basis of abstract state evolution semigroup for (1.1).

3. General necessary conditions for $\mathscr{F}$-approximate controllability

In this section it is shown that by some natural hypothesis on $\mathscr{F}$, satisfied by all commonly used spaces, $\mathscr{F}$-approximate controllability implies multipoint or weak multipoint controllability. Consider a family $\mathscr{F}(a, b; R^n)$ of linear spaces, parametrized by (a, b, n), with elements $f\colon [a, b]\to R^n$ and endowed with a norm $\|\cdot\|_{a,b}$. Let us list some hypotheses on $\mathscr{F}(a, b; R^n)$ to which we shall refer in the sequel.

(H1) Assume the closed intervals $[a_i, b_i]\subset[a, b]$ are disjoint and the functions $f_i\colon [a_i, b_i]\to R^n$ are constant. Then there exists a function $f\in\mathscr{F}(a, b; R^n)$ which coincides with f_i on $[a_i, b_i]$ for prescribed finitely many indexes i.

(H2) If $[a, b]\subset[c, d]$ and $f\in\mathscr{F}(c, d; R^n)$ then the restriction $f|_{[a,b]}\in\mathscr{F}(a, b; R^n)$ and $\|f\|_{a,b}\leqslant\|f\|_{c,d}$ where we omitt the restriction symbol under the norm.

(H3) Let S_t be the shift operator, i.e. $(S_t f)(s)=f(s+t)$. Then $f\in\mathscr{F}(a, b; R^n)$ iff $S_t f\in\mathscr{F}(a-t, b-t; R^n)$ and $\|f\|_{a,b}=\|S_t f\|_{a-t,b-t}$ for any real t.

[H4] $\|f_k\|_{a,b}\to 0$ with $k\to\infty$ and $q\in R^n$ implies that $\|\tilde{q}' f_k\|_{a,b}\to 0$, where $(\tilde{q}' f_k)(t)=q' f_k(t)$ and denotes transposition.

[H5] $\|f_k\|_{a,b}\to 0$ implies $f_k(t)\to 0\in R^n$ for all t in $[a, b]$.

The following theorems take place.

THEOREM 3.1. Let $\mathscr{F}(a, b; R^n)$ satisfy (H1) through (H4). Denote $\mathscr{F}=\mathscr{F}(T-h, T; R^n)$. If the system (1.1) is $\mathscr{F}$-approximately controllable on $[0, T]$ then it is weakly multipoint controllable on $[0, T]$.

Proof. Suppose, on the contrary, that (1.1) is not weakly multipoint controllable on $[0, T]$. Then by Definition 1.3 there exist an integer r and real numbers $t^1, \ldots, t^r$ from $(T-h, T]$ such that not all vectors $x^i\in R^n$ are reachable at t^i simultaneously by one control. Hence the set of all reachable r-tuples of vectors form a proper subspace in R^{nr}. In other words, there is a nonzero row vector $q' \in (R^{nr})'$, $q' = (q_1', \ldots, q_r')$, such that for all controls u

$$q_1' x(t^1) + \ldots + q_r' x(t^r) = 0. \tag{3.1}$$

By stationarity the same equality holds when replacing each t^i by t^i-s for arbitrary $s \in [0, t^0)$, $t^0 = \min(t^1, \ldots, t^r)$. Choose $s^0 > 0$ such that $t^0 - s^0 > T - h$ and the intervals $I_j = [t^j - s^0, t^j]$, $j = 1, \ldots, r$ are disjoint. By hypothesis (H1) one may construct a function $f \in \mathscr{F}$ such that $f(t) = q_j$ for $t \in I_j$.

Suppose now that there exists control sequence $u^{(k)}$, $k = 1, 2, \ldots$ such that corresponding sequence of attainable states x_T^k tends to f in the norm topology, $\|\cdot\| = \|\cdot\|_{T-h, T}$. Denote by g^k a function $[-s^0, 0] \to R^1$ defined as

$$g^k(s) = q_1'\big(x^k(t^1 + s) - f(t^1 + s)\big) + \ldots + q_r'\big(x^k(t^r + s) - f(t^r + s)\big) \tag{3.2}$$

By hypotheses (H2), (H3), (H4) and triangle inequality we get

$$\|g^k\|_{-s^0, 0} \leqslant \|q_1'(x^k - f)\|_{t^1 - s^0, t^1} + \ldots + \|q_r'(x^k - f)\|_{t^r - s^0, t^r}$$

$$\|q_1'(x^k - f)\|_{T-h, T} + \ldots + \|q_r'(x^k - f)\|_{T-h, T}.$$

Since $\|x_T^k - f\| \to 0$ we get by (H4) that the last sum tends to zero and therefore $\|g^k\|_{-s^0, 0} \to 0$. On the other hand, since (3.1) holds with t^i replaced by $t^i + s$, $s \in [-s^0, 0]$, and since, by construction, $f(t) = q_j$ on I_j one obtains that the function g^k is constant, nonzero on $[-s^0, 0]$. In fact, $-g^k(s) = q_1' q_1 + \ldots + q_r' q_r = |q|^2 > 0$ as it is seen from (3.1) and (3.2).Hence $\|g^k\|_{-s^0, 0} \nrightarrow 0$.

The obtained contradiction proves the implication stated in Theorem 3.1.

Theorem 3.2. Let $\mathscr{F}(a, b; R^n)$ satisfy hypotheses (H1) and (H5). If the system (1.1) is $\mathscr{F}$-approximately controllable on $[0, T]$, where $\mathscr{F} = \mathscr{F}(T-h, T; R^n)$, then it is multipoint controllable on $[0, T]$.

Proof. The proof is quite similar to the proof of Theorem 3.1 so we shall only sketch the crucial steps. Suppose the system (1.1) is $\mathscr{F}$-approximately controllable but not multipoint controllable on $[0, T]$. First notice that equation (3.1) holds for some t^i, q_i, $i = 1, \ldots, r$, $(q_1', \ldots, q_r') \neq 0$ and all controls u. Without loss of generality assume $T - h = t^1 < t^2 < \ldots < t^r = T$ and choose a function f in the following way (see (H1)).

$$f(t) = \begin{cases} q_i \text{ on } [t^i - (t^i - t^{i-1})/3,\ t^i + (t^{i+1} - t^i)/3] \cap [T-h, T] \\ \text{arbitrary otherwise.} \end{cases}$$

If x_T^k, $k = 1, 2, \ldots$, is an approximating sequence in $\mathscr{A}(T)$, i.e. $\|x_T^k - f\| \to 0$, then by hypothesis (H5) the corresponding $x^k(t^i)$ satisfy

$$q_i'\big(x^k(t^i) - f(t^i)\big) \to 0 \text{ with } k \to \infty,\ i = 1, \ldots, r. \tag{3.3}$$

On the other hand, by definition of f and by (3.1)

$$\sum_{i=1}^{r} q_1'\big(x^k(t^i) - f(t^i)\big) = -\sum_{i=1}^{r} q_i' q_i = -|q| < 0$$

and does not depend on k. This, clearly, contradicts to (3.3) and completes the proof.

REMARK 3.1. Note that assumptions (H1)—(H4) are not restrictive in applications. They are satisfied by all commonly used function spaces L^r, $C^{(i)}$, W_j^r, $1\leqslant r, j\leqslant\infty$, $0\leqslant i\leqslant\infty$. The most restrictive is the hypothesis (H5) (it is not satisfied by the L^r-type norm). This is also seen by comparing assumptions in Theorem 3.1 and Theorem 3.2. Assuming (H5) we were able to omitt (H2), (H3) and (H4).

GENERALIZATION 3.1. The results of Theorem 3.1 and 3.2 can be immediately generalized, without any change in proofs, to the case of past dependence in control action. We have in mind the nonhomogenous term in eq. (1.1) of the form. $\Sigma B_i\, u(t-h_i)$ or $\int_{-h}^{0} [d_s\, N(s)]\, u(t+s)$. The reason we restrict ourselves to the model (1.1) is that for systems with delays in control the Definition 1.2 of approximate controllability is not adequate of stabilization problem is considered. Then the true state of a system is a set (1.2) plus essential past values of control. Approximate controllability of the true state is a more complex and open problem.

4. Characterization of multipoint controllability

In this section we shall completely characterize multipoint controllability of a system with multiple lumped delays (2.1). For simplicity we shall deal with systems with one delay $h>0$,

$$\dot{x}(t)=A_0\, x(t)+A_1\, x(t-h)+Bu(t),\ t\geqslant 0, \tag{4.1}$$

however the proofs admit immediate generalizations to systems of the form (2.1) and this will be remarked thereafter. The basic result of this section is Theorem 4.1. below.

THEOREM 4.1.

(a). The system (4.1) is weakly multipoint controllable on $[0, T]$ iff it is weakly 1-point controllable on $[0, T]$ i.e. iff it is R^n-controllable on $[0, T_1]$ for each $T_1>T-h$.

(b). The system (4.1) is multipoint controllable on $[0, T]$ iff for each $q_1, q_2\in R^n$ there exists $x_T\in\mathscr{A}(T)$ such that

$$x_T(0)=x(T)=q_1,\ x_T(-h)=x(T-h)=q_2\,.$$

Proof.

(a). Starting the proof as in case of Theorem 3.1 we come to the conclusion that the system (4.1) is not weakly multipoint controllable on $[0, T]$ iff there exist an integer r, a nonzero vector $q'=(q_1', \ldots, q_r')$ and instants $t^1, \ldots, t^r\in(T-h, T)$ such that (3.1) holds for all controls. After substituting $x(t^i)$ from representation formula (1.3) and taking into account that, by definition, $X(t)=0$ for $t<0$ we get from (3.1) an equivalent relation.

$$\sum_{i=1}^{r} q_i' \int_0^T X(t^i-t)\, Bu(t)\, dt=0 \quad \text{for all } u.$$

This is equivalent to the following

$$\sum_{i=1}^{r} q_i' X(t^i - t) B = 0 \text{ for all } t \in [0, T]. \tag{4.2}$$

Recall that for systems of type (2.1) the fundamental matrix solution $X(t)$ is piecewise analytic [4] and for system (4.1) it is analytic on each interval $[(k-1)h, kh]$, $k=1, 2, \ldots$. This and the fact that $X(t)=0$ for $t<0$ enables one to obtain, by unique analytic extension of zero function, the following equivalent characterization of (4.2)

$$q_i' X(t^i - t) B = 0 \quad \forall t \in [0, t^i] \ \forall i = 1, \ldots, r. \tag{4.3}$$

Since $q_i \neq 0$ for at least one index i, then (4.3) is equivalent, by standard argument [4], that the system (4.3) is not R^n-controllable on $[0, t^i]$ for at least one t^i, $T \geqslant t^i > T-h$. This means, by antithesis, that weak multipoint controllability on $[0, T]$ is equivalent to R^n-controllability on each $[0, t^i]$, $t^i > T-h$ (the contraint $t^i \leqslant T$, in view of stationarity, is immaterial).

(b) The proof is analogous as of part (a). We get (4.2) as a characterization of uncontrollability in multipoint sense with only change that now one of the instants t^i may be equal to $T-h$. We distinguish three cases:

(b1) $t^i < T$ for all i. One may proceed to (4.3) and conclude that the system is not R^n-controllable on $[0, t^i]$ for some $T > t^i \geqslant T-h$.

(b2) $t^i > T-h$ for all i. The conclusion as above with $T \geqslant t^i > T-h$.

(b3) $t^1 = T-h$ and $t^r = T$. This leads to the relations

$$\begin{aligned} &q_1' X(T-h-t) B + q_r' X(T-t) B = 0 \quad \forall t \in [0, T], \\ &q_i' X(t^i - t) B = 0 \ \forall t \in [0, t^i], \ \forall i = 2, \ldots, r-1. \end{aligned} \tag{4.4}$$

So multipoint uncontrollability means that either the system is not R^n-controllable on $[0, t^i]$, $t^i \geqslant T-h$ or not all pairs $(x(T-h), x(t))$ are reachable. Since the latter is implied by the first property we get, by antithesis, that multipoint is equivalent to 2-point controllability on $[0, T]$.

Theorem 4.1 enables us to proceed immediately to fully algebraic and computable criteria expressed in terms of parameters A_0, A_1, B.

THEOREM 4.2. Multipoint controllability of system (4.1) on $[0, T]$ is equivalent to each of the following:

(i) The $2n$-dimensional system

$$\begin{aligned} &\dot{x}(t) = A_0 x(t) + A_1 y(t) + Bu(t), \\ &\dot{y}(t) = A_0 y(t) + A_1 y(t-h) + Bu(t-h), \end{aligned} \tag{4.5}$$

is R^{2n}-controllable on $[0, T]$.

(ii) The set of columns of the matrices

$$\begin{bmatrix} \Gamma(i-j+1, j-1) B \\ \Gamma(i-j, j) B \end{bmatrix} \begin{matrix} i=0, 1, \ldots, 2n-1, \\ jh<T, \ j=0, 1, \ldots \end{matrix} \tag{4.6}$$

has rank $2n$ i.e. spans the whole R^{2n}, where the matrices $\Gamma(i,j)$ are defined by recurrence relations

$$\Gamma(i,j)=A_0\,\Gamma(i-1,j)+A_1\,\Gamma(i,j-1) \tag{4.7}$$

with initial values

$$\Gamma(0,0)=I\,\text{(identity)},\ \Gamma(i,j)=0 \text{ if } i<0 \text{ or } j<0. \tag{4.8}$$

Proof.

(i) Setting zero initial conditions $u(t)=0$, $x(t)=0$, $y(t)=0$ for $t\leq 0$ it is easily seen that and y satisfy $y(t)=x(t-h)$ for all $t\geq 0$. Then the proof follows immediately by Theorem 4.1 (b).

(ii) By Theorem 4.1 (b) one may begin with (4.4), where $(q_1', q_r')\neq(0,0)$, as characterization of uncontrollability. Recall that the fundamental matrix has the form [6], [15]

$$X(t)=X_0(t)+X_1(t-h)+\ldots+X_k(t-kh)+\ldots, \tag{4.9}$$

where $X_j(t)=0$ for $t<0$ and

$$X_j(t)=\sum_{i=0}^{\infty}(t^i/i)\,\Gamma(i-j,j) \text{ for } t\geq 0,\ j=0,1,\ldots. \tag{4.10}$$

Substituting this into (4.4) yields

$$[q_1', q_r']\begin{bmatrix}\Gamma(i-j+1,j-1)\,B\\ \Gamma(i-j,j)\,B\end{bmatrix}=0 \text{ for } jh<T,\ i=0,1,\ldots.$$

It remains to prove that in the sequence above we may restrict ourselves to indices $i=0,1,\ldots,2n-1$ but this follows from the generalized Cayley—Hamilton theorem [4], [6] applied to the pair of matrices

$$\begin{bmatrix}A_0 & A_1\\ 0 & A_0\end{bmatrix} \text{ and } \begin{bmatrix}0 & 0\\ 0 & A_1\end{bmatrix}.$$

Hence, since $(q_1', q_r')\neq(0,0)$, we get that for multipoint controllability (ii) is an equivalent characterization.

It is worth to note that (ii) follows also directly from controllability criteria, given in [4], when applied to system (4.5).

GENERALIZATION 4.1. Proceeding to system (2.1) with multiple delays it is easily seen that all arguments for proving Theorem 4.1 remain valid. The only difficulty of rather technical nature is the discovering for what points t^i the relation (4.2) does not imply (4.3) and what we shall obtain instead of that. The detailed analysis of this problem shows that multipoint controllability of (2.1) on $[0,T]$ is equivalent to reachability of arbitrary N-tuples of n-vectors $(x^1,\ldots,x^N)$ where $x^i=x(t^i)$, $t^1=T-h$, $t^N=T$ and for $1<i<N$, $T-h<t^i<T$ and there exist nonnegative integers $j_1,\ldots,j_{l-1}\geq 0$ such that $T-t^i=j_1h_1+\ldots+j_{l-1}h_{l-1}$. Clearly, these conditions define the number N uniquely. Similar conclusion holds for weak multipoint controllability where only $t^1=T-h$ and x^1 are omitted. Let us summarize this in the theorem.

THEOREM 4.3.

(a). System (2.1) is weakly multipoint controllable on $[0, T]$ iff it is weakly $(N-1)$-point controllable where the integer $N-2$ is the number of different reals s^i, $0<s^i<h$, $i=2, \dots, N-1$ for which there exist nonnegative integers $j_1, \dots, j_{l-1} \geqslant 0$ such that $s^i = j_1 h_1 + \dots + j_{l-1} h_{l-1}$. Moreover it suffices to examine weak $(N-1)$-point controllability just for the instants $t^i = T - s^i$, $i=2, \dots, N-1$ and $t^N = T$.

(b). System (2.1) is multipoint controllable on $[0, T]$ iff it is N-point controllable at fixed instants $t^1 = T-h$, t^i specified above for $i=2, \dots, N-1$ and $t^N = T$.

The explicite characterization in the form of Theorem 4.2 is also feasible for system (2.1). We shall not write down appriopriate generalizations of system (4.5) and formula (4.6). However, if needed, it can be easily done by the reader after using shifted by $T-t^i$ system equations (2.1), where t^i are determined as in Theorem 4.3. Thus one gets a generalization of (4.5). To obtain a formula analogous, to (4.6) one may utilize appropriate criteria of [4] applied to generalization of (4.5).

5. Adjoint space characterization of approxi matecontrollability

This section is devoted to characterizing $\mathscr{F}$-approximate controllability with $\mathscr{F}$ being one of the spaces described in paragraph 0.3, that is C, $M^r_{\alpha\beta}$, W^r_1. The characterization will be given in terms of adjoint topological spaces. This enables us to formulate, in subsequent section, equivalent observability problems.

First we state the following general lemma.

LEMMA 5.1. Let $\mathscr{F}$ be a linear normed space of n-vector valued functions defined on $[T-h, T]$. The system (1.1) is $\mathscr{F}$-approximately controllable on $[0, T]$ iff for any $f^* \in \mathscr{F}$ the equality $f^*(x_T)=0$ holding identically for all $x_T \in \mathscr{F} \cap \mathscr{A}(T)$ implies $f^*=0$, in other words, iff the annihilator of $\mathscr{F} \cap \mathscr{A}(T)$ is trivial.

Proof. Suppose $f^*(x_T)=0$ on $\mathscr{F} \cap \mathscr{A}(T)$ for some nonzero $f^* \in \mathscr{F}$. Hence $\mathscr{F} \cap \mathscr{A}(T) \subset \ker f^*$. Since $\ker f^*$ is a closed proper subspace in $\mathscr{F}$ the closure of $\mathscr{F} \cap \mathscr{A}(T)$ is in $\ker f^*$ and is not all of $\mathscr{F}$. This was for necessity. For sufficiency, suppose the closure $\overline{\mathscr{F} \cap \mathscr{A}(T)} = \mathscr{B}$ is a proper (closed) subspace of $\mathscr{F}$. Then by one of the well known corollaries to Hahn-Banach theorem (see e.g. Rudin [24], Theorem 3.5) there exists a functional $f^* \in \mathscr{F}$ such that $f^*(\mathscr{B})=0$ and $f^*(y)=1$ for a prescribed vector $y \in \mathscr{F} \setminus \mathscr{B}$. Thus the proof is complete.

Let us specialize Lemma 5.1 to individual cases of C, $M^r_{\alpha\beta}$ and W^r_1 spaces where $1 \leqslant r < \infty$.

THEOREM 5.1. Given a real r, $1 \leqslant r < \infty$ let $\bar{r}$ satisfy $1 (r+1) r = 1$ and let $f \in L^r(T-h, T; R^n)$. Assume $g\colon [T-h, T] \to R^n$ is left-continuous on $(0, h)$, of bounded variation, $g(T-h)=0$ and let $q_1, q_2 \in R^n$. The $\mathscr{F}$-approximate controllability of system (1.1) on $[0, T]$ is characterized by the following statements corresponding to individual cases.

(i) $\mathscr{F}=C$. The equality

$$\int_{T-h}^{T} [dg'(t)]\, X(t-s)\, B=0 \text{ for a a.s} \in [0, T] \tag{5.1}$$

implies $g=0$. Here $X(t)$ satisfies (1.4).

(ii) $\mathscr{F}=M^r_{\alpha\beta}$. The equality

$$\int_{T-h}^{T} f'(t)\, X(t-s)\, B\, dt+\alpha q_1'\, X(T-h-s)\, B+\beta q_2'\, X(T-s)\, B=0 \quad \text{for a.a.s} \in [0, T] \tag{5.2}$$

implies $f=0$ and $q_1=q_2=0$.

(iii) $\mathscr{F}=W^r_1$. The equality

$$q_1'\, X(T-h-s)\, B+\tilde{f}'(s)\, B+ \int_{T-h}^{T} f'(t)\, X(t-s)\, B\, dt=0 \text{ for a.a.s} \in [0, T] \tag{5.3}$$

or

$$q_1'\, X(T-s)\, B+\tilde{f}'(s)\, B+ \int_{T-h}^{T} f'(t)\, X(t-s)\, B\, dt=0 \text{ for a.a.s} \in [0, T] \tag{5.4}$$

implies $q_1=0$ and $f=0$ where we have put by definition $\tilde{f}(s)=0$ for $s \in [0, T-h)$ and $\tilde{f}(s)=f(s)$ for $s \in [T-h, T]$.

Proof.

(i) Note that $\mathscr{A}(T) \subset C(T-h, T; R^n)$. Given $f^* \in C^*$ the equation $f^*(x_T)=0$ $\forall x_T \in \mathscr{A}(T)$ of Lemma 5.1 may be written, with the aid of representation (0.4) and formulas (1.3), (1.6), as

$$\int_{T-h}^{T} [dg'(t)] \int_{0}^{t} X(t-s)\, Bu(s)\, ds=0 \ \forall u \in L^p \tag{5.5}$$

where g satisfies all assumptions required in the theorem. Since, by definition, $X(t)=0$ for $t<0$ we may put T as upper bound for both integrals in (5.5). By Fubini theorem [33] the order of integration can be changed provided that the suitable integral is understood in Lebesgue—Stieltjes sense or else a decomposition of the function $g(t)$ similar as for $A(s)$ in Remark 1.1 should be done to assure the existence of Riemann—Stieltjes integral. After these manipulation sit obtains.

$$\int_{0}^{T} \left(\int_{T-h}^{T} [dg'(t)]\, X(t-s)\, B \right) u(s)\, ds=0 \ \forall u \in L^p. \tag{5.6}$$

Since $X(t)$ is absolutely continuous in t except the jump $X(0)-X(0-)=I$ at $t=0$ it follows that the map

$$[0, T] \ni s \rightarrow \int_{T-h}^{T} [dg'(t)\, X(t-s)\, B \in L^\infty \subset L^{\bar{p}}$$

where $1/\bar{p}+1/p=1$, $1\leqslant p\leqslant\infty$. This is easily seen after writing separately the terms corresponding to jump part of g and then integrating by parts the remaining integral in order to obtain usual Riemann integral.

Now we can interpret (5.6) as though a linear bounded functional from $(L^p)^*=L^{\bar{p}}$ takes zero value on the whole domain $L^p=L^p(0, T;\ R^m)$. Hence this functional must be trivial i.e. (5.1) holds. Applying Lemma 5.1 we complete the proof.

(ii) Taking representation (0.9) one may utilize the same argument as for part (i).

(iii) It can be assumed without loss of generality that $p\geqslant r$ since otherwise the set $\mathscr{A}(T)\cap W_1^r$ consists of all final states x_T attainable with L^r controls. So it can be assumed $\mathscr{A}(T)\cap W_1^r$. With the use of Lemma 5.1 and representation (0.12) the following equation, analogical to (5.5), is obtained.

$$\int_0^{T-h} q_1' X(T-h-s)Bu(s)\,ds+\int_{T-h}^{0} f'(t)\left(d/dt\int_0^t X(t-s)Bu(s)\,ds\right)dt=0\ \forall u\in L^p,$$

where $f\in L^{\bar{r}}$. After differentation in the second integral using Fubini theorem and other manipulations we arrive at

$$\int_0^T\left(q_1' X(T-h-s)+\tilde{f}'(s)+\int_{T-h}^{T} f'(t)\dot{X}(t-s)\,dt\right)Bu(s)\,ds=0\qquad \forall u\in L^p\qquad (5.7)$$

where $\tilde{f}$ is an extension of f defined by setting $\tilde{f}(s)=0$ for $s\in[0, T-h)$ and $\dot{X}(0)=\dot{X}(0+)$ is put. Hence and from assumption above that $p\geqslant r$ it follows that $\bar{p}\leqslant\bar{r}$ where $1/\bar{r}+1/r=1$, $1/\bar{p}+1/p=1$ and $f\in L^{\bar{r}}(0, T;\ R^n)\subset L^{\bar{p}}(0, T;\ R^n)$. Since $X(T-h-s)$ is absolutely continuous in s except at $s=T-h$ and $\dot{X}(t-s)$ is at least bounded measurable in s the identity (5.7), similarly as for (5.6), is understood as a linear bounded functional vanishing on $L^p(0, T;\ R^m)$. This is clearly equivalent to (5.3). To obtain (5.4) start with the second form of representation (0.12).

All arguments preserve for the case $p=\infty$ since $L^{\bar{p}}=L^1\subset(L^\infty)^*$.

COROLLARY 5.1. (L^r-approximate controllability). System (1.1) is L^r-approximately controllable on $[0, T]$ iff the equality

$$\int_{T-h}^{T} f'(t)X(t-s)B\,dt=0\qquad \forall s\in[0, T]\qquad (5.8)$$

for some $f\in L^{\bar{r}}(T-h, T;\ R^n)$, $1/\bar{r}+1/r=1$, $1\leqslant r<\infty$ implies $f=0$.

Proof. By substituting $\alpha=\beta=0$ in Theorem 5.1 (ii).

COROLLARY 5.2. System (1.1) is $M_{\alpha\beta}^r$-approximately controllable on $[0, T]$, $1\leqslant r<\infty$, iff it is $M_{\gamma\delta}^r$-approximately controllable on $[0, T]$ where $\gamma=\operatorname{sgn}\alpha$, $\delta=\operatorname{sgn}\beta$ and $\operatorname{sgn}\alpha=\alpha/|\alpha|$ for $\alpha\neq 0$, $\operatorname{sgn} 0=0$.

Proof. Follows trivially from Theorem 5.1 (ii).

On the basis of Lemma 5.1 further conclusions concerning interrelations between C-, $M^r_{\alpha\beta}$- and W^r_1-approximate controllability can be drawn. We consider them in t he form of the following diagram of implications.

COROLLARY 5.3. The following implications are valid

$$\begin{array}{ccccc} & & M^{r'}_{10} & & \\ & & \Uparrow & & \\ & & M^{r}_{10} & & \\ & \nearrow & & \searrow & \\ W^r_1 \Rightarrow W^{r'}_1 \Rightarrow C \Rightarrow M^r_{11} & & & & L^r \Rightarrow L^{r'} \\ \Downarrow & \searrow & & \nearrow & \\ M^{r'}_{\alpha\beta} & & M^r_{01} & & \\ & & \Downarrow & & \\ & & M^{r'}_{01} & & \end{array}$$

for any $1 \leqslant r' < r < \infty$ and $\alpha \geqslant 0$, $\beta \geqslant 0$. The symbols of spaces represent here appropriate notions of approximate controllability of system (1.1) on a fixed interval $[0, T]$.

Proof.

$W^{r'}_1 \Rightarrow C$. We, equivalently, prove that of system (1.1) is not approximately controllable on $[0, T]$ in the space C then it is not in $W^{r'}_1$ either. In fact, uncontrollability in C implies by Lemma 5.1 and representation (0.4) that there exists a nonzero n-vector valued function f of bounded variation, left-continuous on $(T-h, T)$, $f(T-h)=0$ and such that

$$\int_{T-h}^{T} x'(t)\, df(t) = 0 \text{ for all } x_T \in \mathscr{A}(T).$$

This integral can be represented, after integration by parts, as

$$f'(T)\,x(T) - f'(T-h)\,x(T-h) - \int_{T-h}^{T} f'(t)\, dx(t) = f'(T)\,x(T) - \int_{T-h}^{T} f'(t)\, x(t)\, dt$$

where either $f(T) \neq 0$ or $f(t)$ is nonzero on some subinterval of $(T-h, T)$. Comparing this with representation (0.12) nad by Lemma 5.1 we get a conclusion that the system (1.1) is not $W^{r'}_1$-approximately controllable (recall that every function of bounded variation is of class $L^\infty \subset (L^r)^*$.

$C \Rightarrow M^r_{11}$. the proof follows by similar argument as above and is based on the following manipulation

$$q_1'\, x(T-h) + q_2'\, x(T) + \int_{T-h}^{T} f'(t)\, x(t) = \int_{T-h}^{T} x'(t)\, dg(t)$$

for any $q_1, q_2 \in R^n$, $f \in L^r(T-h, T; R^n)$ and g which is absolutely continuous on each closed subinterval of $(T-h, T)$ with $\dot{g}(t) = f(t)$ on $(T-h, T)$, $g(T-h+) - g(T-h) = q_1$ and $g(T) - g(T-) = q_2$.

$M^r_{11} \Rightarrow (M^r_{10}$ and $M^r_{01})$, $M^r_{10} \Rightarrow L^r$ and $M^r_{01} \Rightarrow L^r$ follow trivially from Theorem 5.1 (ii).

$W_1^r \Rightarrow W_1^{r'}$ and other implications with $r'<r$ involved follow from the fact that every function of class $L^{\bar{r}'}$ is also of class $L^{\bar{r}}$, where $\bar{r}<\bar{r}'$ satisfy $1/\bar{r}+1/r=1$, $1/\bar{r}'+1/r'=1$. Formal argument as for $W_1^r \Rightarrow C$.

Now proceed to the case $r=\infty$, that is, $M_{\alpha\beta}^{\infty}$- and W_1^{∞}-controllability. It is at the first sight an unexpected result that in one of these function spaces the system (1.1) is never approximately controllable. It is so for the space $M_{\alpha\beta}^{\infty}$.

THEOREM 5.2. System (1.1) is never $M_{\alpha\beta}^{\infty}$-approximately controllable.

Proof. Try to approximate by x_T the following function

$$f(t)=\begin{cases} 0, & t\in[T-h,\, T-h/2],\\ q, & t\in(T-h/2,\, T],\end{cases} \tag{5.9}$$

with q being arbitrary nonzero n-vector. Let $x_T\in\mathscr{A}(T)$. Since $t\to x(t)=x_T(t-T)$ is continuous there exists $s>0$ such that $|x(t)-x(T-h/2)|\leqslant|q|/3$ for $T-h/2-s<t<T-h/2+s$. Hence for $t\in(T-h/2-s,\, T-h/2]$

$$|x(t)-f(t)|=|x(t)-x(T-h/2)+x(T-h/2)|\geqslant|x(T-h/2)|-|q|/3,$$

and similarly for $t\in(T-h/2,\, T-h/2+s)$

$$|x(t)-f(t)|\geqslant|x(T-h/2)-q|-|q|/3.$$

Therefore for the case of $M_{\alpha\beta}^{\infty}$ norm we have

$$\|x_T-f\|\geqslant\max\big(|x(T-h/2)|,\ |x(T-h/2)-q|\big)-|q|/3\geqslant|q|/2-|q|/3=|q|/6.$$

This shows that whatever the state x_T (the control u) is one cannot reach arbitrarily small neighbourhood of f in $M_{\alpha\beta}^{\infty}$ topology.

In case of W_1^{∞} space we do not have such completely negative result although the theorem below shows that one can reach a state x_T arbitrarily close to a given arbitrary function in W_1^{∞} only if all functions in W_1^{∞} can be reached exactly. This is also less than one might have expected.

THEOREM 5.3. System (1.1) is W_1^{∞}-approximately controllable on $[0, T]$ iff it is W_1^{∞}-controllable on $[0, T]$, that is, iff rank $B=n$.

Proof. Choose for approximation a function $g\in W_1^{\infty}$ whis is the unique solution to

$$g(T-h)=0,\ \dot{g}(t)=f(t),\ t\in(T-h,\, T) \tag{5.10}$$

where f is given by (5.9). Let $x^i(t)$, $i=1, 2, \ldots$, be an approximating sequence in W_1^{∞} topology, that is

$$\operatorname*{ess\,sup}_{t\in[T-h,T]}|x^i(t)-\dot{g}(t)|\to 0 \text{ with } i\to\infty. \tag{5.11}$$

Suppose rank $B<n$. Then there exists a nonzero vector $y\in R^n$ such that $y'B=0$. This implies that for any solution $x(t)$

$$y'\,\dot{x}(t)=y'\,Lx_t+y'\,Bu(t)=y'\,Lx_t. \tag{5.12}$$

Substitution of this and (5.10) into (5.11) (premultiplied by y') gives

$$\operatorname*{ess\,sup}_{t\in[T-h,T]} |y' Lx_t^i - y' f(t)| \to 0. \tag{5.13}$$

Observe now that $y' Lx_t$ is continuous in $t \in [T-h, T]$, $T>h$. In fact it is easy to show that for $x \in C(-h, T; R^n)$ the map $t \to x_t \in C(-h, 0; R^n)$ is continuous in sup norm topology for $t \in [0, T]$ (see also Hale [1], Lemma 3.1). Hence, since L is bounded and therefore $|y' Lx_t - y' Lx_s| \leqslant \|y' L\| \, \|x_t - x_s\|$ we get continuity of $y' Lx_t$ in t. This and the additional assumption that the vector q in (5.9) is chosen such that $y' q \neq 0$ enables one to follow the proof of the preceding theorem thus to obtain the final inequality

$$\operatorname*{ess\,sup}_{t\in[T-h,T]} |y' Lx_t^i - y' f(t)| \geqslant |y' q|/6$$

contradicting to (5.13). This proves the necessity of condition rank $B=n$ for W_1^∞-approximate controllability (and clearly for W_1^∞-controllability too). For sufficiency restrict ourselves, without loss of generality, to case $m=n$ (square matrix B). If rank $B=n$ then one can reach exactly any function $g \in W_1^p(T-h, T; R^n) \supset \supset W_1^\infty(T-h, T; R^n)$ with the aid of a control $u \in L^p$. It is true because one can reach any function $x \in W_1^p(0, T; R^n)$ such $x(0)=0$ by setting $u(t)=B^{-1}(\dot{x}(t) - Lx_t)$ and because $T-h>0$.

Remark 5.1. We have shown, by the way, the sufficiency of the condition rank $B=n$ for W_1^r-controllability of system (1.1) on $[0, T]$ provided $r \geqslant p$. It is easy to prove that this condition is also necessary; it follows from the proof of Theorem 5.3 (see also [3], Theorem 3.1). Actually, if rank $B<n$ then (5.12) is valid for some nonzero vector y. Since, as it was shown in the proof above, $y' Lx_t$ is continuous in t (5.12) implies that for any reachable trajectory of (1.1) the projection of the derivative $x(t)$ onto the line through y is always continuous in t while in case of W_1^r-controllability the set of attainable derivatives should contain the class L^r.

Since the case $r=\infty$ has been discussed completely we shall assume in subsequent considerations that $1 \leqslant r < \infty$.

6. Dual observability problems

6.1. Main result

It is a well known relation between controllability of linear system without delay and observability of its dual. For instance the system (4.1) with $A_1=0$ is R^n-controllable if and only if the free-force system $\dot{x}(t)=A_0' x(t)$ with output $y(t)=B' x(t)$ is observable. For systems with delays a very little has been done to clarify the relations between various notions of controllability and observability. Delfour and Mitter [18] stated duality result between R^n-controllability of a system similar to (1.1) and observability of dual system. It has to be pointed out that in

[18] the system is defined to be observable if all initial functions of type $x(0)\neq 0$, $x(t)=0$, $t<0$, can be determined from output measurement. The same definition was used by Gabasov at al. [21] and the observability results obtained for systems with one delay are of the form of rank conditions for a sequence of matrices. It was not stated by the authors but (by matrix transposition) the results of [21] are dual to R^n-controllability criteria of [7]. To the authors knowledge no other duality results using observability definition more suitable for applications were published. Fortunately we are able to extend the results mentioned above and establish duality between approximate controllability of (1.1) and observability of the following dual (transposed) system

$$\dot{x}(t)=\int_{-h}^{0}[d_s A'(s)]\, x(t+s),\quad t\geqslant 0, \tag{6.1}$$

$$y(t)=B'\, x(t). \tag{6.2}$$

We shall use the following definition of observability.

Definition 6.1. Let $\mathscr{G}$ be a given class of initial functions $x_0\colon [-h, 0]\to R^n$ for system (6.1). The observed system (6.1), (6.2) is said to be $\mathscr{G}$-observable on $[0, T]$ iff for each nonzero initial function $x_0\in\mathscr{G}$ the output $y(t)$ does not vanish identically on $[0, T]$.

It should be pointed out that the Definition 6.1 might be inadequate to real problems. For instance, in the system $\dot{x}(t)=A_1 x(t-h)$, $\ker A_1\neq 0$, the initial condition $x_0(t)\in\ker A_1$, $t\in[-h, 0]$, $x_0(0)=0$, is equivalent to zero function is a sense that both yield $x(t)=0$, $t\geqslant 0$. Therefore the notion of "state $\mathscr{G}$-observability" should be understood as the existence of a map $y(\cdot)\to x(\cdot)$ provided that initial conditions are in $\mathscr{G}$ and $y(\cdot)$ and $x(\cdot)$ are defined on $[0, T]$ (see [22]).

Testable algebraic criteria of state L^r- and C-observability for systems with one delay were derived by Olbrot [22] (for extension to the case of output delay see Lee [23]).

Before proving duality result we state:

Lemma 6.1. The fundamental matrix solution $X(t)$ of the free-force ($u=0$) system (1.1) satisfies the following commutation property

$$\int_{-h}^{0}[d_s A(s)]\, X(t+s)=\int_{-h}^{0} X(t+s)\, d_s A(s)\ \text{for a.a. } t\geqslant 0. \tag{6.3}$$

Proof. The proof is based on two fundamental results which can be found in books of Hale [1] or Halanay [30]. First is the equation for fundamental matrix solution to adjoint system

$$Y(t)=I-\int_{t}^{0} Y(s)\, A(t-s)\, ds,\ t\leqslant 0 \ \text{and}\ Y(t)=0,\ t>0 \tag{6.4}$$

where the kernel $A(s)$ has been normalized such that

$$A(s)=0 \ \text{for}\ s\geqslant 0 \ \text{and}\ A(s)=A(-h)\ \text{for}\ s\leqslant -h. \tag{6.5}$$

Second is the property that

$$X(t)=Y(-t) \text{ for all } t. \tag{6.6}$$

Substituting (6.6) into (6.4) and shifting by t the bounds of integration one

$$X(t)=I-\int_0^t X(t-s)A(-s)\,ds,\ t\geq 0.$$

Hence by differentation

$$\dot{X}(t)=-A(-t)-\int_0^t \dot{X}(t-s)\,A(-s)\,ds \text{ a.e. in } t.$$

Applying now integration by parts

$$-\int_0^t \dot{X}(t-s)\,A(-s)\,ds=\int_0^t [d_s X(t-s)]\,A(-s)=X(0)\,A(-t)-$$
$$-X(t)\,A(0)-\int_0^t X(t-s)\,d_s A(-s)$$

obtains and substituting (6.5) yields

$$\dot{X}(t)=-\int_0^h X(t-s)\,d_s A(-s)=\int_{-h}^0 X(t+s)\,d_s A(s).$$

Comparing this with (1.4) we get (6.3).

COROLLARY 6.1. The transpose $X'(t)$ of the fundamental matrix solution to homogenous ($u=0$) part of (1.1) is itself a fundamental matrix solution to the following homogeneous system

$$\dot{x}(t)=\int_{-h}^0 [d_s A'(s)]\,x(t+s) \text{ a.e. in } t\geq 0. \tag{6.7}$$

Proof. Follows trivially by transposing (1.4) and applying Lemma 6.1.

COROLLARY 6.2. The equation (1.4) for $X(t)$ can be rewritten in the form

$$\dot{X}(t)=\int_{-h}^0 X(t+s)\,d_s A(s) \text{ a.e. for } t\geq 0,\ X(0)=I, \tag{6.8}$$

$X(t)=0$ for $t<0$.

Proof. Obvious from Lemma 6.1.

REMARK 6.1. Lemma 6.1 is a generalization of a similar result (Bellman and Cooke [34], Lemma 10.1) given for systems of type (2.1).

Now we are in a position to prove the duality theorem characterizing approximate controllability in function spaces C, $M^r_{\alpha\beta}$ and W^r_1. All the results of Theorem

6.1 below are formulated in terms of dual observed system (6.1), (6.2) or, alternatively, in terms of the following dual controlled-observed system

$$\dot{x}(t)=\int_{-h}^{0}[d_s A'(s)\,x(t+s)+w(t), \quad t\in[0,T], \tag{6.9}$$

$$x(t)=0 \text{ for } t<0, \; w(t)=0 \text{ for } t\in(h,T], \tag{6.10}$$

$$y(t)=B'\,x(t), \quad t\in[0,T] \tag{6.11}$$

where the class to which w belongs depends on what space is considered.

THEOREM 6.1. The following statements are valid.

(i) C-approximate controllability of system (1.1) on $[0, T]$ is equivalent to $\mathscr{G}$-observability of system (6.1), (6.2) on $[0, T-h]$ where $\mathscr{G}$ is the space of all functions of bounded variation and with values in $\ker B'$.

(ii) $M^r_{\alpha\beta}$-approximate controllability of system (1.1) on $[0,T]$ is equivalent to each of the following

(ii)$'$. For system (6.9), (6.10), (6.11) with

$$w(t)=\alpha q_1\,\delta(t-h)+\beta q_2\,\delta(t)+v(t) \tag{6.12}$$

where $\delta(t)$ is Dirac's distribution and v is of class $L^{\bar{r}}$ on $[0, h]$, $1/\bar{r}+1/r=1$, the condition $y(t)=0$ identically on $[0,T]$ implies $\alpha q_1=\beta q_2=0$, $v(t)=0$ a.e. on $[0, h]$. Here the effect of $\alpha q_1\,\delta(t-h)$, $\beta q_2\,\delta(t)$ on system behaviour is understood as jumps of the trajectory $x(0)-x(0-)=\beta q_2$, $x(h)-x(h-)=\alpha q_1$

(ii)$''$. System (6.1), (6.2) is $\mathscr{G}$-observable on $[0, T-h]$ where $\mathscr{G}=\{x\in W^{\bar{r}}_1(-h, 0; \ker B')\colon x(-h)=\beta q_2\}+\{x\colon x(0)=\alpha q_1\in\ker B',\, x(t)=0,\; t\in[-h, 0)\}$.

(iii). W^r_1-approximate controllability of system (1.1) on $[0, T]$ is equivalent to each of the following

(iii)$'$ For system (6.9), (6.10). (6.11) with $w\in L^{\bar{r}}$, $\bar{r}$ as above, the equality

$$y(q,0)(t)+d/dt\,y(0,w)(t)=0 \text{ for a.a. } t\in[0,T] \tag{6.13}$$

implies $q=0$, $w(t)=0$ a.e. where it is denoted by $y(q.\,w)(t)$ the output (6.11) provided that initial condition is $x(0)=q$.

(iii)$''$ The equality

$$y(q,0)(t-h)+d/dt\,y(0,w)(t)=0 \text{ for a.a. } t\in[0,T] \tag{6.14}$$

satisfied in system (6.9), (6.10), (6.11) with $w\in L^{\bar{r}}$ implies $q=0$ and $w(t)=0$ a.e.

(iii)$'''$ $\mathscr{G}$-observability on $[0, T-h]$ of the following dual system

$$\begin{aligned}\dot{x}^1(t)&=\int_{-h}^{0}[d_s A'(s)]\,x^1(t+s),\\ \dot{x}^2(t)&=\int_{-h}^{0}[d_s A'(s)]\,x^2(t+s),\\ y(t)&=B'\big(\dot{x}^1(t)+x^2(t)\big),\end{aligned} \tag{6.15}$$

where $\mathcal{G}=\{(x_0^1, x_0^2)\colon x_0^1 \in \overline{W_1^r}(-h, 0; \ker B'), x_0^1(-h)=0,\ x_0^2(0) \in R^n,\ x_0^2(s)=0$ for $s \in [-h, 0)\}$.

Proof.

(i). For the proof of this and next parts we use Theorem 5.1 as a starting point. Let rewrite equation (5.1) in the following form (after transposition and change of integration variable)

$$B' \int_0^h X'(t-s)\, dv(s)=0 \text{ for a.a. } t \in [0, T]. \tag{6.16}$$

where $v(s)=g(T-s)$. From representation formula (1.3) and Corollary (6.1) it is seen that if v is absolutely continuous ($dv(t)=w(t)\,dt,\ w \in L^1$) then left hand of (6.16) is equal to output $y(t)$ of the system (6.9), (6.10), (6.11) with initial condition $x(0)=0$. In general, however, v is of bounded variation and therefore the function $t \to x(t)$ where

$$x(t)=\int_0^h X'(t-s)\, dv(s),\ t \in [0, T] \tag{6.17}$$

is only of bounded variation on $[0, h]$. Strictly speaking the integral (6.17) is well defined in RS sense for all $t \in [0, T]$ but at most countably many points of $[0, h]$ (compare Remark 1.1), namely, the jump points of v. With the exception of these points (6.17) is equivalent on $(0, h)$ to

$$x(t)=\int_0^t X'(t-s)\, dv(s),\ t \in (0, h). \tag{6.18}$$

Now observe that proving the following steps yields the complete proof for part (i).

(**A**). The operator defined by (6.18) and by equality

$$x(0)=v(0) \tag{6.19}$$

takes $BV(0, h; R^n)$ onto itself.

(**B**). The identity (6.16) taken for $t \in [0, h]$ implies that $x(t)$ defined by (6.18) equals a.e. on $[0, h]$ to a function from $BV(0, h; \ker B')$.

(**C**). Formula (6.17), taken for $t \in [h, T]$, defines a solution to (6.1) on $[h, T]$ with initial condition $x(t)$, $t \in [0, h]$, given by (6.18) and (6.19).

Proof of (A): It is seen that operator (6.18), (6.19) takes $BV(0, h; R^n)$ into itself. Actually, it is more evident after integration by parts of (6.18)

$$x(t)=v(t)-X'(t)\, v(0)+\int_0^t \dot{X}'(t-s)\, v(s)\, ds\,.$$

Suppose now that x is an arbitrary function of bounded variation and define

$$v(t)=x(t)+(1-\delta_{0t})\, x(0)-\int_0^t L' x_s\, ds \tag{6.20}$$

where $\delta_{st}=1$ for $t=s$, $\delta_{st}=0$ otherwise and

$$L' x_s = \int_{-h}^{0} [d_\theta A'(\theta)]\, x(s+\theta). \tag{6.21}$$

Here in (6.21) and in subsequent formulas it is assumed that $x(t)=0$ for $t<0$. The integral (6.21) is understood in LS sense or else Remark 1.1 applies in order to assure that $v(t)$ is well defined for all $t \in [0, h]$.

We want to show that substitution of (6.20) to (6.18) and (6.19) yields an identity. In fact, (6.20) implies (6.19) and substituting

$$dv(s) = dx(s) - d\delta_{0s}\, x(0) - L' x_s\, ds$$

into right-hand side of (6.18) we get for $t \in (0, h)$

$$\int_0^t X'(t-s)\, dx(s) + X'(t)\, x(0) - \int_0^t X'(t-s)\, L' x_s\, ds.$$

After integration by parts of the first term this is equivalent to $x(t)+R(t)$ where

$$R(t) = \int_0^t \dot{X}'(t-s)\, x(s)\, ds - \int_0^t X'(t-s)\, L' x_s\, ds. \tag{6.22}$$

It remains to prove that $R(t)=0$ for all $t \in [0, h]$ and all $x \in BV(0, h; R^n)$ which is itself an interesting property of fundamental matrix solution. Perhaps the simplest proof of this identity is via Laplace transform method. Extending meanwhile the definition of $x(s)$ onto the whole real axis by setting $x(s)=0$ for $s \notin [0, h]$ we get for Laplace transform $\hat{R}(z)$, $\hat{X}(z)$, $\hat{x}(z)$ of appropriate functions the relation.

$$\hat{R}(z) = (z\hat{X}'(z) - I)\, \hat{x}(z) - \hat{X}'(z) \int_{-h}^{0} d_\theta A'(\theta)]\, e^{(\theta z)}\, \hat{x}(z) \tag{6.23}$$

The existence of the above transforms is guaranteed by exponential boundedness and local integrability of $X'(t)$ in $t \geqslant 0$. The columns of $X'(t)$ for $t \geqslant h$ are solutions to (6.1) with continuous initial function on $[0, h)$ and therefore general estimates for such solutions are applicable (Myshkis ([13], Chapter III, Theorem 11). The form of the Laplace transform of the map $t \to Lx_t$ follows by order interchanging in appropriate integrals (see also [13], Chapter III). From transposed version of (6.8) or (1.4) it follows that

$$\hat{X}'(z) = \left(zI - \int_{-h}^{0} e^{\theta z}\, d_\theta A'(\theta) \right)^{-1}. \tag{6.24}$$

Substitution of (6.24) into (6.23) yields $\hat{R}(z)=0$ which means that $R(t)=0$ a.e. in t. By continuity (see (6.22)) we get $R(t)=0$ for all $t \geqslant 0$ which completes the proof of (A).

Proof of (B): Follows trivially from (A).

Proof of (C): Evaluate the derivative of $x(t)$ defined by (6.17).

$$\dot{x}(t)=\int_0^h \dot{X}'(t-s)\,dv(s)=\int_0^h \left[\int_{-h}^{0} d_\theta A'(\theta)\right] X'(t-s+\theta)]\,dv(s),\ t\geqslant h. \quad (6.25)$$

On the other hand after substituting (6.17) into right-hand side of (6.1) we get for $t\geqslant h$

$$x(t)=\int_{-h}^{0} d_\theta A'(\theta)\int_0^h X'(t-s+\theta)\,dv(s)$$

which is equal to right-hand side of (6.25) by Fubini theorem. This proves that (6.17) verifies (6.1) on $[h, T]$ with initial conditions (6.18), (6.19) on $[0, h]$.

(ii). We proceed with the same argument as for part (i), that is, in lieu of (6.16), with the following identity

$$B'\int_0^h X'(t-s)\,v(s)\,ds+\alpha X'(t-h)\,q_1+\beta X'(t)\,q_2=0 \quad (6.26)$$

a.e. in $t\in[0, T]$ which is an equivalent version of (5.2) with $v(s)=f(T-s)$. Considering the function

$$x(t)=\int_0^h X'(t-s)\,v(s)+\alpha X'(t-h)\,q_1+\beta X'(t)\,q_2 \quad (6.27)$$

we check easily that it satisfies (6.1) on $[h, T]$ with initial conditions also defined by (6.27) for $t\in[0, h]$. The set of these initial conditions

$$\{x: x \text{ satisfies (6.27) on } [0, h], v\in L^{\bar{r}}, q_1, q_2\in R^{\bar{n}}\}$$

is equal to

$$\{x: x=\tilde{x}+\delta_{th}\,\alpha q_1, \tilde{x}\in W_1^{\bar{r}}(0, h, R^n), \tilde{x}(0)=\beta q_2;\ q_1\ q_2\in R^n\}. \quad (6.28)$$

To prove this it suffices to show that the map

$$v\to x(t)=\int_0^h X'(t-s)\,v(s)\,ds,\ t\in[0, h], \quad (6.29)$$

takes $L^{\bar{r}}$ onto $\{x\in W_1^{\bar{r}}: x(0)=0\}$ and $t\to\beta X'(t)\,q_2$ is of class $W_1^\infty\subset W_1^{\bar{r}}$. The latter is obvious from (1.4)

$$|\dot{X}(t)|\leqslant\int_{-h}^{0}[d_s|A(s)|]\,|X(t+s)|\leqslant \mathrm{Var}_{[-h,0]}|A(\cdot)|)\,(\sup_{[0,h]}|X(\cdot)|)<\infty$$

for all but countably many points $t\in[0, h]$. The first is verified by direct substitution of

$$v(t)=\dot{x}(t)-L'\,x_t,\ x\in W_1^{\bar{r}}(0, h; R^n),\ x(0)=0$$

where it is understood that $x(t)=0$, $t<0$ in order that Lx_t have sense. After similar manipulations as in part (i) we get that (6.29) takes the value $x(t)+R(t)$, where $R(t)$ is defined by the expression (6.22) and, as it was proved, vanishes identically.

Now, taking (6.26) for $t \in [0, h+\varepsilon]$, $\varepsilon > 0$ arbitrarily small, yields that $x(t) \in \ker B'$ for x defined by (6.27) and hence $\tilde{x}(t) \in \ker B'$, $\alpha q_1 \in \ker B'$ in (6.28). Ba Theorem 5.1, with (6.26) in lieu of (5.2), we obtain part (ii)''. The part (ii)' follows easily from the fact that (6.27) can be considered as a solution to (6.9) (6.10) with w defined by (6.12).

(iii). Utilizing the identity (5.4) of Theorem 5.1 we shall prove the statements (iii)'' and (iii)'''. The proof of (iii)', when starting from (5.3), is similar to that of (iii)''.

As in preceding cases rewrite first the basic relation (5.4) in the equivalent form.

$$B'\left(X'(t-h)\,q + d/dt \int_0^t X'(t-s)\,w(s)\,ds\right) = 0 \text{ a.e. in } t \in [0, T] \qquad (6.30)$$

where $q = q_1 \in R^n$ and $w(t) = f(T-t)$ for $t \in [0, h]$ and $w(t) = 0$ for $t \in (h, T]$. Since $X'(t-h)\,q$, $t \in [0, h]$ represents the solution $x(t-h)$ to (6.9) with $w = 0$ and initial conditions $x(0) = q$, $x(t) = 0$, $t < 0$, (actually, compare (6.1) and transposed (6.8)) and the second term of (6.30) is equal to $y(t)$ where y is defined by (6.9), (6.10), (6.11) with $x(0) = 0$ (compare (1.3)) we conclude that (6.30) and (6.14) are equivalent. By Theorem 5.1 the proof of (iii)'' is complete.

In order to prove (iii)''' note that the set of all solutions to (6.9), (6.10) on $[0, h]$ provided that $x(0) = 0$, $w \in L^{\bar{r}}$, equals to $\{x \in W_1^{\bar{r}}(0, h; R^n) : x(0) = 0\}$ (the proof by direct substitution of suitable chosen w) and that (6.14) taken for $t \in [0, h)$ is equivalent to the property that the values of $x(t)$ above are in $\ker B'$ a.e. Shifting the time we may consider (6.14) on $[-h, T-h)$ and translate easily (iii)'' into equivalent form (iii)'''.

From part (ii) of Theorem 6.1 it follows trivially the dual observability characterization of L^r-approximate controllability ($L^r = M_{00}^r$).

Corollary 6.3. System (1.1) is L^r-approximately controllable on $[0, T]$, $1 \leqslant r < \infty$, iff (a) or equivalently (b) holds.

(a). For system (6.9), (6.10), (6.11) with w of class $L^{\bar{r}}$, 1 ($\bar{r} + 1/r = 1$, and $x(0) = 0$ the condition $y(t) = 0$ identically on $[0, T]$ implies $w(t) = 0$ a.e. on $[0, h]$.

(b). System (6.1), (6.2) is $\mathscr{G}$-observable on $[0, T-h]$ where $\mathscr{G} = \{x \in W_1^{\bar{r}}(-h, 0; \ker B') : x(-h) = 0\}$.

Remark 6.2. In Theorem 6.1 the part (i) corresponds (in style) to part (ii)'' or (iii)'''. The reason we cannot establish an equivalent form of (i) corresponding to, say, (ii)' is that we are not able to perform (6.18) as a solution to (6.9) even if admitting distributions for w. It is possible only in case the continuous part of v in (6.18) is absolutely continuous. In general, however, v may contain nonzero singular part, that is, a nonzero continuous function the derivative of which is zero a.e.

REMARK 6.3. It is possible to formulate the statement (iii)^{IV} as a conclusion from (iii)$'$. Its form is a slight modification of (iii)$'''$ with a modified initial function $x_0^2(t)=X(t+h)\,q$. We do not include this here as it would make the formulation of Theorem 6.1. unnecessarily too long.

Now let us examine briefly the case rank $B=n$ or, equivalently, ker $B'=0$. As it was shown in Theorem (5.3) this condition is necessary and sufficient for W_1^∞-approximate controllability. From Theorem 6.1 it is seen that this is also sufficient for C-, $M_{\alpha\beta}^r$ and W_1^r-approximate controllability.

COROLLARY 6.4. Assume rank $B=n$. Then system (1.1) is approximately controllable on $[0, T]$ in each of the spaces C, $M_{\alpha\beta}^r$, W_1^r, $1\leqslant r<\infty$.

Proof. For the spaces C and $M_{\alpha\beta}^r$ the proof is immediate from (i) and (ii)$''$ of Theorem 6.1. In case of W_1^r observe that if ker $B'=0$ then the space $\mathscr{G}$ in (iii)$'''$ of Theorem 6.1 reduces to the set $\{(x_0^1, x_0^2): x_0^1=0,\ x_0^2(0)\in R^n,\ x_0^2(t)=0,\ t\in[-h, 0)\}$. With such initial conditions the solution to (6.15) is

$$x^1(t)=0,\ x^2(t)=X'(t)\,x_0^2(0),\ t\in[0, T-h].$$

Therefore the condition $y(t)=0$, $t\in[0, T-h]$ in (6.15) implies that $y(0)=B'\,x_0^2(0)=0$ and hence $x_0^2(0)=0$. From Theorem 6.1 (iii) we obtain W_1^r-approximate controllability.

In applications we merely deal with systems for which the condition rank $B=n$ holds. However it is extremely difficult to give less restrictive sufficient condition suitable for checking numerically approximate controllability of general system (1.1). This will be possible for systems with discrete delays (in section 7). We are able to formulate instead a simple necessary condition.

COROLLARY 6.5. Suppose that there exists a nonzero n-vector q such that $q'B=0$ and $q'A(s)=$constant for all $s\in[-h, \bar{s}]$ where $-h<\bar{s}\leqslant 0$. Then system (1.1) is not approximately controllable on $[0, T]$ in each of the spaces C, $M_{\alpha\beta}^r$ and W_1^r.

Proof. The equations $q'B=0$ and $q'A(s)=$constant are, respectively, equivalent to $q\in\ker B'$ and $A'(s)q=$constant. Construct and absolutely continuous nonzero scalar-valued function a: $[-h, 0]\to R^1$ such that $a(s)=0$ for $s\geqslant\bar{s}$ and $s=-h$. Choose an initial function for system (6.1) of the form

$$x(t)=a(t)\,q,\ t\in[-h, 0]. \tag{6.31}$$

From the properties of q it follows that the solution to (6.1) with initial condition (6.31) is $x(t)=0$ and hence $y(t)=B'\,x(t)=0$ for all $t\geqslant 0$. Comparing this with Theorem 6.1 (i), (ii)$''$ and (iii)$'''$ it is seen that system (1.1) is not approximately controllable on $[0, T]$ (in case (iii)$'''$ we set $x^1=x$ and $x_0^2(0)=0$).

Taking the assumptions contrary to those of Corollary 6.5 we get a necessary condition for approximate controllability.

6.2. A sufficient condition for stabilizability

One of the most important implications of the main Theorem 6.1 is the result that M_{01}^r-approximate controllability of system (1.1) on some interval $[0, T]$ is sufficient for its pole assignability and stabilizability.

Let us recal the definitions of these notions.

DEFINITION 6.2. System (1.1) is called stabilizable iff there exists $m \times n$ matrix $F(s)$ with elements of bounded variation in $s \in [-h, 0]$ such that after applying a feedback

$$u(t) = \int_{-h}^{0} [d_s F(s)]\, x(t+s) \tag{6.32}$$

the closed loop system

$$\dot{x}(t) = \int_{-h}^{0} [d_s (A(s) + BF(s))]\, x(t+s) \tag{6.33}$$

is asymptotically stable. System (1.1) is pole assignable iff for any real c there exist a feedback (6.32) such that the closed loop system (6.33) has no eigenvalues in the closed right half-plane $\mathrm{Re}\lambda \geqslant c$.

Clearly, pole assignability implies stabilizability.

Recall also the following result due to Pandolfi [35] and for systems with one delay also discovered by Bhat and Koivo [36].

LEMMA 6.2. (Pandolfi [35]). System (1.1) is stabilizable iff

$$\mathrm{rank} \left[\lambda I - \int_{-h}^{0} e^{\lambda s}\, d_s A(s);\, B \right] = n \tag{6.34}$$

for all complex λ such that $\mathrm{Re}\, \lambda \geqslant 0$. System (1.1) is pole assignable iff (6.34) holds for all complex λ.

Now we are in a position to prove

THEOREM 6.2. If system (1.1) is M_{01}^r-approximately controllable on some interval $[0, T]$ for some $1 \leqslant r < \infty$ then it is pole assignable and hence stabilizable.

Proof. Observe that it is sufficient to check (6.34) for λ being an eigenvalue since otherwise the characteristic matrix of (1.1) (the first n columns of (6.34)) is nonsingular. Suppose that system (1.1) is not pole assignable. Then, by Lemma 6.2, there exist nonzero n-vector q such that $B' q = 0$ and for some eigenvalue λ_0 of (1.1) we have

$$\left[\lambda_0 I - \int_{-h}^{0} e^{\lambda_0 s}\, d_s A'(s) \right] q = 0. \tag{6.35}$$

This implies that $x(t) = e^{\lambda_0 t} q$ is a solution to dual system (6.1) corresponding to nonzero initial function $x(t) = e^{\lambda_0 t} q$, $t \in [-h, 0]$ such that the output (6.2) vanishes identically for $t \geqslant 0$. By Theorem 6.1 (ii)'' system (1.1) is not M^r_{01}-approximately controllable. This completes the proof.

Clearly, approximate controllability in any of spaces W^r_1, C and M^r_{11} is, by Corollary 5.3, also a sufficient condition for pole assignability.

7. Testable algebraic criteria

Although the dual adjoint space or observability characterization can be applied for checking approximate controllability in some simple cases it is extremely difficult to construct on the basis of them a numerical algorithm. In this section we show however that for the class of systems with discrete delays of type (2.1) the Theorem 6.1 can be further developed into a fully geometric-algebraic form where all conditions are numerically checkable in finite dimension. The only additional hypothesis is that the lags h_i, $i=1, \ldots, l$ are commensurable, the condition which is always satisfied for real mathematical models of type (2.1). All the proofs are given, for simplicity, for one-delay system

$$\dot{x}(t)=A_0\, x(t)+A_1\, x(t-h)+Bu(t) \tag{7.1}$$

and the straightforward generalization to more general systems of type (2.1) are indicated.

The dual system corresponding to (7.1) is

$$\dot{x}(t)=A_0'\, x(t)+A_1'\, x(t-h),\ t\geqslant 0 \tag{7.2}$$

$$y(t)=B'\, x(t) \tag{7.3}$$

as a specialization of (6.1), (6.2). The similar specialization of (6.9), (6.10). (6.11) yields

$$\dot{x}(t)=A_0'\, x(t)+A_1'\, x(t-h)+w(t),\ t\in[0, T], \tag{7.4}$$

$$x(t)=0 \text{ for } t<0 \text{ and } w(t)=0 \text{ for } t>h, \tag{7.5}$$

$$y(t)=B'\, x(t),\ t\in[0, T]. \tag{7.6}$$

By transforming these dual systems into nondelayed form (using the method of Olbrot [2], [15]) and then utilizing the concept of controlled invariant of Basile and Marro [25] (see also Wonham and Morse [27] and Wonham [26]) we shall obtain the criteria mentioned above. For that purpose we specialize the Corollary 6.5 to system (7.1) and then quote some preliminary results.

COROLLARY 7.1. If the system (7.1) is approximately controllable on $[0, T]$ in the space C (or $M^r_{\alpha\beta}$ or W^r_1) then

$$\operatorname{rank}[A_1; B]=n. \tag{7.7}$$

Proof. Follows from Corollary 6.5 by the observation that the condition rank $[A_1; B]=n$ is equivalent to the existence of nonzero vector $q\in\ker B'\cap\ker A_1'$.

7.1. A nondelayed equivalent system

Consider the system equation (7.4) and conditions (7.5). Let $(k-1)\,h<T\leqslant kh$, k an integer. If $T<kh$ then assume (7.4) holds $[0, kh]$. The system behaviour for $T<t\leqslant kh$ is immaterial to our problems: we include this interval for symmetry. Define the following transformation.

$$x(ih+s)=z_{i+1}(s),\ i=0, 1, \ldots, k-1,\ s\in[0, h], \tag{7.8}$$

$$z'(s)=[z_1'(s), \ldots, z_k'(s)].$$

It is checked easily that (7.4) holds on $[0, kh]$ with initial conditions and w satisfying (7.5) if and only if z satisfies

$$\dot z(s)=\tilde A z(s)+E_1\, w(s),\ s\in[0, h], \tag{7.9}$$

$$z(0)=z^0+Jz(h), \tag{7.10}$$

where $\tilde A$, E_1 and J are respectively $nk\times nk$, $nk\times n$, and $nk\times nk$ matrices; z^0 is an nk-vector as below

$$\tilde A=\begin{bmatrix} A_0' & 0 & \ldots & 0 \\ A_1' & A_0' & \ldots & 0 \\ \vdots & & & \vdots \\ 0 & \ldots & A_1' & A_0' \end{bmatrix}\quad E_1=\begin{bmatrix} I \\ 0 \\ \vdots \\ 0 \end{bmatrix}\quad J=\begin{bmatrix} 0 & 0 & \ldots & 0 \\ I & 0 & \ldots & 0 \\ \vdots & & & \vdots \\ 0 & \ldots & I & 0 \end{bmatrix} \tag{7.11}$$

$$(z^o)'=[x'(0), 0, \ldots, 0]. \tag{7.12}$$

With the same transformation (7.8) the equation (7.2) with some initial conditions can be represented in the form

$$\dot z(s)=\tilde A z(s)+\tilde A_1\, x_0(s-h),\ s\in[0, h], \tag{7.13}$$

$$z(0)=z^o+Jz(h) \tag{7.14}$$

where $x_0(t)=x(t)$, $t\in[-h, 0]$ is the initial function and

$$A_1'=[A_1, 0, \ldots, 0] \tag{7.15}$$

is $n\times nk$ matrix.

7.2. Controlled invariants

Let us introduce the concept of controlled invariant due to Basile and Marro [25] and also met in Wonham and Morse [27].

DEFINITION 7.1. Given a subspace $S\subset R^n$ and $n\times n$ matrix A the subspace $V\subset R^n$ is called (controlled, generalized) (A, S)-invariant if $AV\subset V+S$. The maximal (A, S)-invariant contained in a given subspace W is denoted by $\mathrm{Mic}(A, S; W)$.

The following algorithms are known to determine $\mathrm{Mic}(A, S; W)$.

ALGORITM 7.1. (Wonham and Morse [27], [26]). Denote $W_0 = W$ and

$$W_i = W_{i-1} \cap A^{-1}(W_{i-1} + S), \ i = 1, 2, \ldots .$$

Then Mic $(A, S; W) = W_j$ where $j = \dim W$ or take W_l if $W_{l+1} = W_l$.

ALGORITM 7.2. (Basile and Marro [25]). Denote $V_0 = W^{\perp}$ and

$$V_i = V_0 + A'(V_{i-1} \cap S^{\perp}), \ i = 1, 2, \ldots .$$

Then Mic $(A, S; W) = V_j^{\perp}$ where $j = \dim W$ or take $V_l^{\perp}$ if $V_{l+1} = V_l$.

We shall need in the sequal a characterization of $(\tilde{A}, \operatorname{im} E_1)$-invariant in terms of feedback properties of the system (7.9). The results of this type are summarized in the following lemma.

LEMMA 7.1..

(i). A subspace $V \subset R^{nk}$ is an $(\tilde{A}, \operatorname{im} E_1)$-invariant iff there exist a constant feedback matrix F such that

$$(\tilde{A} + E_1 F) V \subset V. \tag{7.16}$$

(ii). Given for system (7.9) an initial state $z(0)$ belonging to a subspace W there exists an integrable function $t \to w(t)$ such that $z(t) \in W$ for all $t \in [0, h]$ iff $z(0) \in V = \operatorname{Mic}(\tilde{A}, \operatorname{im} E_1; W)$. Moreover, if $z(0) \in V$ then there exists an analytic, as a function of time, trajectory $z(t) = (\exp t(\tilde{A} + E_1 F))\, z(0) \in V \subset W$ corresponding to analytic "control" $w(t) = Fz(t)$ where F and V satisfy (7.16).

(iii). The set of all reachable $z(h)$ by a trajectory starting from $z(0) = 0$ and such that $z(t) \in V$, $t \in [0, h]$, V-being $(\tilde{A}, \operatorname{im} E_1)$-invariant, is equal to the controllable subspace $\{\tilde{A} + E_1 F | V \cap \operatorname{im} E_1\}$ where F is arbitrary satisfying (7.16). If $0 \neq z(0) \in V$ then $z(h) = (\exp h(\tilde{A} + E_1 F))\, z(0) + \tilde{z}$ for some $\tilde{z}$ from controllable subspace.

The part (i) and (iii) of this lemma were proven in [27], the part (ii) in [25] and the case $z(0) \neq 0$ of (iii) in [2].

We shall also need in the sequel another simple result on controllability of non-delayed systems.

LEMMA 7.2. For system (7.9) there exists an absolutely continuous (of class $W_1^{\bar{r}}$) control w satisfying $w(0) = w(h) = 0$ and such that the corresponding trajectory is nonzero and satisfies $z(0) = z(h) = 0$.

Proof. Assume $\dot{w}(t) = v(t)$ where v is of class $L^{\bar{r}}$. Consider this differential equation together with (7.9) as a system controlled by v and with initial conditions $w(0) = 0$, $z(0) = 0$. Take arbitrary, nonzero on $[0, h/2]$, control v such that the resulting $z(h/2) \neq 0$. Since the pair $z(h/2)$, $w(h/2)$ belongs to controllable subspace it is possible to choose an integrable function $v \in L^{\bar{r}}(h/2, h; R^n)$ such that $z(0) = 0$, $w(0) = 0$.

7.3. Algebraic criteria

The equivalence between (7.4), (7.5) and (7.10) and also between (7.2) and (7.13), (7.14) enables one to obtain, with the help of Lemma 7.1, the algebraic results described in the sequel.

An interesting feature of systems with discrete delays is that C-, M_{11}^{r}- and W_{1}^{r}-approximate controllability are equivalent (M_{11}^{r} can be as well substituted by $M_{\alpha\beta}^{r}$, $\alpha>0$, $\beta>0$ which was shown in Corollary 5.2). We prove appropriate algebraic criterion for the spaces C and M_{11}^{r} first.

THEOREM 7.1. Assume $kh<T\leqslant(k+1)\,h$, k a positive integer. Denote by $\mathcal{N}$ and $\mathcal{M}$ the following subspaces

$$\mathcal{N}=\ker B'\times\ldots\times\ker B'\ (k\text{-times}) \tag{7.17}$$

$$\mathcal{M}=\mathrm{Mic}\,(\tilde{A},\ \mathrm{im}\,\tilde{A}_1 N_1;\mathcal{N}) \tag{7.18}$$

where the matrices $\tilde{A}$, $\tilde{A}_1$ are defined by (7.11) and (7.15) respectively and the columns of N_1 form a basis in $\ker B'$.

(i). If $T=(k+1)\,h$ the necessary and sufficient condition for C- (and equivalently for M_{11}^{r}-) approximate controllability of system (7.1) on $[0, T]$ is that (7.7) and the following conditions hold

$$\mathcal{M}\cap\tilde{A}_1\ker B'=0, \tag{7.19}$$

$$\mathrm{rank}\,J'\,[I-J\exp\big(h(\tilde{A}+\tilde{A}_1 N_1 F)\big)]\,M=\mathrm{rank}\,M \tag{7.20}$$

where we set $M=0$ if $\mathcal{M}=0$ and otherwise the solumns of M form a basis in $\mathcal{M}$ and F is an arbitrary matrix satisfying

$$(\tilde{A}+\tilde{A}_1 N_1 F)\,\mathcal{M}\subset\mathcal{M}\,. \tag{7.21}$$

Moreover, under conditions (7.7) and (7.19) the restriction of F to $\mathcal{M}$ in unique.

(ii). If $kh<T<(k+1)\,h$ then system (7.1) is C- (or equivalently M_{11}^{r}-) approximately controllable on $[0, T]$ iff the conditions (7.7), (7.20) and

$$\tilde{\mathcal{M}}\cap\tilde{A}_1\ker B'=0 \tag{7.22}$$

hold where

$$\tilde{\mathcal{M}}=\mathrm{Mic}\,(A,\ \mathrm{im}\,A_1 N_1;\tilde{\mathcal{N}}) \tag{7.23}$$

and

$$\tilde{\mathcal{N}}=\ker B\times\ldots\times\ker B'\times R^n\subset R^{nk} \tag{7.24}$$

Proof.

(i). By theorem 6.1 (i) and (ii)'' of preceding section we have to characterize observability of system (7.2), (7.3) on $[0, kh]$ the initial conditions for which are supposed to be of bounded variation in case of C space or of class $\bar{W}_{1}^{r}$ on $(-h, 0]$ except at $t=0$ in case M_{11}^{r} and, moreover, the values of $x(t)$ are in $\ker B'$ for all $t\in[-h, 0]$. Passing to equivalent system (7.13), (7.14) we may eliminate the latter constraint by substituting $x_0\,(s-h)=N_1\,v\,(s)$ where $v\,(\cdot)$ is now arbitrary of bounded variation or absolutely continuous with a possible jump at $s=h$ respectively. Therefore (7.13) we rewrite as

$$\dot{z}(s)=Az(s)+A_1 N_1 v(s),\ s\in[0,h]. \tag{7.25}$$

Now, assuming that necessary condition (7.7) holds, the condition for unobservability in the sense of Theorem 6.1 (i) or (ii)'' expressed in terms of system (7.25), (7.14) sounds as

(**A**). There exist a function $N_1 v\colon [0,h]\to\ker B'$ (of class $W_1^{\bar{r}}$ or of bounded variation respectively) and vectors $z(0)$ and z^o in (7.14) such that trajectory of (7.25) is nonzero, satisfies (7.14) and the condition $z(s)\in\mathscr{N}$ for all $s\in[0,h]$. By Lemma 7.1 (ii) a nonzero trajectory $z(s)$ of (7.25) completely belonging to $\mathscr{N}$ exists if and only if $\mathscr{M}\neq 0$. Thus the conditions (7.7) and $\mathscr{M}=0$ are sufficient for (*C*-) M_{11}^r-approximate controllability. It can also be shown that the condtion $\mathscr{M}\cap\tilde{A}_1\ker B'=0$ is a necessary one. In fact, if we suppose the contraty, that is, $\mathscr{M}\cap\tilde{A}_1(\mathrm{Ker}\,B')\neq 0$ then by Lemma 7.1 (iii) the set $\mathscr{P}$ of all vectors $z(t)\in\mathscr{M}\subset\mathscr{N}$, $t>0$, reachable from $z(0)$ by trajectory in $\mathscr{N}$ is a nonzero controllable subspace $\mathscr{P}=\{\tilde{A}+\tilde{A}_1 N_1 F\,|\,\mathscr{M}\cap\cap\tilde{A}_1\ker B'\}\subset\mathscr{M}\subset\mathscr{N}$. Set $v(t)=Fz(t)+N_2 v_2(t)$ where N_2 is such that im $\tilde{A}_1 N_1 N_2=\mathscr{M}\cap\tilde{A}_1\ker B'$. The system (7.25) takes now the form

$$\dot{z}(t)=(\tilde{A}+\tilde{A}_1 N_1 F)\,t(t)+\tilde{A}_1 N_1 N_2 v_2(t).$$

Applying to this system Lemma 7.2 we conclude that there exists an absolutely continuous (of class W_1^r) function v_2 such that the nonzero trajectory of the above system completely belongs to $\mathscr{P}\subset\mathscr{N}$ and satisfies $z(0)=z(h)=0$. Clearly v is also of class $W_1^{\bar{r}}$. Hence, by statement (A) above, the observability in the sense of Theorem 6.1 (both (i) and (ii'')) does not hold. This proves necessity of (7.19).

Finally, to complete the proof, suppose that $\mathscr{M}\neq 0$, (7.7) and (7.19) hold and there exists a nonzero solution $z(t)\in\mathscr{N}$ to (7.25), (7.14). Since we have $\mathscr{P}=0$ the only possibility for that is, by Lemma 7.1 (iii), that $z(0)\in\mathscr{M}$ and that $z(h)\overset{\mathrm{df}}{=}\exp\big(h(\tilde{A}+\tilde{A}_1 N_1 F)\big)\,z(0)$ satisfies (7.14) for some z^o. This can be fulfilled iff $J'\,[I-J\exp h(\tilde{A}+\tilde{A}_1 N_1 F)]\,z(0)=0$ for some nonzero $z(0)\in\mathscr{M}$, which is seen from the fact that a vector $z\in R^{kn}$ is of type (7.12) iff $J'z=0$. Taking the contrary case one obtains (7.20).

The uniqueness of F under conditions (7.7) and (7.19) follows from the fact that if F_1 and F_2 are any two matrices satisfying (7.21) then this and (7.19) implies that

$$\tilde{A}M=MQ-\tilde{A}_1 N_1 F_1 M \text{ and } \tilde{A}M=MQ-\tilde{A}_1 N_1 F_2 M$$

where Q is unique. Hence $\tilde{A}_1 N_1 (F_1-F_2) M=0$. Since $\ker\tilde{A}_1=\ker A_1'$ and from (7.7) $\ker A_1'\cap\ker B'=0$ and im $N_1=\ker B'$ this implies $N_1(F_1-F_2)M=0$. By definition, the columns of N_1 are linearly independent. Therefore $F_1 M=F_2 M$.

(ii) Similarly as above the necessary and sufficient condition for that system (7.1) not to be approximately controllable on $[0,T]$ can be expressed in terms of system (7.25), (7.14) and has the form (provided (7.7) holds).

(B). There exist a function v in (7.25) of suitable class and vectors $z(0)$ and z^o such that the trajectory of (7.25) is nonzero, satisfies (7.14) and the conditions $z(s) \in \mathcal{N}$ for $s \in [0, \bar{s}]$ and $z(s) \in \tilde{\mathcal{N}}$ for $s \in [\bar{s}, h]$ where $(k+1)h - T = \bar{s}$.

Assuming (7.7) similarly as in part (i) we show that $\tilde{\mathcal{M}} = 0$ is sufficient and (7.22) is a necessary condition. Note that $\mathcal{N} \supset \tilde{\mathcal{N}}$ and therefore $\tilde{\mathcal{M}} \supset \mathcal{M}$. Thus (7.22) implies (7.19). Suppose now that $\tilde{\mathcal{M}} \neq 0$ and that (7.22) (and (7.19)) holds. If so one may construct a matrix F such that both (7.21) and

$$(\tilde{A} + \tilde{A}_1 N_1 F)\tilde{\mathcal{M}} \subset \tilde{\mathcal{M}} \tag{7.26}$$

are satisfied. In fact, by Definition 7.1. of controlled invariant

$$\tilde{A}\mathcal{M} \subset \mathcal{M} + \operatorname{im} \tilde{A}_1 N_1 \text{ and } \tilde{A}\tilde{\mathcal{M}} \subset \tilde{\mathcal{M}} + \operatorname{im} \tilde{A}_1 N_1 \tag{7.27}$$

where the right-hand sides are, by (7.19) and (7.22), the direct sums. Choose a basis $m_1, \ldots, m_{\tilde{p}}$ for $\tilde{\mathcal{M}}$ such that $m_1, \ldots, m_p$, $p \leqslant \tilde{p}$ is a basis for $\mathcal{M}$. The unique decomposition of $\tilde{A}m_i$ gives

$$\tilde{A}m_i = Mc_i + d_i,\ i = 1, \ldots, p \text{ and } \tilde{A}m_i = \tilde{\mathcal{M}}c_i + d_i,\ i = p+1, \ldots, \tilde{p}$$

where M and $\tilde{\mathcal{M}}$ are $nk \times p$ and $nk \times \tilde{p}$ matrices for which m_i constitutes i-th column and $d_i \in \operatorname{im} \tilde{A}_1 N_1$. A matrix F satisfies (7.27) is and only if for $i = 1, \ldots, p$

$$Fm_i = v_i \text{ for some } v_i \text{ such that } \tilde{A}_1 N_1 v_1 = -d_i. \tag{7.28}$$

Since the vectors m_i are linearly independent these equations define a matrix F with properties required. As before, it can be shown that FM is unique if (7.7) and (7.22) hold. Furthermore, we conclude as in part (i) that a trajectory $z(t)$ lying in $\mathcal{M}$ for $t \in [0, \bar{s}]$ and in $\tilde{\mathcal{M}}$ for $t \in [\bar{s}, h]$ is nonzero if and only if $z(t) = \exp\big(t(\tilde{A} + \tilde{A}_1 N_1 F)\big) z(0)$ for some nonzero $z(0) \in \mathcal{M}$ and (7.14) holds for some z^o. In fact, for the interval $[0, s]$ it is the same argument as before and for $[\bar{s}, h]$ the trajectory, by Lemma 7.1 (iii), has to have the form $z(t) = \exp\big((t-\bar{s})(\tilde{A} + \tilde{A}_1 N_1 F)\big) z(\bar{s})$ where F satisfies the relation (7.26) and, as was shown above, it may satisfy (7.21). Thus given $z(0) \in \mathcal{M}$ we have unique trajectory $z(t)$ completely belonging to $\mathcal{N}$. Substituting the final state $z(h)$ into (7.14) we get again (7.20) as a necessary and sufficient condition of approximate controllability provided that (7.7) and (7.22) hold. Thus the proof of Theorem 7.1 is complete.

For the case of M_{10}^r space the conditions for approximate controllability are weaker than those of Theorem 7.1. This is due to the fact that in the M_{10}^r topology the sequence of final states x_T^i, $i = 1, 2, \ldots$, approximating a given function x on $[T-h, T]$ may have the property that $|x^i(T) - x(T)|$ is not necessarily convergent to zero.

THEOREM 7.2. Assume $kh < T \leqslant (k+1)h$ and denote

$$\mathcal{M}_{\tilde{A}} = \mathcal{M} \cap \tilde{A}^{-1} \mathcal{M} \tag{7.29}$$

where $\mathcal{M}$ is defined by (7.18) and (7.17).

(i) $T=(k+1)h$. System (7.1) is M^r_{10}-approximately controllable on $[0,(k+1)h]$ iff the conditions (7.7). (7.19) and

$$\operatorname{rank} J'[I-J\exp(h(\tilde{A}+\tilde{A}_1 N_1 F))]\, M_{\tilde{A}}=\operatorname{rank} M_{\tilde{A}} \tag{7.30}$$

hold where columns of $M_{\tilde{A}}$ are basis vectors for $\mathcal{M}_{\tilde{A}}$ ($M_{\tilde{A}}=0$ if $\mathcal{M}_{\tilde{A}}=0$).

(ii) $kh<T<(k+1)h$. System (7.1) is M^r_{10}-approximately controllable on $[0,T]$ iff the conditions (7.7), (7.22) and (7.30) hold.

Proof. The proof of necessity of (7.19) in part (i) and of (7.22) in part (ii) is identical as in Theorem 7.1. Assuming (7.7) and, respectively, (7.19) or (7.22), hold we show as before that the trajectory of (7.25) completely belonging to $\mathcal{N}$ has to have the form

$$z(t)=\exp(t(\tilde{A}+\tilde{A}_1 N_1 F))\, z(0),\; z(0)\in\mathcal{M}. \tag{7.31}$$

Hence the only case that system (7.2), (7.3) is not $\mathcal{G}$-observable in sense of Theorem 6.1 (ii)'' is that the trajectory (7.31) satisfies (7.14) for some z^o and also (since $\beta=0$ and elements of $\mathcal{G}$ satisfy $x(-h)=0$) that the "control" v in (7.25) vanishes at $s=0$ i.e. $v(0)=0$. Since v is of feedback form $v(s)=Fz(s)$ (compare (7.31) and (7.25)) condition $v(0)=0$ is equivalent to

$$Fz(0)=0,\; z(0)\in\mathcal{M}. \tag{7.32}$$

From the relations (7.21) and (7.26) satisfied by F it is easily seen that (7.32) implies

$$z(0)\in\mathcal{M}_A \tag{7.33}$$

where $\mathcal{M}_{\tilde{A}}$ is defined by (7.29). Since FM is unique (7.33) is the only additional restriction for $z(0)$ when comparing with the proof of Theorem 7.1. Hence sufficiency of conditions (7.7) and $\mathcal{M}_{\tilde{A}}=0$ is clear. It is also clear by the same argument as in suitable part of the proof of Theroem 7.1 that (7.30) is equivalent to approximate controllability provided the other conditions of Theorem 7.2 hold.

We complete the considerations of this paragraph with two theorems which represent necessary and sufficient conditions for the attainable subspace $\mathcal{A}(T)$ to be dense in M^r_{01} and $L^r=M^r_{00}$ respectively.

THEOREM 7.3.

(i) $T=(k+1)h$. System (7.1) is M^r_{01}-approximately controllable on $[0,(k+1)h]$ iff the conditions (7.7), (7.19) and

$$\operatorname{rank}[I-(J+E_F)\exp(h(\tilde{A}+\tilde{A}_1 N_1 F))]\, M=\operatorname{rank} M \tag{7.34}$$

hold where the $nk\times nk$ matrix E_F is of the form

$$E'_F=[F' N'_1, 0, \ldots, 0] \tag{7.35}$$

and the other terms are as in Theorem 7.1.

(ii) $kh<T<(k+1)h$. System (7.1) is M^r_{01}-approximately controllable on $[0,T]$ iff (7.7), (7.22) and (7.34) are satisfied.

Proof. For necessity of (7.19) or (7.22) the argument of Theorem 7.1 applies. Assuming (7.7) and respectively, (7.19) or (7.22) we prove that (7.34) is equivalent to the condition that null function is the only trajectory of (7.25) satisfying (7.14), the condition $N_1 v(h)=x(0)=z_1(0)$ and $z(t)\in\mathcal{N}$ on $[0,h]$ (or $z(t)\in\mathcal{N}$ on $[0,(k+1)h-T]$ and $z(t)\in\tilde{\mathcal{N}}$ on $[(k+1)h-T,h]$) and this, in view of transformation (7.8) and Theorem 6.1 (ii)'', is equivalent to M^r_{01}-approximate controllability. In fact the contrary case to (7.34) is equivalent to the condition that there exists a nonzero vector $z(0)\in\mathcal{M}$ such that

$$[I-(J+E_F)\exp\big(h(\tilde{A}+\tilde{A}_1 N_1)\big)]\,z(0)=0.$$

This means that condition (7.14) is satisfied with

$$z(h)=\exp\big(h(\tilde{A}+\tilde{A}_1 N_1 F)\big)\,z(0)$$

and

$$z^o=E_F\,z(h) \tag{7.36}$$

corresponding to a nonzero trajectory of type (7.31) generating by a feedback $v(t)=Fz(t)$. The condition (7.36) is equivalent to equality $x(0)=z_1(0)=N_1 v(h)$.

Theorem 7.4.

(i) $T=(k+1)h$. System (7.1) is L^r-approximately controllable on $[0,(k+1)h]$ iff the condition (7.7), (7.19) and

$$\operatorname{rank}[I-(J+E_F)\exp\big(h(\tilde{A}+\tilde{A}_1 N_1 F)\big)]\,M_{\tilde{A}}=\operatorname{rank} M_{\tilde{A}} \tag{7.37}$$

are satisfied.

(ii) $kh<T<(k+1)h$. System (7.1) is L^r-approximately controllable on $[0,T]$ iff (7.7), (7.22) and (7.37) are satisfied.

All the terms in the appropriate formulas are as defined before.

Proof. Follows immediately by combining the proofs of Theorem 7.2 and Theorem 7.3.

We complete the considerations of this paragraph proving the equivalence between W^r_1-, M^r_{11}- and C-approximate controllability. We shall use Theorem (iii)''' (6.1) to which there corresponds a dual system

$$\begin{aligned}
\dot{x}_1(t)&=A_0'\,x^1(t)+A_1'\,x^1(t-h),\\
\dot{x}_2(t)&=A_0'\,x^2(t)+A_1'\,x^2(t-h),\\
y(t)&=B'\big(A_0'\,x^1(t)+A_1'\,x^1(t-h)+x^2(t)\big).
\end{aligned} \tag{7.38}$$

The W^r_1-approximate controllability of system (7.1) on $[0,T]$ is equivalent, by Theorem 6.1 (iii)''', to the following property of system (7.38):

(C_1). The condition $y(t)=0$ for all $t\in[0,T-h]$ provided that initial conditions satisfy $\dot{x}^1(t)=N_1 v(t)$, $t\in[-h,0]$, $x^1(-h)=0$, $x^2(t)=0$ on $[-h,0)$, $x^2(0)=q$ where $v\in\bar{L}^r$ and the columns of N_1 are basis vectors in ker B', implies that $v=0$ and $q=0$.

In order to obtain algebraic criteria in terms of maximal controlled invariants it is helpful to introduce a new equivalent system without delay corresponding to system (7.38), Firstly, by defining the following matrices

$$G_0=\begin{bmatrix} A_0' & 0 \\ 0 & A_0' \end{bmatrix},\quad G_1=\begin{bmatrix} A_1' & 0 \\ 0 & A_1' \end{bmatrix},\quad H_1=\begin{bmatrix} N_1 \\ 0 \end{bmatrix}, \tag{7.39}$$

$$C_0=[B'\, A_0';\, B'],\quad C_1=[B'\, A_1';\, 0], \tag{7.40}$$

the system (7.38) can be written as

$$\dot{x}(t)=G_0\, x(t)+G_1\, x(t-h), \tag{7.41}$$

$$y(t)=C_0\, x(t)+C_1\, x(t-h), \tag{7.42}$$

where $x'(t)$ stands for $(x^{1\prime}(t), x^{2\prime}(t))$ and, in connection with statement (C_1) above, the initial conditions for (7.41) are

$$\dot{x}(t)=H_1\, v(t) \text{ on } [-h, 0),\ x(-h)=0,\ x(0)-x(0-)=(0, q')' \tag{7.43}$$

Consider now (7.41) on $[0, T-h]$, $(k-1)\,h<T\leqslant kh$, where k is a positive integer. Setting $x(ih+s)=z_{i+2}(s)$, $=-1, 0, 1, \ldots, k-2$, $z'(s)=[z_1'(s), \ldots, z_k'(s)]$ we write (7.41) with continuity conditions $x(ih+)=x(ih-)$, $i=1, 2, \ldots, k-2$ and initial conditions (7.43) as

$$\dot{z}(s)=Gz(s)+Hv(s),\ s\in[0, h], \tag{7.44}$$

$$z(0)=Jz(h)+E_4\, q, \tag{7.45}$$

where G, H and E_4 are $2nk\times 2nk$, $2nk\times \operatorname{rank} N_1$ and $2nk\times n$ respectively and of the form

$$G=\begin{bmatrix} 0 & 0 & \ldots 0 \\ G_1 & G_0 & \ldots 0 \\ 0 & & 0 \\ \vdots & & \vdots \\ 0 & \ldots G_1 & G_0 \end{bmatrix},\quad H=\begin{bmatrix} H_1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix},\quad E_4=\begin{bmatrix} 0 \\ 0 \\ 0 \\ I \\ \vdots \\ 0 \end{bmatrix} \tag{7.46}$$

and J is now $2nk\times 2nk$ but still of structure (7.11).

The condition $y(t)=0$, $t\in[0, T-h]$ in terms of system (7.44) takes the form

$$z(s)\in\ker C,\ s\in[0, h] \text{ if } T=kh, \tag{7.47}$$

$$z(s)\in\ker C,\ s\in[0, \bar{s}[\text{ and } z(s)\in\ker\tilde{C},\ s\in[\bar{s}, h]$$

for

$$T=(k-1)\,h+\bar{s} \tag{7.48}$$

where both C and $\tilde{C}$ are $mk\times 2nk$ as below

$$C=\begin{bmatrix} 0 & 0 & \ldots 0 \\ C_1 & C_0 & \ldots 0 \\ \vdots & & \vdots \\ 0 & \ldots C_1 & C_0 \end{bmatrix},\quad \tilde{C}=\begin{bmatrix} 0 & 0 & \ldots 0 \\ C_1 & C_0 & \ldots 0 \\ \vdots & & \vdots \\ 0\ldots C_1 & C_0 & 0 \\ 0\ldots 0 & 0 & 0 \end{bmatrix}. \tag{7.49}$$

With this new notation the statement (C_1) can be reformulated as

(C_2). The system (7.1) is W_1^r-approximately controllable on $[0, T]$ iff for system (7.44), (7.46) with some $N_1 v \in \bar{L}^r(0, h; \ker B')$, $q \in R^n$ the condition (7.47) for $T=kh$ (or (7.48) for $T<kh$) implies $z(s)=0$ for all $s \in [0, h]$ (which is equivalent to $v=0$ and $q=0$). This formulation is fully analogical to the contrary case of statement (A) in the proof of Theorem 7.1 (i). By utilizing the concept of maximal controlled invariant and the arguments similar as for Theorem 7.1 we arrive at

THEOREM 7.5. The W_1^r-, M_{11}^r- and C-approximate controllability of system (7.1) on $[0, T]$ are equivalent.

Proof. In view of Theorem 7.1 and Corollary 5.3 it is sufficient to show that if system (7.1) is not W_1^r- then it is not M_{11}^r-controllable. In order to obtain this result we prove firstly the following

LEMMA 7.3. If system (7.1) is not W_1^r-approximately controllable on $[0, T]$, that is, if there exist a nonzero pair $(q, v) \in R^n \times \bar{L}^r$ such that the solution $z(t)$ to (7.44), (7.45) is nonzero and satisfies (7.47) (or (7.48) respectively) then there exist a function $v_1 \in \bar{W}_1^r$ such that the pair (q, v_1) posses the same properties.

Proof of Lemma 7.3. For the trivial case $q \neq 0$, $v=0$ the lemma is evidently valid so assume $v \neq 0$. Denote $\mathcal{N}_W = \ker C$, $\tilde{\mathcal{N}}_W = \ker \tilde{C}$, $\mathcal{M}_W = \mathrm{Mic}(G, \mathrm{im}\, H; \mathcal{N}_W)$, $\tilde{\mathcal{M}}_W = \mathrm{Mic}(G, \mathrm{im}\, H; \tilde{\mathcal{N}}_W)$. Let F be arbitrary matrix satisfying

$$(G+HF)\mathcal{M}_W \subset \mathcal{M}_W \text{ and } (G+HF)\tilde{\mathcal{M}}_W \subset \tilde{\mathcal{M}}_W. \tag{7.50}$$

If $\mathcal{M}_W=0$ for $T=kh$ or respectively $\tilde{\mathcal{M}}_W=0$ for $T \neq kh$ (clearly the latter implies the first since $\mathcal{N}_W \subset \tilde{\mathcal{N}}_W$) then the only trajectory satisfying (7.44) and (7.47) (resp. (7.48)) is $z(t)=0$ corresponding to $v=0$. Therefore suppose now $\mathcal{M}_W \neq 0$ (resp $\tilde{\mathcal{M}}_W \neq 0$).

If $\mathcal{M}_W \cap \mathrm{im}\, H \neq 0$ (resp. $\tilde{\mathcal{M}}_W \cap \mathrm{im}\, H \neq 0$) then apply Lemma 7.2 to the following system

$$\dot{z}(t) = (G+HF)\, z(t) + HD\bar{v}(t) \tag{7.51}$$

which is a result of substitution of $v(t) = Fz(t) + D\bar{v}(t)$ into (7.44) where $\mathrm{im}\, HD = \mathcal{M}_W \cap \mathrm{im}\, H$ (resp. $\tilde{\mathcal{M}}_W \cap \mathrm{im}\, H$) and F satisfy (7.50). The existence of F is proved analogically as for inclusions (7.27). As a conclusion from Lemma 7.2 we get the existence of a nonzero function $\bar{v}_1 \in \bar{W}_1^r$ such that the corresponding nonzero solution to (7.51) satisfies $z(0)=z(h)=0$ (resp. $z(t)=0$ for $t \in [0, \bar{s}]$ and $t=h$). Hence the corresponding $v_1(t) = Fz(t) + D\bar{v}_1(t)$ is nonzero and of class $\bar{W}_1^r$ and the corresponding trajectory $z(t)$ satisfies (7.44), (7.45) and (7.47) (resp. (7.48)). If $\mathcal{M}_W \cap \mathrm{im}\, H = 0$ (resp. $\tilde{\mathcal{M}}_W \cap \mathrm{im}\, H = 0$) then we are able to prove, as in appropriate part of the proof of Theorem 7.1, that the only trajectory satisfying conditions required in Lemma 7.3 has to have the form

$$z(t) = \exp\big(t(G+HF)\big)\, z(0)$$

for some nonzero $z(0) \in \mathcal{M}_W$ and is attainable with the control $v(t) = Fz(t)$ analytic as a function of t and therefore of class $W_1^{\bar{r}}$. Thus in any case Lemma 7.3 is valid.

Now, continuig the proof of Theorem 7.5, recall what was the meaning of function $N_1 v$ in preceding sections. It corresponds, in case of uncontrollability, to a functional $(q, f) \in R^n \times L^{\bar{r}}(T-h, T; R^n) = (W_1^r)^*$ being an annihilator of the attainable subspace $\mathcal{A}(T)$, namely $N_1 v(t) = f(T-t)$, $t \in [0, h]$. So Lemma 7.3 claims that if there exists an annihilator $(0,0) \neq (q, f) \in R^n \times L^{\bar{r}}$ then there exists $f_1 \in W_1^{\bar{r}}$ such that (q, f_1) is a nonzero annihilator of $\mathcal{A}(T)$ as well. Then the fact that system (7.1) is not W_1^r-approximately controllable on $[0, T]$ implies, by Lemma 5.1 and the above conclusion, that for some nonzero $(q, f), f \in W_1^{\bar{r}}$, $1/\bar{r} + 1/r = 1$, and for all attainable trajectories of (7.1)

$$q' x(T) + \int_{T-h}^{T} f'(t)\, x(t)\, dt = 0 .$$

Integrating by parts yields

$$(q' + f'(T))\, x(T) - f'(T-h)\, x(T-h) - \int_{T-\eta}^{T} f'(t)\, \dot{x}(t)\, dt = 0$$

for all $x_T \in \mathcal{A}(T)$. This means that a nonzero functional from $(M_{11}^r)^*$ takes zero value on $\mathcal{A}(T)$ or, equivalently, by Lemma 5.1 system (7.1) is not M_{11}^r-approximately controllable on $[0, T]$. This completes the proof.

7.4. Remarks, corollaries and interrelations

REMARK 7.1. In the considerations of this section the dual system (7.2), (7.3) for C and $M_{\alpha\beta}^r$ spaces has been basically exploited. An alternative way is to utilize (7.4), (7.5), (7.6) and respectively, the specialization of (6.17). This leads to equivalent formulations for all theorems of this section the only disadvantage of which is that the larger dimension $n(k+1)$ in lieu of nk would appear for appropriate vectors and matrices.

REMARK 7.2. It is seen easily that all the criteria developed in this section are checkable numerically. First, the maximal invariants $\mathcal{M}$, $\tilde{\mathcal{M}}$ can be represented by their basis matrices which can be completed by a matrix version of Algorithm 7.1 or 7.2. The rule for converting these algorithms into matrix form is the following: Given two subspaces $\mathcal{U}_1, \mathcal{U}_2 \subset R^n$ and their respective basis matrices U_1, U_2 the basis matrix for theirs intersection is constructed accordingly to the expression $\mathcal{U}_1 \cap \mathcal{U}_2 = (\mathcal{U}_1^\perp + \mathcal{U}_2^\perp)^\perp$; that is by constructing bases for orthogonal complements e.g. with the help of pseudoinverse [19] or by computing the bases for ker U_1' and ker U_2' [37], then eliminating some columns to form a basis for $\mathcal{U}_1 + \mathcal{U}_2$ and then again via orthogonal complement. Second, the conditions which were not stated in matrix rank form (for compactness of notation) can be, into such form, converted. E.g. for (7.19) we have

$$\mathcal{M}^{\perp}+(\tilde{A}_1' \ker B')^{\perp}=R^n \text{ or } \operatorname{rank}[M^{\perp}; (\tilde{A}_1 N_1)^{\perp}]=n$$

as equivalent characterizations where by $M^{\perp}$ we mean the basis matrix for $\mathcal{M}$. Third, the matrix F need not be computed in applications although it appears explicitly in the criteria. This is due to the fact that, for instance, in the Theorem 7.1 the conditions (7.19) and (7.21) imply that $(\tilde{A}+\tilde{A}_1 N_1 F) M=MQ$ where Q is uniquely determined and to be computed. Hence

$$\exp(h(\tilde{A}+\tilde{A}_1 N_1 F)) M=M\exp(hQ). \tag{7.54}$$

Despite testability the computations of maximal controlled invariants might sometimes by cumbersome especially for large n and large ratio T/h. Therefore the simplification of the criteria presented here becomes an important problem from the point of view of applications. We are able to simplify them only by sacrificing the completeness of characterization.

Corollary 7.2. Write the following conditions:

(a) rank $B=n$;
(b) $\tilde{A}\mathcal{N}\cap(\mathcal{N}+\tilde{A}_1 \ker B')=0$, (7.7) and $\mathcal{N}\cap\ker\tilde{A}=0$;
(c) $\mathcal{M}=0$ and (7.7);
(d) system (7.1) is approximately controllable on $[0, T]$, $T\geqslant(k+1)h$, for each of the spaces W_1^r, C, $M_{\alpha\beta}^r$;

where all the terms are as in Theorem 7.1. Then (a)$\Rightarrow$(b)$\Rightarrow$(c)$\Rightarrow$(d).

Proof. It follows from (a) that $\mathcal{N}=0$ so (a) implies (b). The second implication is proved to be valid by observing that $\mathcal{M}\subset\mathcal{N}$ and, furthermore, the only subspace satisfying (b) and the defining inclusion (7.27) is zero subspace. Condition (d) follows from (c) in view of Theorems 7.1 through 7.5.

It is seen from the results of preceding section that for system (7.1) the property of being approximately controllable in some function space is to some extent of algebraic character; it equalizes many topologies. For instance all the criteria given above does not depend on the number r, $1\leqslant r<\infty$, so that it suffices to consider the spaces $M_{\alpha\beta}^1$, W_1^1 in lieu of general case of $M_{\alpha\beta}^r$, W_1^r. Furthermore, by Theorems 7.1 and 7.5 also the spaces C, M_{11}^r and W_1^r are equivalent in that sense. It is conjectured that similar equivalences hold for general system (1.1) but we are able to prove only those implications which are indicated in Corollary 5.3 and the proof of appropriate converses would probably require more technical details.

Let us supplement the diagram of Corollary 5.3.

Corollary 7.3. The following implications are valid

$$W_1^r\Leftrightarrow W_1^1\Leftrightarrow C\Leftrightarrow M_{11}^1, M_{11}^r \begin{matrix} \nearrow & M_{10}^r\Leftrightarrow M_{10}^1 & \searrow \\ \searrow & M_{01}^r\Leftrightarrow M_{10}^1 & \nearrow \end{matrix} L^1\Leftrightarrow L^r,\ M_{11}^1\Rightarrow M_{\alpha\beta}^r$$

for any $1 \leq r < \infty$. The symbols of spaces represent here appropriate notions of approximate controllability of system (7.1) on a fixed interval $[0, T]$.

Proof. All implications which do not follow from Corollary 5.3 can be immediately obtained from Theorem 7.1 through 7.5.

GENERALIZATION 7.1. Consider the system with finitely many discrete delays of type (2.1). Assuming the lags are commensurable, i.e.

$$h_i = k_i \bar{h}, \; i = 1, \ldots, l$$

for some $\bar{h} > 0$ and some integers k_i. Applying transformation (7.8) with h replaced with $\bar{h}$ one can construct immediately nondelayed systems of type (7.9), (7.10) and (7.13), (7.14) which correspond to appropriate specializations of (6.9), (6.10), (6.11) and (6.1), (6.2) respectively. Algebraic criteria analogical to Theorems 7.1 through 7.4 can be derived in similar manner as in paragraph 7.3. Theirs main disadvantage is that the dimensions of matrices, proportional to Entier $[T/k_i]$, might become too large if arge T is to be considered.

8. Further special cases

8.1. Systems without delays

After setting $A_1 = 0$ in (7.1) the speaking on h as on system delay makes no sense. Therefore we interpret the trajectory $x(t)$, $t \in [T-h, T]$, of the system

$$\dot{x}(t) = A_0 x(t) + Bu(t), \; t \geq 0 \tag{8.1}$$

as a result of tracking of a given function f: $[T-h, T] \rightarrow R^n$ while the tracking error is measured by a norm $\|x-f\|_{T-h, T}$ in a dunction space $\mathscr{F}$. With this interpretation of $\mathscr{F}$-approximate trajectory controllability the following result is obtained as an immediate conclusion from Corollaries 6.4 and 7.1.

COROLLARY 8.1. The system without delay (8.1) is C-$(W_1^r$-, $M_{\alpha\beta}^r$-$)$ approximately trajectory controllable iff rank $B = n$.

$\mathscr{F}$-approximate trajectory controllability in system (8.1) can also be interpreted as the ability to compensate any integrable additive disturbance (which is known (measured)) with arbitrarily small (in topology of $\mathscr{F}$) error. Thus it is rather strong property and it is known from multivariable control theory that for this as many controls as independent disturbances is needed. In many practical problems the number of independent disturbances is rather less than the dimension n of state vector.

8.2. The case of $A_0 = 0$

The system (7.1) with $A_0 = 0$ is extremely simple. The maximal invariant $\mathscr{M} =$ Mic $(\tilde{A}, \operatorname{im} \tilde{A}_1 N_1; \mathscr{N})$ can be computed analytically. Indeed, the application of Algorithm 7.2 yields (a). For the maximal invariant $\mathscr{M}$ of Theorem 7.1 (i) (the case $T=(k+1)h$) $V_0=(\operatorname{im} B)^k$, the product of k subspaces. Furthermore,

$$V_{k-1}=V_{k-1+i}=(\operatorname{im} B+A_1 \operatorname{im} B+\ldots+A_1^{k-1} \operatorname{im} B)\times\ldots\times(\operatorname{im} B+A_1 \operatorname{im} B)\times \operatorname{im} B \quad (8.2)$$

for $i=1, 2, \ldots$. Hence $\mathscr{M}^\perp=V_{k-1}$. The condition (7.19) holds iff

$$V_{k-1}+(\tilde{A}_1 \ker B')^\perp=R^{kn}$$

or equivalently

$$\operatorname{im}[B; A_1 B; \ldots; A_1^{k-1} B]+(A_1' \ker B')^\perp=R^n.$$

This can be easily shown to be equivalent to

$$\operatorname{rank}[B; A_1 B; \ldots; A_1^k B]=n. \quad (8.3)$$

From the form of $\tilde{A}$ and $\tilde{A}_1$ (see (7.11), (7.15)) it follows that the defining condition for maximal invariant $\mathscr{M}$ is equivalent to $\tilde{A}\mathscr{M}\subset\mathscr{M}$. This gives $F=0$ in (7.21) and, as a consequence, a simplification of (7.20). In fact, the matrix $I-J\exp(h\tilde{A})$ is of block lower triangle form with identity matrices on the main diagonal; therefore its kernal is zero subspace. Furtherfore, from this triangle form and from the form of

$$\mathscr{M}=V_{k-1}^\perp=\mathscr{M}_1\times\mathscr{M}_2\times\ldots\times\mathscr{M}_k$$

implied by (8.2) it follows that $\big(I-J\exp(hA)\big)\mathscr{M}\cap\ker J'=0$ iff $\mathscr{M}_1=0$ which, by the form of V_{k-1}, is equivalent to

$$\operatorname{rank}[B; A_1 B; \ldots; A_1^{k-1} B]=n. \quad (8.4)$$

(b). For the maximal invariant $\tilde{\mathscr{M}}$ of Theorem 7.1 (ii) $\big(kh<T<(k+1)h\big)$

$$V_0=(\operatorname{im} B)^{k-1}\times\{0\}\subset R^{kn} \text{ and for } i=1, 2, \ldots$$

$$V_{k-2}=V_{k-2+i}=(\operatorname{im} B+A_1 \operatorname{im} B+\ldots+A_1^{k-2} \operatorname{im} B)\times\ldots\times(\operatorname{im} B+A_1 \operatorname{im} B)\times \operatorname{im} B\times\{0\}.$$

Similarly as above the condition (7.22) is equivalent to (8.4).

Summarizing (a) and (b) and observing that (8.4) implies (7.7) we see that necessary and sufficient conditions of Theorem 7.1 are equivalent to (8.4). Furthermore, the fact that $\tilde{A}\mathscr{M}\subset\mathscr{M}$ implies that $\mathscr{M}_{\tilde{A}}=\mathscr{M}$ and that condition (7.30) is equivalent to (7.20). Therefore, taking into account this and also Theorem 7.5 we have

COROLLARY 8.2. System (7.1) with $A_0=0$ is approximately controllable on $[0, T]$, $kh<T\leqslant(k+1)h$, in any of function spaces C, M_{11}^r, M_{10}^r, W_1^r iff (8.4) holds.

Considering the spaces M_{01}^r and L^r we have, as previously, the necessary conditions (8.3) and (8.4) corresponding to (7.19) $\big(T=(k+1)h\big)$ and to (7.22) $\big(kh<T<$ $<(k+1)h\big)$ respectively. Since $F=0$ and therefore $E_F=0$ we see as previously that $\ker\big(I-J\exp(h\tilde{A})\big)=0$ so that (7.34) and (7.37) hold automatically. Thus we get

COROLLARY 8.3. System (7.1) with $A_0=0$ is approximately controllable on $[0, (k+1)h]$ in any of spaces M^r_{01}, L^r iff (8.3) holds. For the case $kh<T<(k+1)h$ the appropriate necessary and sufficient condition is (8.4).

Since (8.4) is also a necessary and sufficient condition for R^n-controllability of the system considered on $[0, T_1]$, $(k-1)h<T_1\leqslant kh$ it can be said that this system is approximately controllable in spaces C, M^r_{11}, M^r_{10}, $\bar{W}^r_1$ on $[0, T]$ iff it is R^n-controllable on $[0, T-h]$ and for the cases of M^r_{01}, L^r the characterization is by R^n-controllability on each $[0, T_1]$, $T_1>T-h$ or, equivalently, by weak 1-point controllability on $[0, T]$ (compare Theorem 4.1).

REMARK 8.1. The conditions above are the weakest in a sense that, by Theorems 3.1. and 3.2 weak 1-point controllability and, respectively, R^n-controllability on $[0, T]$ are necessary for approximate controllability in a general function space satisfying suitable hypothesis of section 3.

REMARK 8.2. Corollary 8.3 overlaps with Zmood's [11] result for L^2-approximate controllability. For this type of systems ($A_0=0$) a criterion for function space controllability in the sense of Korytowski [8] is of the form (8.4) with $k=n$.

8.3. The case of $\operatorname{im} B\supset \operatorname{im} A_1$ and $h<T\leqslant 2h$

If $\operatorname{im} B\supset\operatorname{im} A_1$ then one may choose $u=u_1+u_2$ where u_1 is arbitrary and u_2 compensates the term $A_1 x(t-h)$ in (7.1) .Therefore $\mathscr{A}(T)$ consist of pieces of trajectories attainable for system (8.1) and the conclusion is as in Corollary 8.1, that is, rank $B=n$ is a necessary and sufficient condition for approximate controllability in any of the spaces considered in section 7. The same conclusion we get immediately for general system (1.1) satisfying $\operatorname{im} B\supset\operatorname{im} d_s A(s)$ for all $s\in[-h, 0)$ or, after normalization $A(0-)=0$, the condition $\operatorname{im} B\supset\operatorname{im} A(s)$, $s\neq 0$.

For the case $h<T<2h$ apply Theorem 6.1. Take an initial condition for system (6.1) as an absolutely continuous function (of class $\bar{W}^r_1$) satisfying $x(t)=0$ for $t\in[-h, T-2h]$ and $t=0$ and $0\neq x(t)\in\ker B'$ on $(T-2h, 0)$ where $\ker B'$ is supposed to be nonzero. This yields $x(t)=0$ and $y(t)=0$ for $t\in[0, T-h]$ in (6.1), (6.2). From Theorem 6.1 (i) and (ii)'' we conclude that system (1.1) is nor C-neither $M^r_{\alpha\beta}$-controllable on $[0, T]$. Similar conclusion holds for the space W^r_1 after taking suitable initial conditions for x^1 in (6.15) and $x^2_0=0$. Thus the condition rank $B=n$ is necessary, and also by Corollary 6.4, sufficient for approximate controllability. Therefore we can state

COROLLARY 8.4. Assume that, in system (1.1), $A(0-)=0$ and either $\operatorname{im} B\supset\operatorname{im} A(s)$ for $s\neq 0$ or $h<T<2h$. Then system (1.1) is approximately controllable on $[0, T]$ in any of spaces C, $M^r_{\alpha\beta}$, W^r_1 iff rank $B=n$.

Suppose finally that $T=2h$. Following the notation of section 7 we have $k=1$, $\tilde{A}=A'_0$, $\tilde{A}_1=A'_1$, $J=0$ and

$$\mathscr{M}=\operatorname{Mic}(A'_0, A'_1\ker B'; \ker B'). \tag{8.5}$$

Condition (7.19) takes the form

$$\mathcal{M} \cap A_1' \ker B' = 0 . \tag{8.6}$$

Condition (7.20) holds iff $\mathcal{M}=0$ which clearly implies (8.6). In Theorem 7.2, (7.30) is equivalent to $\mathcal{M}_{\bar{A}}=0$ or, equivalently,

$$\mathcal{M} \cap (A_0')^{-1} \mathcal{M} = 0 . \tag{8.7}$$

The conditions (7.34) and (7.37) are simplified slightly by substitution $E_F = N_1 F$ and other matrices simplified above. F is any matrix satisfying

$$(A_0' + A_1' N_1 F) \mathcal{M} \subset \mathcal{M} . \tag{8.8}$$

Summarizing, we get

COROLLARY 8.5. System (7.1) is approximately controllable on $[0, 2h]$ iff rank $[A_1; B]=n$ and additionally:
(a) $\mathcal{M}=0$ for the spaces C, M_{11}^r and W_1^r,
(b) conditions (8.6) and (8.7) hold for the space M_{10}^r,
(c) conditions (8.6) and

$$\operatorname{rank} [I - N_1 F \exp (h (A_0' + A_1' N_1 F))] = \operatorname{rank} M \tag{8.9}$$

hold for the case of M_{01}^r space,
(d) conditions (8.6) and

$$\operatorname{rank} [I - N_1 F \exp (h (A_0' + A_1' N_1 F))] M_{\bar{A}} = \operatorname{rank} M_{\bar{A}} \tag{8.10}$$

hold in case of L^r space.

The columns of N_1, M and M_A are, by definition, basis vectors for $\ker B'$, $\mathcal{M}$ and $\mathcal{M}_{\bar{A}} = \mathcal{M} \cap (A_0')^{-1} \mathcal{M}$ respectively and any of these matrices is set to be zero if corresponding subspace equals zero. F is any matrix satisfying (8.8) and if M is chosen the matrix FM is unique provided rank $[A_1; B]=n$ and (8.6) holds.

9. Examples and counterexamples

The examples presented below show how the theory developed in this paper works and also show that some implications between approximate controllability in various spaces and stabilizability does not hold.

EXAMPLE 9.1. The evolution of a system is described by eqs. [16]

$$\dot{x}_1(t) = x_1(t) + x_3(t-h),$$

$$\dot{x}_2(t) = x_2(t) + x_3(t),$$

$$\dot{x}_3(t) = u(t).$$

By constructing the corresponding matrices A_0, $\bar{A}_1$, B we check that the pair (A_0, B) is not controllable but rank $(B, A_0, B, A_1 B)=3$ so that the system is R^3-

controllable on $[0, T]$ for $T>h$ (see [4], [7]). However this system is not multipoint controllable on $[0, T]$, $T>h$, since it is not 2-point controllable (see Theorem 4.2). The maximal dimension of the subspace spanned by the columns of (4.6) equals $4<6$. From Theorem 3.2 and the diagram of Corollary 7.3 the system is not approximately controllable on any interval $[0, T]$ in any of the spaces W_1^r, C, M_{11}^r. This is also immediately visible from the fact that necessary condition (7.7) is not satisfied. The last conclusion extends therefore on any space $M_{\alpha\beta}^r$.

Remark 9.1. It is interesting to note that in the original version of this example [16] it was shown for the formal controllability matrix $[B; A(\lambda)B; \ldots; (A(\lambda))^{n-1} B]$, ($A(\lambda)=A_0+\lambda A_1$, λ-delay operator) that its $R[\lambda]$-linear span ($R[\lambda]$- the ring of real polynomials over λ) neither equals the free module $(R[\lambda])^n$ ($R[\lambda]$-controllability) nor is isomorphic to $(R[\lambda])^n$ (weak $R(\lambda)$-controllability). The main result of [16] is that $R[\lambda]$-controllable systems are pole assignable (and vice versa) with feedback $u(t)=K(\lambda)x(t)$ where the elements of matrix $K(\lambda)$ belong to $R[\lambda]$. The system of Example 9.1 is not stabilizable since $x_1(t)-x_2(t-h)=(x_1(h)-x_2(0))\exp(t-h)$ for $t\geqslant h$. It was suggested by Morse that weak $R[\lambda]$-controllability may be essential in stabilizability problem. The result established above that the system is not approximately controllable clarifies better the situation. It is intuitively clear that for pole assignability a type of approximate null controllability is needed. This is also supported by a result of Theorem 6.2 that M_{01}^r-approximate controllability implies pole assignability. So multipoint controllability as necessary for approximate controllability might appear necessary for assignability problem but it requires more investigations to obtain rigorous results.

Example 9.2. Another interesting example is the following

$$\dot{x}_1(t)=u(t),$$
$$\dot{x}_2(t)=x_1(t)-x_1(t)-x_1(t-h).$$

This system and the corresponding matrices A_0, A_1, B have the following easy to check properties.

(i). Each of the pairs (A_0, B), (A_1, B) is controllable.

(ii). The system is R^2-controllable on $[0, T]$, $T>0$ and hence, by Theorem 4.1 (a), weakly multipoint controllable on $[0, T]$, $T>h$.

(iii). By Theorem 4.2 the system is multipoint controllable on $[0, T]$, $T\geqslant h$.

(iv). The condition (7.7), necessary for approximate controllability, is satisfied.

(v). Checking the approximate controllability on $[0, 2h]$ according to Corollary (8.5) we get $\mathcal{M}^{\perp}=A_1'\ker B'=\operatorname{im} B\neq 0$ and hence $\mathcal{M}=\ker B'\neq 0$. Therefore the system is not approximately controllable in spaces C, M_{11}^r, W_1^r. For other spaces we check that the necessary condition (8.6) holds. Furthermore we have $(A_0')^{-1}\mathcal{M}=0$ so that (8.7) is valid. By Corollary 8.5 (b) and by Corollary 7.3 the system is M_{10}^r- and L^r-approximately controllable on $[0, 2h]$. Finally, to check (8.9) we compute $A_0' M = M = N_1 = -A_1' N_1 = -A_1' N_1 FM$ where $N_1' = [0, 1]$ and $F = [0, 1]$. This gives

$$N_1 F = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \; A_0' + A_1' N_1 F = 0.$$

It is seen now that the 2×1 matrix on the left side of (8.9) has zero rank and (8.9) is not valid. Therefore the system is not M_{01}^r-approximately controllable on $[0, 2h]$.

(vi). Take $T = (k+1) h$ and check approximate controllability. With the use of Algorithm 7.2 we compute easily that $V_2 = V_1 = \mathcal{M}^{\perp}$ where the basis matrix M for $\mathcal{M}$ is $kn \times 1$ of the form $M = [0, 1, 0, 1, ..., 0, 1]$. With N_1 as above we get $F = [0, 1, 0, 0, ..., 0]$ and $(\tilde{A} + \tilde{A}_1 N_1 F) M = 0$. This enables one to check immediately that neither (7.20) nor (7.34) holds. The system is not approximately controllable in any of spaces $C, M_{11}^r, M_{01}^r, W_1^r$ on any interval $[0, T]$.

(vii). The system is not asymptoticaly stabilizable with even general type of feedback $u(t) = \int_{-\bar{h}}^{0} d_s K(s) x(t+s)$, $\bar{h}$ arbitrary positive, since the eigenvalue $\lambda = 0$ remains insensitive (see [30] and [34] for eigenvalues and stability).

(viii). It follows from (vii) and Morse's theorem [16] on pole assignment that the system is not $R[\lambda]$-controllable, which can be also easily verified by definition.

(ix). The system is weakly $R[\lambda]$-controllable in the sense of Morse.

The importance of this example is evident; it allows to claim that the diagram of Corollary 7.3 is complete. In fact, it shows that, in general, neither L^r- nor M_{10}^r-implies M_{11}^r- or M_{01}^r-approximate controllability. That the implications $L^r \Rightarrow M_{10}^r$, $M_{01}^r \Rightarrow M_{10}^r$ and $M_{01}^r \Rightarrow M_{11}^r$ are not valid it follows from Corollaries 8.3 and 8.2 (the case $T = (k+1) h$). Therefore we have

COROLLARY 9.1. The diagram of Corollary 7.3 is complete, that is, none of the implications $L^r \Rightarrow M_{01}^r$, $L^r \Rightarrow M_{10}^r$, $M_{10}^r \Rightarrow M_{01}^r$, $M_{10}^r \Rightarrow M_{11}^r$, $M_{01}^r \Rightarrow M_{10}^r$ and $M_{01}^r \Rightarrow M_{11}^r$ is valid in general for system (7.1).

The next important conclusion which is evident in view of properties (v) and (vii) is the following

COROLLARY 9.2. Neither L^r- nor M_{10}^r-approximate controllability is sufficient in general for pole assignability of system (7.1).

REMARK 9.2. The property (vi) that the system is not approximately controllable on any interval in any of the spaces W_1^r, C, M_{11}^r and M_{01}^r follows immediately from property (vii), Theorem 6.2 and Corollary 7.3. But it is not a typical situation that condition (6.34) serve as a necessary checkable condition for approximate controllability. We have not appropriate numerical algorithm. Fortunately it is rather on the contrary; the theorems of section 7 and its specializations given in Corollaries 8.2, 8.3 and 8.5 serve as constructible sufficient conditions for stabilizability (pole assignability) problem. Fortunately, because stabilizability problem appears to be more important in real problems than approximate controllability itself and because constructibility can be motivated in the following way. There are many real problems in which a multivariable plant containing internal and input delays has to be stabilized but not all inputs are fixed. There is a freedom in choosing some of them and the problem is how to choose inputs so that the system obtained be stabilizable. With the use of Theorems 6.2 and 7.3 or Corollary 8.5 any choice of inputs can be tested with respect to system stabilizability.

EXAMPLE 9.3. Let the system equation be

$$\dot{x}_1(t) = x_2(t),$$

$$\dot{x}_2(t) = x_2(t-h) + u(t).$$

This system has the property of R_2-controllability on $[0, T]$, $T>0$, since the corresponding pair (A_0, B) is controllable. It is also multipoint controllable on $[0, T]$, $T>h$ which can be easily checked by Theorem 4.2. Furthermore, the system is not approximately controllable on any $[0, T]$, $T>h$, in any of the spaces of section 7 since the necessary condition (7.7) is not valid. Finally, we check that the system is pole assignable since the condition (6.34) holds for all complex λ. In fact, if it were not there exists a nonzero vector q orthogonal to all columns of (6.34) for some λ. The equation $q' B=0$ implies $q'=[a, 0]$, $a\neq 0$ and hence $q' A_1=0$ and the orthogonality condition takes the form $[a\lambda, -a]=0$ for some λ which is, clearly, not satisfied. The above properties enables one to draw the following general conclusion.

COROLLARY 9.3. Neither multipoint controllability nor pole assignability implies in general approximate controllability in any of the spaces W_1^r, C, $M_{\alpha\beta}^r$.

Now we consider the question whether controllability of the pair (A_1, B) can be a sufficient (or a necessary) condition of approximate controllability for systems of somewhat more general form than the case $A_0=0$ in section 8. We are led by the fact that, by Corollaries 8.2 and 8.3, the controllability of the pair (A_1, B) is both necessary and sufficient for approximate controllability of system (7.1) with $A_0=0$ on any interval $[0, T]$, $T>nh$ and in any of spaces W_1^r, C, $M_{\alpha\beta}^r$. However, in general case of $A_0\neq 0$ we get the following negative result.

COROLLARY 9.4.

(i). The controllability of the pair (A_1, B) is not necessary, in general, for W_1^r-approximate controllability of system (7.1) on some interval $[0, T]$.

(ii). The condition

$$\operatorname{rank}[B; A_1 B; \ldots; A_1^k B]=n \tag{9.1}$$

is, in general, not sufficient for L^r-approximate controllability of system (7.1) on $[0, (k+1)h]$.

The proof is by counterexamples.

EXAMPLE 9.4. Consider the system

$$\dot{x}_1(t)=u(t),$$

$$\dot{x}_2(t)=x_1(t)+x_2(t-h).$$

We check easily that the condition rank $[A_1; B]=n$ and condition (a) of Corollary 8.5 are satisfied and hence the system is W_1^r-approximately controllable on $[0, 2h]$ although the condition $A_1 B=0$, holds implying uncontrollability of the pair (A_1, B). This proves part (i) of Corollary 9.4.

EXAMPLE 9.5. Let us write the following equations

$$\dot{x}_1(t)=(1/h)\,x_2(t)-x_3(t-h),$$

$$\dot{x}_2(t)=x_3(t)+x_4(t-h),$$

$$\dot{x}_3(t)=u_1(t),$$

$$\dot{x}_4(t)=u_2(t).$$

Take $T=2h$. We check that

$$\operatorname{rank}[B; A_1 B]=4, \tag{9.2}$$

that is condition (9.1) holds with $k=1$. Condition (9.2) implies that rank $[B; A_1]=4$ and that (8.6) holds. In order to prove part (ii) of Corollary 9.4 we shall show that condition (8.10) of Corollary 8.5 is not valid. Indeed, we check by substitution to defining relation (8.5) that $\mathcal{M}=\mathcal{N}=\ker B'$. Hence

$$N_1'=M'=\begin{bmatrix}1&0&0&0\\0&1&0&0\end{bmatrix},\quad F=\begin{bmatrix}0&1&0&0\\0&0&0&0\end{bmatrix},\quad (\tilde{A}+\tilde{A}_1 N_1)\,M=MQ,$$

where

$$Q=\begin{bmatrix}0&0\\1/h&0\end{bmatrix}.$$

Substituting this and $M_{\tilde{A}}'=[1, 0, 0, 0,]=[1, 0]\,M'$ into (8.10) yields

$$\operatorname{rank}[I-N_1 F\exp(h(A+A_1 N_1 F))]\,M_{\tilde{A}}=0<1=\operatorname{rank} M_{\tilde{A}}.$$

Therefore the system is not L^r-approximately controllable on $[0, 2h]$.

Remark 9.3. The conclusions of Corollary 9.4 can be completed by taking into consideration the implications of Corollary 7.3.

10. Applications and concluding remarks

10.1 Applications

The results of this paper concerning approximate controllability of linear hereditary systems and dual observability problems are important from the point of view of general theory of such systems. The more important for applications are, however, the concrete checkable criteria for approximate controllability of sections 7 and 8 and the sufficient condition for stabilizability and pole assignability (Theorem 6.2). Let us indicate some of possible applications.

Optimal control to a target set. It is known that rather strong conditions are required in order to establish a nontrivial maximum principle for optimal control of time lag systems with target set given by equality constraints. For instance, Banks and Kent [38] derived a type of maximum principle for nonlinear problems with delays and fixed final state x_T. They could not, however, prove the nontriviality of adjoint variables. Jacobs and Kao [39] obtained nontrivial necessary conditions for optimality but under sever assumption that the controls are unconstrained and the system is W_1^2-controllable. In case of linear systems the latter is equivalent to the condition rank $B=n$ (see Remark 5.1). Kurcyusz [40] investigated equality constraints on final state and proved nontriviality of his maximum principle while assuming that the subspace of attainable values for linearized constraint function is not a prover subspace dense in constraint space and if additionally this subspace was assumed to be closed normality ($\psi_0 \neq 0$) was established. However, the conditions for closedness of the attainable subspace are also restrictive. According to Kurcyusz and Olbrot [29] the attainable subspace $\mathscr{A}(T)$ for system (7.1), provided that we assume L^p controls, is never closed in any of spaces W_1^r, C, L^r except for the space W_1^p. In this exceptional case the necessary and sufficient condition for closedness is that

$$A_1 \{A_0|B\} \subset \operatorname{im} B. \tag{10.1}$$

One unpleasant feature of condition (10.1) is that it is absolutely sensitive (not generic) unless rank $B=n$. This means that if im $B \neq R^n$ then for any pair of matrices (A_0, A_1) one may find arbitrarily small variations of their elements such that (10.1) is not valid for perturbed system.

From the other hand, in light of the above mentioned references [39] and [40], the density property of $\mathscr{A}(T)$ looks like the worst case for control problems with function space equality constraints. But this is not the whole truth if practical aspect is taking into consideration. As it was pointed out by Olbrot [20] even for problems with fixed final state $x_T = \xi$ we do not reach in practice the function exactly. Therefore it is strongly motivated that the final constraints should be substituted by

$$\|x_T - \xi\| \leqslant \varepsilon \tag{10.2}$$

where the normed function space for final states and the accuracy ε are chosen by the user. It is now evident that approximate controllability is necessary in order that an admissible solution satisfying (10.2) exists for all ξ from a given function space and all $\varepsilon > 0$. It was also shown in [41] that necessary nontrivial optimality conditions can be easily derived for problems with discrete delays both in state and in control variables under assumptions which are typical for nondelayed problems with finite-dimensional constraints while for state space any of spaces C, W_1^r, $M_{\alpha\beta}^r$ is chosen. The most regular results were shown for W_1^2 and $M_{\alpha\beta}^2$.

Optimal and suboptimal stabilization. Consider a special kind of optimal control problems considered above, namely, the problem of steering the system (1.1) from a given nonzero initial state x_0 to a neighbourhood of zero state at time T, i.e. to a state x_T satisfying

$$\|x_T\| \leqslant \varepsilon \|x_0\| \tag{10.3}$$

(where ε is sufficiently small positive) such that a given performance index (e.g. of quadratic type) attains minimum. This will be called optimal stabilization problem. One may also consider optimal stabilization as minimizing the norm $\|x_T\|$ under some constraints on control values. It is seen that approximate null controllability is the property needed for such problems to be solvable for any $\varepsilon > 0$. Since, practically, we are looking for sufficient conditions for solvability the conditions for approximate controllability may be useful in view of Corollary 1.1. However, not all commonly used norms are suitable when the control has to be generated by state feedback and asymptotic stability of closed loop system is required (compare Example 9.2). In view of Theorem 6.2 it seems that the norms of type M_{01}^2 and M_{11}^2 when applied to (10.3) may lead to a stable optimal controller or its suboptimal approximation. Clearly, after taking C or W_1^2 norm in (10.3) and applying the resulting controller sequentially on $[0, T]$, $[T, 2T]$, etc. we get $x(t) \to 0$ as $t \to \infty$ provided that $0 \leqslant \varepsilon < 1$. This follows from the fact that the convergence in these spaces is unoform.

Algebraic methods for linear feedback stabilization. There was recently a considerable progress in applications of algebraic methods to feedback stabilization of linear systems with delays (Pandolfi [35], Bhat and Koivo [36], Morse [16], Kamen [41], Sontag [42]). However, appropriate stabilizability and pole assignability criteria are merely computable although there exist general procedures for computing feedback gains. After we have established Theorem 6.2 our criteria for approximate controllability of sections 7 and 8 may serve as a preliminary test to assure the designer that pole placement can be achieved by general linear state feedback. An open problem is how approximate controllability problems are related to problems of pole assignability with the use of proportional and unit delay elements in feedback gain [16] and additionally with the use of feedback from previous control values [42].

10.2. Concluding remarks

Approximate controllability of general linear autonomous hereditary systems has been widely examined. It has been shown for general function spaces that approximate controllability implies weak multipoint controllability and also multipoint controllability if the convergence in function space norm implies uniform convergence. Algebraic criteria for weak multipoint and multipoint controllability has been derived. The function spaces C, W_1^r and $M_{\alpha\beta}^r$ has been considered in details. Adjoint space characterization for approximate controllability in these spaces has been given and, on the basis of them, dual observability problems has been formulated for a dual system obtained from a given one in a simple way by matrix transposition. The interrelations for approximate controllability in the spaces C, W_1^r, $M_{\alpha\beta}^r$, $1 \leqslant r$, $r' < \infty$, $\alpha \geqslant 0$, $\beta \geqslant 0$ has been indicated on the diagram. This diagram of implications, in case of system with one delay, has been extended and proven to be complete. It has been shown that the general system considered is never $M_{\alpha\beta}^{\infty}$- (in particular L^{∞}-) approximately controllable and that for W_1^{∞} space the necessary and sufficient condition is rank $B=n$ (the number of independent controls is equal to the number of state variables). This condition has been proven to be sufficient for approximate controllability in any of spaces C, W_1^r, $M_{\alpha\beta}^r$, $1 \leqslant r < \infty$. An equally simple checkable general necessary condition has been proven (Corollaries 6.5 and 7.1). The important result that approximate controllability in any of spaces W_1^r, C, M_{11}^r, M_{01}^r implies pole assignability by general linear state feedback has been established. For system with one delay algebraic numerically checkable criteria for approximate controllability in each of spaces C, W_1^r, $M_{\alpha\beta}^r$ has been derived. The criteria extend immediately to systems with finitely many commensurable delays and, on the other hand, for some special cases they have been much simplified. Numerical examples has been given showing how the criteria obatained can be practically applied and also being counterexamples for that some implications between approximate controllability, stabilizability and other properties of a system do not hold in general. Applications to optimal control, optimal stabilization and feedback pole assignment has been indicated. Let us complete the considerations of this paper with the following conclusion of practical interest.

The property of system (1.1) *being approximately controllable is generic.* What we mean is that under arbitrary but sufficiently small variations of parameters the system remains approximately controllable. We have to admit openly that such statement as above requires usually a rigorous proof. Actually we are not able to prove this for W_1^r state space. However, for the spaces C, $M_{\alpha\beta}^r$ the conclusion is immediate from existing theorems on continuous dependence of solutions with respect to parameters. In fact if a final state x_T of system (1.1) is sufficiently close to a given function f in the space C (resp. $M_{\alpha\beta}^r$) then it is known [1], [13] that the state $\tilde{x}_T$ of a perturbed system corresponding to the same control is sufficiently close to x_T in sup norm topology if the perturbation is sufficiently small in the space of parameters. From triangle inequality, the sup norm $\|\tilde{x}_T - f\|$ can be made arbitrary small by minimizing $\|x_T - f\|$ and $\|\tilde{x}_T - x_T\|$. Since the uniform convergence implies convergence in $M_{\alpha\beta}^r$ the argument is valid also in this space. The extension to W_1^r can be made in view of Corollary 7.3 but only for systems of type (7.1).

An additional bibliographical note. After completing this work it was pointed out to this author by Professor E. B. Lee to whom the author wishes to express his appreciation that the following work is devoted to study of M_{01}^2-approximate controllability of system (7.1) *A. Manitius, R. Triggiani, Function Space Controllability of Linear Retarded Systems: A Derivation from Abstract Operator Conditions, Tech. Report. CRM*-605, *Université de Montréal.* The approach of this reference is different from ours and the results are also of different nature except for the simplest cases such as in our paragraph 8.2 when they overlap. In contradistinction to our paper the authors above do not obtain duality results; they work with abstract nondelayed system equivalent to (7.1) and use the technique of finite Laplace transform. The results are obtained in terms of matrices depending on complex parameter λ and are, in fact, similar in construction to those of Popov [9].

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Sterowanie układów z opóźnieniami przy ograniczeniach w przestrzeni funkcyjnej. Część 2. Sterowalność aproksymacyjna

Rozważany jest problem aproksymacyjnej sterowalności dla liniowych układów opisywanych równaniem różniczkowo-funcjonalnym w R^n z opóźnionymi argumentami. Rezultaty niniejszej pracy stanowią konkretne warunki istnienia sterowania dopuszczalnego dla niektórych problemów poruszonych w części pierwszej pracy (Control and Cybernetics, No 3, 1976). Z drugiej strony stanowią również pewien wkład do ogólnej teorii liniowych stacjonarnych układów z opóźnieniami. Wśród najważniejszych wyników znajduje się warunek konieczny aproksymacyjnej sterowalności w ogólnych przestrzeniach funkcyjnych w postaci tzw. wielopunktowej lub słabej wielopunktowej sterowalności. Warunek ten, w przypadku dyskretnych opóźnień, scharakteryzowano algebraicznie, poprzez kryterium rzędu dla macierzy zbudowanej w oparciu o współczynniki układu. Dalsze wyniki uzyskano po przyjęciu jednej z przestrzeni C, W_1^r, $M_{\alpha\beta}^r$ jako przestrzeni stanów ($M_{\alpha\beta}^r$ jest uogólnieniem $L^r = M_{00}^r$). Uzyskano charakteryzację sterowalności aproksymacyjnej poprzez funkcjonały z przestrzeni dualnej do przestrzeni stanu a stąd wyprowadzono dualne problemy obserwowalności, gdzie układ dualny buduje się w prosty sposób przez transpozycję macierzy układu pierwotnego. Podano proste, albo wystarczające, albo konieczne warunki sterowalności aproksymacyjnej dla ogólnego układu z opóźnieniami skupionymi i rozłożonymi, natomiast dla przypadku współmiernych opóźnień skupionych podano pełną charakteryzację poprzez sprawdzalne warunki algebraiczne. W przypadku jednego opóźnienia przestrzenie C, W_1^r, M_{11}^r, $1 \leqslant r < \infty$, okazują się równoważne ze względu na własność aproksymacyjnej sterowalności. Pokazano, że ogólny układ nie jest L^∞-aproksymacyjnie sterowalny. Najważniejszym z punktu widzenia zastosowań wynikiem jest to, że aproksymacyjna sterowalność ogólnego układu w jednej z przestrzeni C, W_1^r, M_{11}^r, M_{01}^r implikuje przesuwalność wartości własnych przy pomocy sprzężenia zwrotnego od stanu a więc również stabilizowalność. Podane przykłady numeryczne ilustrują wyniki teoretyczne i pokazują, że pewne współzależności między stabilizowalnością a aproksymacyjną sterowalnością nie zachodzą w ogólnym przypadku.

Управление систем с запаздыванием при ограничении в функциональном пространстве

Рассматривается вопрос аппроксимационной управляемости линейных систем описываемых дифференциально-функциональным уравнением в R^n с запаздывающими аргументами, Итогом этой работы являются конкретные условия существования допускаемого управления для некоторых задач рассмотреных в первой части работы (Control and Cybernetics No 3, 1976). Работа является также определенным вносом в общую теорию линейных стационарных систем с запаздыванием. К основным результатам следует отнести необходимое условие аппроксимационной управляемости в общих функциональных пространствах в виде так называемой многоточечной или слабомноготочечной управляемости. Это условие для случая дискретных запаздываний выражается алгебраически просредством критерия ранга матрицы построенной на основании параметров системы.

Дальнейшие результаты получены для одного из пространств C, W_1^r, $M_{\alpha\beta}^r$ как пространства состояний ($M_{\alpha\beta}^r$ является обобщением $L^r = M_{00}^r$). Получено определение аппроксимационной управляемости через функционалы в пространстве дуальном к пространству состояний. Исходя из этого получаются дуальные задачи наблюдаемости при чем дуальная система построена простым образом путем транспонирования матрицы первичной системы.

Приведены простые достаточные или необходимые условия аппроксимационной управляемости дла общего случая системы с сосредоточенными и распределенными запаздываниями. Для случая соизмеримых сосредоточенных запаздываний дано полное определение через проверямые алгебраические условия.

Для случая одного запаздывания пространства C, W_1^r, M_{11}^r ($1 \leqslant r < \infty$) оказываются эквивалентными по отношению к условию аппроксимационной управляемости.

С точки зрения применений самым главным итогом является факт, что аппроксимационная управляемость общей системы в одном из пространств C, W_1^r, M_{11}^r, M_{01}^r влечет за собой передвижимость собственных значений с помощью обратной связи от состояния что означает возможность стабилизации. Приведенные численные примеры иллюстрируют теоретические результаты и показывают что в общем случае не соблюдается определенная зависимость между возможностью стабилизации и аппроксимационной управляемостью.

Erratum and Comment to

Control of reterded systems with function space constraints
Part I. Necessary optimality conditions

Erratum

The last term in eq. (7.12) should read: $+c\,(\mathrm{ch}\,(a)-\mathrm{ch}\,a\,(t-1))]$.

In the eq. (7.14) some symbols were missing by the author. The correct form is as follows

$$\varepsilon^2=2\,(x\,(2))^2+\frac{x\,(2)}{p_a}\,[(1+x\,(2))\,a\,\mathrm{sh}\,a+c\,(\mathrm{ch}\,a-1)]+$$
$$+\frac{a^2}{16p_a^2}\left[a\left(1+(x\,(2))^2\,(2a+\mathrm{sh}\,2a)+\frac{c^2}{a}\,(-2a+\mathrm{sh}\,2a)+\right.\right.$$
$$\left.\left.+2c\,(1+x\,(2))\,(\mathrm{ch}\,(2a)-1)\right)\right]. \qquad (7.14)$$

Therefore the numerical solution to parameters $a, p_a, c, x(2)$ is somewhat different than stated in the paper, namely

$$a=0.924,\ 2p_a=-5.8388,\ x\,(2)=0.2314,\ c=-1.0709$$

and the optimal control is

$$u\,(t)=-1.0709\,\mathrm{ch}\,(0.924\,t)+1.1378\,\mathrm{sh}\,(0.924\,t)-1,\quad t\in[0,1]$$
$$u\,(t)=-1.0221\,\mathrm{ch}\,(0.924\,t-1))+0.9896\,\mathrm{sh}\,(0.924\,(t-1)),\quad t\in(1.2].$$

A comment

The author wishes to thank Doc. K. Malanowski for showing an example suggesting that Theorem 4.1 of this work can be improved, that is, it can be proven that the adjoint variable $p\,(t)$ is continuous in t.

In fact, Theorem 4.1 may preserve its structure but the following modifications are possible:

(i) $\mu\,(\cdot)$ is nonincreasing and right-continuous on $[0, T+h_s)$, equals zero on $[T, T+h_s]$ and constant on $[0, T-h_s)$ and on subintervals on which $|x^o\,(t)-\xi\,(t)|<\varepsilon$.

(ii) The adjoint variable p is absolutely continuous on $[0, T+h_s]$.

The remaining conditions including the equation and boundary conditions for p and maximum conditions for hamiltonian are as previously.

Sketch of the proof: It can be verified by direct substitution that the following modification of Theorem 2.1 holds.

Modification of Theorem 1: If a nontrivial quantuple $(\alpha_0, \alpha_1, \alpha_2, \lambda, \psi)$ satisfies necessary optimality conditions of Theorem 2.1 for the solution (u^o, x^o) then the quantuple $(\alpha_0, \alpha_1, \alpha_2, \Lambda, \Psi)$, where $\Lambda(t)=\lambda(t)+c$, c arbitrary vector from R^l, $\Psi(t)=\psi(t)+g_x(x^o(t), t)\,c$ satisfies similar but slightly modified conditions, namely

(a_1) The value of $\lambda(t_2)=c$ is arbitrary.

(b_1) In the nontriviality condition $|\Lambda(t_1)-\Lambda(t_2)|$ is substituted in lien of $\lambda(t_2)$.

(c_1) Both hamiltonian and adjoint equation for Ψ are of the same form as for ψ.

(d_1) Boundary condition for $\Psi(t_1)$ analogical but we add to $\Psi(t_2)$ a term $g_x(x^o(t_2), t_2)\,\Lambda(t_2)=g_x(^o)\,c$.

(e_1) Maximum condition for hamiltonian of the same form.

This modification enables one for changing boundary conditions for subvectors $\psi_i(0)$, $\psi_i(h)$ in Lemma 4.1. By introducing an artificial constraints $|x(t)-\xi(T-h_s)|^2 \leqslant K \;\forall\; t \in [0, T-h_s]$ where $K \geqslant \sup\{|x^o(t)-\xi(T-h_s)|^2 : t \in [0, T-h_s,\}$, we obtain the existence of multipliers $\lambda_i(\cdot)$, $i=1, \ldots, k$, for which the values $\lambda_i(h)=c_i$ can be arbitrarily chosen and such that $\lambda_i(\cdot)$ is a constant for $i=1, \ldots, k-k_s$. Setting $c_k=0$, $c_{i-1}=\lambda_i(0)$, $i=2, \ldots, k$, yields $\psi_k(h)=0$, $\psi_{i-1}(h)=\psi_i(0)$, $i=2, \ldots, k$. Defining p and μ as previously $p((i-1)h+t)=\psi_i(t)$, $\mu((i-1)h+t)=\lambda_i(t)$, $t \in [0, h]$, $i=1, \ldots, k$, we check that $p(t)$ is absolutely continuous on $[0, T+h_s]$ and μ is right continuous and nonincreasing on this interval, constant on $[0, T-h_s]$ and on subintervals of $[T-h_s, T]$ for which $|x^o(t)-\xi(t)|<\varepsilon$, and zero function on $[T, T+h_s]$.

kazówki dla Autorów

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4. References should be put on the separate page. Numbers in the text identified by references should be enclosed in brackets. This should contain the surname and the initials of Christian names, of the author (or authors), the complete title of the work (in the original language) and, in addition:

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