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Stochastitc Optimal Control of a Multi-Facility, Multi-Product Production Scheduling with Rondom Times of Supplies

by

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A multi-facility, multi-product production scheduling problem with random times of supplies of raw materials and purchased parts is considered over a finite planning horizon. The supplies occur at random time points but for each raw material and purchased part the supply which replenish its beginning shortage occurs not later than at a given time point. The horizon consists of a discrete production periods during each of which at most one product can be assigned to each facility. Product cumulative demands for the entire planning horizon are known in advance. All demands must be met without allowing backorders. The problem objective is to determine an assignment of products to facilities over the horizon which maximizes the facilities utilization, in particular minimizes the expected completion time. The problem is formulated as a stochastic control problem for which efficient solution algorithm produced by combination of heuristic and dynamic programming strategies is given.

1. Introduction

The problem of determining a minimum-cost schedule of production has been studied extensively in the management sciences literature over the past two decades. The terminology production scheduling or planning is often employed when the cost structure of the model includes only direct production and inventory cost elements, e.g. [3, 6]. If the model further includes costs associated with changes in either production levels or the rate of production, the analysis is usually refered to as a production "smoothing" problem, e.g. [18]. When smoothing costs are tied to employment levels of direct labour the model is sometimes called "work force balancing" or "employment smoothing", e.g. [7].

Most of the reported research has been critically reviewed by Baker [2, 3], Conway et al. [5], Elmaghraby [6], Holt et al. [7], Potrzebowski [13]. Almost all of these studies are devoted for deterministic production scheduling problems and the computational algorithms are based on discrete programming methods.

Probabilistic production planning models with exogenous stochastic inputs (demand, prices), convex performance criteria and compact, convex feasible control regions were developed by Kleindorfer and Glover [8], Pekelman [12], Sobel [18, 19]. Results based on dynamic programming [1] and the stochastic maximum principle [10] are the only approach to date.

Kleindorfer et al. [9] have shown that many of the previous models which had appeared in the literature on production planning (scheduling, smoothing, and work force balancing) could be characterized as special cases of the control theory problem formulation. This more general model relaxed a number of assumptions required by other formulations in capturing a wide spectrum of production planning policies. Moreover, optimal control approach to production planning problems provides the new and efficient algorithms, eg. [4], [15]. Finally, recent advances in stochastic optimal control theory offer important avenue of application to probabilistic production planning problems.

The purpose of this paper is to present an optimal control approach to the probabilistic problem, of determining a production schedule which maximizes the facilities utilization, in particular minimizes the completion time of production (otherwise known as total processing time, make-span, elapsed time, maximum flow time, total duration etc.) under random supply times of raw materials and purchased parts. A computational procedure based on the composite algorithm produced by combination of heuristic and dynamic programming strategies is also provided and the solution results are indicated for numerical example.

2. Problem Development of Formulation

Consider an industrial process made up of m facilities in network where there are n different products (fabricated parts, subassemblies and finished products) to be produced over a finite planning horizon H (week, month, year for example). The horizon is made up of N production periods which in general have unequal duration h_k, $k=1, \ldots, N$, where $\sum_{k=1}^{N} h_k \leqslant H$. During any production period the assignment of products to the facilities is considered fixed, and at most one product can be scheduled on each facility. Set-ups are assumed to occur between production periods. The cumulative forecasted demand for all products for the entire planning horizon is perfectly known in advance and equals x_i^f units of each product i. The beginning inventory (z^0) of raw materials and purchased parts is assumed to be lower than total requirement generated by the demand for all products, and the supplies occur at random time points of the planning horizon. For each raw material or purchased part the beginning shortage is completely replenished by one supply which occurs not later than at a given time point. For the reason of uncertain supplies and varying rate of production the inventory level of raw materials and purchased parts fluctuates. The lack of synchronization between the raw materials supplier and the production process schedule may cause the raw materials inventory to become lower than planned. The effect of the stock-out of raw materials or purchased parts is the out-of-stock lost of production time resulting from:

— the additional set-ups when it is necessary to change the production process over to another products until the needed material arrives, and then change back;

— the process interruption when we run out of critical material without which our production process must cease.

The purpose of the production scheduling is to determine the total number N of production periods, they time duration h_k ($k=1, ..., N$), and the assignment of specific operations to specific facilities in each period so as to meet all product cumulative demands and maximize the expected utilization of facilities.

The industrial process is assumed to be a complex of n different operations O_l ($l=1, ..., n$), each in the form of mapping:

$$O_l: L \to l \quad (l=1, ..., n) \tag{1}$$

where: $L=\{L_1, ..., L_n\}$, L_l — a set of raw materials, purchased parts and products processed directly into the product l.

The total number n of different operations O_l is equal to the total number of different products so that only one operation corresponds to each product. The relationships between operations are characterized by the (n, n) consumption matrix D representing simple (direct) requirements for the fabricated parts and subassemblies:

$$D=[d_{il}], \quad (i=1, ..., n;\ l=1, ..., n) \tag{2}$$

$$d_{il} \begin{cases} \geqslant 0, & i=1, ..., n-p;\ l=1, ..., n \\ =0, & i=n-p+1, ..., n;\ l=1, ..., n \end{cases}$$

where the nonzero element d_{il} [unit of product i/unit of product l] is defined as the number of units of product i required to produce one unit of product l; p denotes the total number of different finished products, $(n-p)$ — the total number of different fabricated parts or subassemblies.

Raw materials and purchased parts direct requirement is represented by (s, n) matrix G:

$$G=[g_{rl}], \quad (r=1, ..., s;\ l=1, ..., n) \tag{3}$$

where: g_{rl} is the amount of raw material or purchased part r used directly to make one unit of product l; s denotes the total number of different kinds of raw materials and purchased parts.

Successive supplies of raw materials and purchased parts are considered as random input variables

$$w^k=[w_1^k, ..., w_s^k] \ w^k \in R^s, \ k=1, ..., N \tag{4}$$

where W_r^k is the amount of raw material or purchased part r supplied in period k.

The m facilities are characterized by the (n, m) matrix P of production rates:

$$P=[p_{ij}], \quad (i=1, ..., n;\ j=1, ..., m) \tag{5}$$

where the element p_{ij} [units of product i/unit time]$\geqslant 0$ denotes number of units of product i produced by facility j per unit time.

In a similar way we can represent by zero-one matrix Q, the technological restrictions of the m facilities:

$$Q=[q_{ij}], \quad (i=1, \dots, n[\; j=1, \dots, m) \tag{6}$$

where the element q_{ij} is defined as follows:

$$q_{ij}=\begin{cases} 1 \text{ if facility } j \text{ can be used to produce product } i \\ 0 \text{ otherwise} \end{cases}$$

From the technological restrictions described by the above matrix Q results that at most

$$\sum_{j=1}^{m} q_{ij}=m_i \tag{7}$$

facilities can be used to produce product i simultaneously, and at most

$$\sum_{j=1}^{m} \max_{i \in I_k} (q_{ij})=m_{I_k}, \; k \in \left\{1, \dots, \sum_{j=2}^{m} \binom{n}{j}\right\} \tag{8}$$

facilities can be used to produce simultaneously all products $i \in I_k \subset I$, where I_k denotes the nonempty subset of at most m products, I — the finite set of the first n integers.

The assignment of products to facilities in each period k is described by the (n, m) matrix Q_k:

$$Q_k=[q_{ij}^k], \tag{9}$$

where

$$q_{ij}^k \in \{0, 1\}, \; q_{ij}^k \leqslant q_{ij}, \; \sum_{i=1}^{n} q_{ij}^k \leqslant 1, \; \forall i, j, k. \tag{10}$$

The matrix Q_k is known as the incidence matrix for the covering problem.

Each assignment matrix Q_k defines a column vector u^k of production capacity allocation:

$$u^k=[u_1^k, \dots, u_n^k], \; u^k \in R^n, \; k=1, \dots, N \tag{11}$$

where

$$u_i^k=\sum_{j=1}^{m} q_{ij}^k p_{ij} \tag{12}$$

is the total production rate of the facilities assigned to product i in period k.

3. State Variable Description of Production-Inventory System

The discussion of the production scheduling control problem can be notationally and conceptually simplified by adopting a vector space notation.

The state of the production process at time

$$t_k=\sum_{j=1}^{k} h_j \tag{13}$$

is defined as the column vector x^k of the state variables:

$$x^k = [x_1^k, \dots, x_n^k],\ x^k \in R^n,\ k = 1, \dots, N \tag{14}$$

where x_i^k is the cumulative production of product i up to time t_k. The variables $x_1^k, \dots, x_{n-p}^k$ represent the cumulative production of the fabricated parts and subassemblies, and the variables $x_{n-p+1}^k, \dots, x_n^k$ represent the cumulative production of finished products.

The state of in-process inventory (fabricated parts and subassemblies) and finished products inventory is described by the output vector y^k (Fig. 1):

$$y^k = [y_1^k, \dots, y_n^k],\ y^k \in R^n,\ k = 1, \dots, N \tag{15}$$

where y_i^k is the ending inventory of product i for period k.

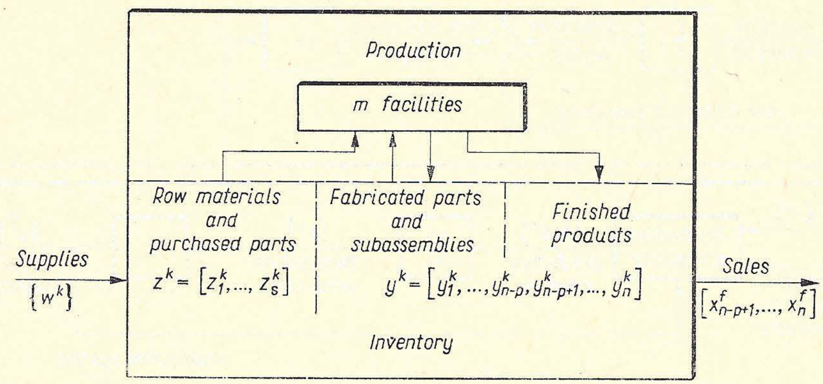

Fig. 1. Production-inventory system

Similarly, the state of the inventory of raw materials and purchased parts at time t_k is defined as the column vector

$$z^k = [z_1^k, \dots, z_s^k],\ z^k \in R^s,\ k = 0, 1, \dots, N-1 \tag{16}$$

where z_r^k is the beginning inventory of raw material or purchased part r for period $k+1$.

The control variables of the problem are the total production rates u_i^k of the facilities assigned to each product i in each period k.

The state and output equations of the production-inventory system (Fig. 2) can be written as follows:

$$x^k = x^{k-1} + h_k u^k,\ k = 1, \dots, N \tag{17}$$

$$z^k = z^{k-1} - G(x^k - x^{k-1}) + w^k,\ k = 1, \dots, N \tag{18}$$

and

$$y^k = Cx^k \tag{19}$$

where

$$x^o = [0, \dots, 0) \in R^n,\ z^o = [z_1^o, \dots, z_s^o]$$

$$x^k \in X_k \subset R^n,\ u^k \in U \subset R^n$$

U — a finite set of facilities capacity admissible allocations, X_k — a set of admissible states of the production process, determined by the precedence relationships between the operations, and the raw materials inventory constraints;

$$X_k(x^{k-1}, z^{k-1}) = \{x \in R^n: Dx \leqslant x^{k-1}, Gx \leqslant Gx^{k-1} + z^{k-1}, x \geqslant 0\} \tag{20}$$

C — the output matrix defined as

$$C = J - D \tag{21}$$

where J — the identity matrix.

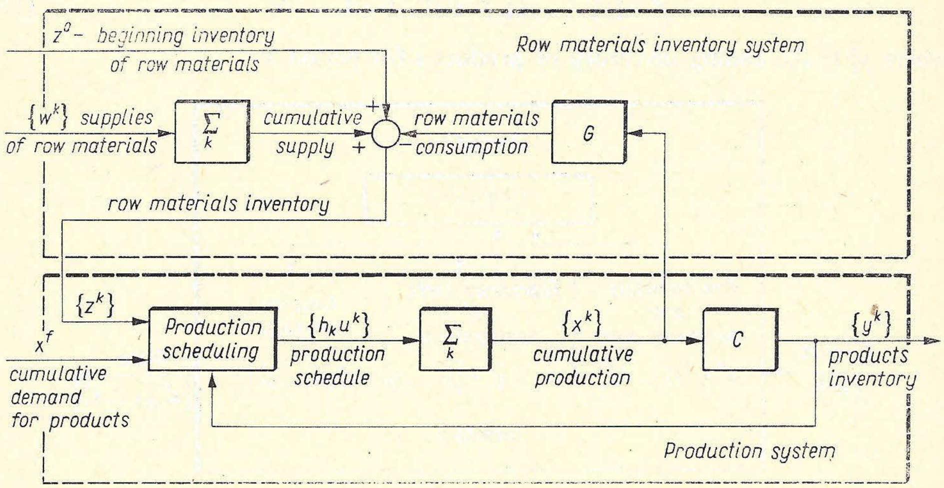

Fig. 2. Flow-diagram: production-inventory system

As we start production at time 0 and want during the planning horizon H to have produced x_i^f units of each product i, the following conditions (22) must be satisfied with probability one:

$$x_i^N = x_i(t_N) = x_i^f \text{ for } t_N = \sum_{k=1}^{N} h_k \leqslant H. \tag{22}$$

For the reason of random times of supplies w^k, the quantities z_r^k, $r = 1, \ldots, s$, are random variables of a discrete s-dimensional Markov process $\{z^k(\omega), k = 1, \ldots, N\}$ with the range Z. The elementary sample ω is the sequence $\{w^1, w^2, \ldots, w^N\}$.

At each time t_{k-1} the state x^{k-1} must lie in the set X_{k-1}. In order that $x^k \in X_k$, the control $u^k(x^{k-1}, z^{k-1}(\omega))$ and duration h_k of period k must be chosen so that

$$x^{k-1} + h_k u^k(x^{k-1}, z^{k-1}(\omega)) \in X_k(x^{k-1}, z^{k-1}(\omega)). \tag{23}$$

The condition (23) implies the following constraints on the value of variables u^k, h_k:

$$u^k \in U_k = \{u \in R^n; u \in U, u_i = 0 \ \forall i \in I_{zk}, u_l = 0 \ \forall l \in L_{yk}\} \tag{24}$$

where: $I_{zk}=\{i: z_r^{k-1}=0, g_{ri}>0\}$ — set of products which cannot be produced in period k because of raw materials or purchased parts stock-out; $L_{yk}=\{l: y_i^{k-1}=0, d_{il}>0\}$ — set of products which cannot be produced in period k, because of fabricated parts or subassemblies stock-out.

$$h_k \leqslant \min (h_{zk}, h_{yk}) \tag{25}$$

where:

$$h_{zk}=\min_r \left\{ \frac{z_r^{k-1}}{\sum_{i=1}^{n} g_{ri}\, u_i^k} \quad \text{for} \quad \sum_{i=1}^{n} g_{ri}\, u_i^k \neq 0 \right\}, \tag{26}$$

$$h_{yk}=\min_i \left\{ \frac{y_i^{k-1}}{\sum_{l=1}^{n} d_{il}\, u_l^k} \quad \text{for} \quad \sum_{l=1}^{n} d_{il}\, u_l^k \neq 0 \right\}. \tag{27}$$

We call admissible any schedule given by the sequence of pairs (u^k, h_k), $k=1, \ldots, N$, in which the controls $u^k(x^{k-1}, z^{k-1}(\omega))$ are defined on the Cartesian product of the set X_k and the range Z of Markov process $z^k(\omega)$, and which satisfy (22), (24), (25).

In the sequel our formulation will deal with the production scheduling problem under the following additional assumptions:

— the m_i facilities are identical in that the product i can be produced by any of m_i facilities at a fixed production rate of p_i units per unit time, i.e.

$$p_{ij}=q_{ij}\, p_i \;\; \forall i; \tag{28}$$

— the delay times due to dependent operations are negligible so that it is possible to consider that all operations can be made simultaneously.

The first assumption makes it possible to consider the facilities allocation vector with integer components as the control variable u. Each component u_i of vector $u \in R^n$ denotes the number of facilities assigned to product i. The finite set $U \subset R^n$ of admissible allocations is given by:

$$U=\Big\{u \in R^n: u_i \in \{0, 1, \ldots, m_i\}, i=1, \ldots, n;$$
$$\sum_{i \in I_k} u_i \leqslant m_{I_k}, I_k \subset I\Big\}. \tag{29}$$

The second assumption allows to approximate the minimum completion time t_f^* (in the deterministic problem) by the following formula, which gives the exact value of t_f^* only in the case of independent operations:

$$t_f^* = \max_{I_k} \frac{\sum_{i \in I_k} x_i^f}{m_{I_k}}. \tag{30}$$

The objective functional representing the utilization of facilities is given by:

$$E\left\{F_N(x^N) + \sum_{k=1}^{N} f_k(x^k, u^k)\right\} \tag{31}$$

where the expectation in (31) is with respect to the joint distribution of $\{w^k, k = 1, ..., N\}$.

Here are four examples of possible measures of the utilization of facilities, and respectively the functions f_k, F_N:

— the completion time:

$$f_k = h_k = \frac{\|x^k - x^{k-1}\|}{\|u^k\|}, \quad F_N = \frac{\|x^N - x^f\|}{m}; \tag{32}$$

— the facilities total production time:

$$f_k = h_k \|u^k\| = \|x^k - x^{k-1}\|, \quad F_N = \|x^N - x^f\|; \tag{33}$$

— the total lost of facilities production time (facilities total idle time):

$$f_k = h_k (m - \|u^k\|) = \|x^k - x^{k-1}\| \left(\frac{m}{\|u^k\|} - 1\right), \quad F_N = m\, \|x^N - x^f\|; \tag{34}$$

— the facilities total set-up time associated with changes of products to facilities assignment [26]:

$$\begin{gathered} f_k = \frac{1}{2}\sigma \left\{\|u^k - u^{k-1}\| - \big|\|u^k\| - \|u^{k-1}\|\big|\right\} \\ F_N = \frac{1}{2}\sigma \left\{\left\| u^N - \frac{x^N - x^f}{\|x^N - x^f\|}\, m\right\| - \big|m - \|u^N\|\big|\right\} \text{ if } x^N \neq x^f, \; F_N \equiv 0 \text{ if } x^N = x^f \end{gathered} \tag{35}$$

where σ — average one facility set-up time.

Vector norm $\|\cdot\|$ on R^n in (32), (33), (34), (35) denotes the l_1 norm and represents the rectangular distance in R^n:

$$\|x\| = \sum_{i=1}^{n} |x_i|, \; x \in R^n. \tag{36}$$

The optimal control for the above stochastic control problem is a function of the present state (x, z) of the production-inventory system. Therefore, the problem to be considered is the minimization of (31) over all state-control trajectories $\{(x^o, z^o, u^1), ..., (x^{N-1}, z^{N-1}, u^N), (x^N, z^N)\}$ for any given (x^o, z^o) satisfying (17), (18).

4. Comments on the Optimization Problem

Notice that if at each period $k = 1, ..., N$ there were a positive probability that $\sum_{j=1}^{k} w_r^j$ equaled 0 for $r \in \{r: \sum_{i=1}^{n} g_{ri}\, x_i^f - z_r^o > 0\}$, it would not be possible to satisfy (22) with probability one. Motivated by this let us assume that the beginning shortage of raw materials and purchased parts

$$b_r = \max\left(\sum_{i=1}^{n} g_{ri}\, x_i^f - z_r^o, 0\right), \; r = 1, ..., s \tag{37}$$

is replenished with probability one up to time τ_r (the end of some period k_r):

$$\operatorname{prob}\left(\sum_{k=1}^{k_r} w_r^k \geqslant b_r,\ t_{k_r}=\sum_{k=1}^{k_r} h_k=\tau_r\right)=1 \tag{38}$$

where τ_r denotes the latest time to replenish the shortage of raw material or purchased part r (e.g. the longest lead time from the vendor). From the time τ_r the inventory level z_r of the raw material r is with probability one not lower than the actual requirement generated by the actual cumulative demand for all products.

It may not be possible to satisfy conditions (22) if the beginning inventory z^o of raw materials and purchased parts is too low. There are worst conditions that:

— supplies of each raw material or purchased part r occur at the latest time τ_r (we assume that $\tau_r=0$ if $b_r=0$):

$$\sum_{k=1}^{k_r-1} w_r^k=0,\ \forall r\in\{r\colon b_r>0\}; \tag{39}$$

— the beginning inventory z^o is used up at maximum usage rate, so that the inventory level goes to zero in the shortest time $\bar{t}$, and at the same time the cumulative production goes to the state $\bar{x}$ given by the solution of the following minimax problem:

$$\min_x\left(\max_{I_k}\frac{\sum_{i\in I_k} x_i}{m_{I_k}}\,|Gx=z^o,\ x\geqslant 0,\ I_k\subset I\right). \tag{40}$$

In order to complete the planned production x^f before time H with probability one, one would have to be able to do this under the worst conditions. This is possible only if

$$v_r=\frac{\sum_{i\in I_r}(x_i^f-x_i^{k_r})}{H-\tau_r}\leqslant\min\left(m_{I_r},\ m-\sum_{j\in J_r} v_j\right) \tag{41}$$

and

$$\max_{I_r^\alpha}\frac{\sum_{i\in I_r^\alpha}(x_i^f-x_i^{k_r})}{m_{I_r^\alpha}}\leqslant H-\tau_r \tag{42}$$

where

$$x_i^{k_r}=x_i(\tau_r)=\bar{x}_i\min\left(\frac{\tau_r}{\bar{t}},\ 1\right),\ \forall i\in I_r=\{i\colon g_{ri}>0\} \tag{43}$$

$\bar{t}=\max_{I^\alpha}\dfrac{\sum_{i\in I^\alpha}\bar{x}_i}{m_{I^\alpha}}$ — the earliest time to use up the beginning inventory z^o of raw materials and purchased parts,

$I^\alpha\subset\bar{I}$ — the nonempty subset of at most m products $i\in\bar{I}=\{i\colon \bar{x}_i>0\}$,

$I_r^\alpha\subset I_r$ — the nonempty subset of at most m products $i\in I_r$,

$m_\beta = \sum_{j=l}^{m} \max_{i \in \beta} (q_{ij})$ — maximum number of facilities simultaneously available for products $i \in \beta$,

$$J_r = \{j; \tau_j < \tau_r\}.$$

Let us say that the production schedule is feasible if conditions (41), (42) hold.

We shall finish our comments on the optimization problem by showing that the terminal conditions (22) imply constraints on the values of variables at intermediate times.

The state of production at time $t_{k_r} = \tau_r$ is $x^{k_r} = \sum_{k=1}^{k_r} h_k u^k \in R^n$. Under the worst conditions it is possible to complete the planned production of products $i \in I_r$ in period $\tau_r \leqslant t \leqslant H$ at production rate not less than v_r. To assure that the entire planned production is completed before time H with probability one, the inequalities

$$\sum_{i \notin I_r} x_i^{k_r} \geqslant \sum_{i \notin I_r} x_i^f - (H - \tau_r) \min (m_{I_{\bar{r}}}, m - v_r) = x_{\tau r}, \forall r \tag{44}$$

must hold, where $m_{I_{\bar{r}}} = \sum_{i \notin I_r}^{m} \max (q_{ij})$ — maximum number of facilities simultaneously available for all products except $i \in I_r$.

The right-hand side of inequalities (44) can be interpreted as the minimal cumulative demand of products $i \notin I_r$ for the subhorizon τ_r.

If:

$$\sum_{i \notin I_r} u_i^k < \frac{\sum_{i \notin I_r} x_i^f - (H - \tau_r) \min (m_{I_{\bar{r}}}, m - v_r) - \sum_{i \notin I_r} x_i^{k-1}}{\tau_r - \sum_{j=1}^{k-1} h_j} = u_{kr}, \; k \leqslant k_r$$

the inequalities (44) give the equivalent constraints on the value of duration h_k:

$$h_k \leqslant h_{xk} = \min_{\left\{ \sum_{i \notin I_r} u_i^k < u_{kr}, k \leqslant k_r \right\}}^{r:} \left[\frac{m_{I_{\bar{r}}} \left(\tau_r - \sum_{j=1}^{k-1} h_j \right) - \left(x_{\tau_r} - \sum_{i \notin I_r} x_i^{k-1} \right)}{m_{I_{\bar{r}}} - \sum_{i \notin I_r} u_i^k} \right]. \tag{45}$$

It follows from the above comments that probabilistic aspect of the optimization problem will be completely determined if probability distribution of supply times for each raw material and purchased part is given.

5. Solution Algorithm

In this section we state the composite algorithm produced by combination of heuristic and dynamic programming strategies. The algorithm will form the basis for computation.

Using (17), (18) we may write the optimal return function suppressing time and conditioning arguments, as

$$V_k(x, z) = \min_{\{(u,h): u \in U_k^o, h \in H_k\}} \{E[V_{k+1}(x+hu, z-Ghu+w^k)|x^{k-1} = \\ = x, z^{k-1}(\omega) = z, u^k = u, h_k = h] + f_k(x, u)\}, \ k = 1, \dots, N \tag{46}$$

$$V_{N+1}(x, z) = E[F_N(x)|z^{N-1}(\omega) = z].$$

The heuristic procedure is based upon the optimal control algorithm derived for deterministic flow-shop problem [15], and is used to determine the sets U_k^o, H_k of facilities allocation in period k, and its duration, respectively.

Formally, the heuristic procedure in period k is as follows:

Step 1. Determine the facilities pseudo-allocation (not integer) vector $\tilde{u}^k \in \operatorname{conv}(U_k)$ of maximum l_1 norm, generating rectilinear, minimal-time trajectory from the given state x^{k-1} to state x^{kf}, where:

$$x_i^{kf} = \begin{cases} x_i^f, \ i \in J_k = \{i: x_i^{k-1} < x_i^f, \ i \notin I_{zk} \cup L_{yk}\}. \\ x_i^{k-1}, \ i \notin J_k \end{cases} \tag{47}$$

Notice that $(x^{kf} - x^{k-1})$ is feasible direction (i.e. direction satisfying the raw materials, purchased parts and inprocess inventory constraints for period k), the "closest" to direction $(x^f - x^{k-1})$ of the trajectory terminating in the desired state x^f.

Step 2. In the set U_k find subset U_k^o of admissible allocations u^{ok} "closest" (according to l_1 norm) to $\tilde{u}^k$, (u^{ok} is often unique).

Step 3. Determine the set H_k of period k admissible durations, taking into consideration the beginning for period k raw materials, purchased parts and in-process inventory level, the neariest supply time τ_r, and the constraint (45) implied by terminal conditions (22).

Sets U_k^o and H_k are defined as follows:

$$U_k^o = \{u^o \in U_k \subset R^n: \|u^o - \tilde{u}^k\| = \min_{\substack{u: u \in U_k \\ \|u\| = m_{J_k}}} \|u - \tilde{u}^k\|\} \tag{48}$$

where:

$$\tilde{u}_i^k = \begin{cases} \dfrac{x_i^f - x_i^{k-1}}{\sum\limits_{i \in J_k} (x_i^f - x_i^{k-1})} m_{J_k}, \ i \in J_k, \\ 0, \ i \in J_k \end{cases} \tag{49}$$

$$m_{J_k} = \sum_{j=1}^{m} \max_{i \notin J_k} (q_{ij}),$$

$$H_k = \left\{ h \geq 0: h = \begin{cases} \gamma \underline{h} \leq h_{k\,max}, \ \gamma = 1, 2, \dots \text{ if } \underline{h} < h_{k\,max}, \\ h_{k\,max} \text{ if } h_{k\,max} \leq \underline{h} \end{cases} \right\}, \tag{50}$$

where:

$$h_{k\,max} = \min\left[h_{xk}, h_{yk}, h_{zk}, \min_r \left(\tau_r - \sum_{j=1}^{k-1} h_j > 0\right)\right], \tag{51}$$

h — minimal duration of production period (e.g. 1 shift duration).

The above heuristic procedure is applied to eliminate successively regions of the set U_k. If, as it often happens, u^{ok} is unique, dynamic programming is used only to determine durations h_k of production periods in such a way as to avoid the excessive lost of production time when some facilities are kept idle awaiting the needed material arrival. In that case the shorter is minimal duration $\boldsymbol{h}$ of production period, the closer to optimal is the schedule obtained by the above algorithm.

In general case, like most dynamic programming algorithms the above procedure is useful only for problems of limited size due to the fact that storage and computation requirements grow very rapidly as the difference $(n-m)$ and the number N of production periods increase.

6. Example

Time-optimal schedule was computed for the following simple example: $n=m=s=2$, $H=100$, $\boldsymbol{h}=5$, $\tau_1=50$, $\tau_2=25$.

The cumulative demands for the planning horizon H:
$x_1^f=80$ [machine-hours of production of product 1],
$x_2^f=100$ [machine-hours of production of product 2].

Raw materials requirement matrix (3):

$$G=\begin{bmatrix} g_1 & 0 \\ 0, & g_2 \end{bmatrix}, \; I_1=\{1\}, \; I_2=\{2\}$$

where:

$g_1=2$ [units of raw material 1/machine hour of production of product 1],
$g_2=5$ [units of raw material 2/machine-hour of production of product 2].
The beginning inventory of raw materials:

$$z_1^o=80, \; z_2^o=100.$$

The product-facility incidence matrix (6):

$$Q=\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \; m_{I_1}=1, \; m_{I_2}=2.$$

The raw materials supply probability:

$$\text{prob}\left(\sum_{j=1}^{k} w_r^j > 0\right) = p_r^k = \min\left(\frac{t_k}{\tau_r}, 1\right), \; r=1, 2$$

$$\text{prob}\left(\sum_{j=1}^{k} w_r^j = 0\right) = 1 - p_r^k = \max\left(\frac{\tau_r - t_k}{\tau_r}, 0\right), \; r=1, 2.$$

The schedule feasibility test (41), (42):

$$\bar{x}_1=\bar{x}_2=40,\ \bar{t}=40,$$

$$x_1(\tau_1)=x_1^{k_1}=\bar{x}_1\min\left(\frac{\tau_1}{\bar{t}},1\right)=40,\ x_2(\tau_2)=x_2^{k_2}=\bar{x}_2\min\left(\frac{\tau_2}{\bar{t}},1\right)=25$$

$$v_2=\frac{x_2^f-x_2^{k_2}}{H-\tau_2}=1\leqslant\min(m_{I_2},m)=2,$$

$$v_1=\frac{x_1^f-x_1^{k_1}}{H-\tau_1}=\frac{4}{5}\leqslant\min(m_{I_1},m-v_2)=1,$$

$$\frac{x_1^f-x_1^{k_1}}{m_{I_1}}=40\leqslant H-\tau_1=50,$$

$$\frac{x_2^f-x_2^{k_2}}{m_{I_2}}=37.5\leqslant H-\tau_2=75.$$

The minimal cumulative demands of products 1, 2 for the subhorizons, respectively τ_2, τ_1 (44):

$$x_1^{k_2}\geqslant x_{\tau_2}=x_1^f-(H-\tau_2)\min(m_{I_1},m-v_2)=5,$$

$$x_2^{k_1}\geqslant x_{\tau_1}=x_2^f-(H-\tau_1)\min(m_{I_1},m-v_1)=40.$$

Time-optimal schedule for this example is:

$$k=1,\ u_1^1=u_2^1=1,\qquad h_1=25$$

$$k=2,\ u_1^2=u_2^2=1,\qquad h_2=15$$

$k=3$, a) if supply of raw material 2 occurred for $0\leqslant t\leqslant 40$:

$$u_1^3=u_2^3=1,\qquad h_3=40,$$

b) if supply of raw material 2 didn't occur for $0\leqslant t\leqslant 40$:

$$u_1^3=1,\ u_2^3=0,\ h_3=\boldsymbol{h}=5,$$

$k=4$, a) if supply of raw material 2 occurred for $0\leqslant t\leqslant 40$:

$u_1^4=0,\ u_2^4=2,\quad h_4=10$. The end of production.

b) if supply of raw material 2 didn't occur for $0\leqslant t\leqslant 45$:

$$u_1^4=1,\ u_2^4=0,\qquad h_4=\boldsymbol{h}=5,$$

c) if supply of raw material 2 occurred for $40<t\leqslant 45$:

$$u_1^4=u_2^4=1,\qquad h_4=35,$$

$k=5$, b) if supply of raw material 2 didn't occur for $0\leqslant t\leqslant 45$:

$$u_1^5=u_2^5=1,\qquad h_5=30$$

c) if supply of raw material 2 occurred for $40<t\leqslant 45$:

$$u_1^5=0,\ u_2^5=2,\qquad h_5=12.5.\ \text{The end of production.}$$

$k=6$, b) if supply of raw material 2 didn't occur for $0\leqslant t\leqslant 45$:

$$u_1^6=0,\ u_2^6=2,\qquad h=15.\ \text{The end of production.}$$

7. Conclusions

A multi-facility, multi-product production scheduling probabilistic problem, formulated as an optimal control problem has been considered. To analyze it under several realistic assumptions, a composite heuristic — dynamic programming algorithm was given, and in a simple case used to obtain the optimal solution. An important step in attacking the problem was to determine the schedule feasibility test, and to show how the terminal conditions implied constraints on intermediate values of control variables. The generalization of the presented model to allow for the inclusion of other realistic random factors appears to be an interesting area for future research.

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Stochastyczne sterowanie optymalne harmonogramem produkcji o wielu stanowiskach i wielu wyrobach z przypadkowymi czasami dostaw

Rozpatruje się zagadnienie harmonogramu produkcji o wielu stanowiskach i wielu wyrobach z przypadkowymi czasami dostaw surowców i zakupionych części przy skończonym horyzoncie planowania. Dostawy mają miejsce w przypadkowych chwilach czasu, ale dla każdego surowca i zakupionej części dostawa uzupełniająca ich początkowy niedobór ma miejsce nie później niż w danej chwili czasu. Horyzont składa się z dyskretnych okresów produkcji, podczas każdego z których co najwyżej jeden wyrób może być przydzielony do każdego stanowiska. Skumulowane zapotrzebowania na wyroby dla całego horyzontu planowania są znane z góry. Wszystkie zapotrzebowania muszą być zrealizowane, niedopuszczalne jest odrzucanie zamówień. Zagadnienie polega na określeniu przydziału w czasie wyrobów do stanowisk, a w szczególności minimalizuje oczekiwany czas zakończenia. Zagadnienie jest sformułowane jako problem sterowania stochastycznego. Podano efektywny algorytm, i jego rozwiązania, otrzymany przez połączenie strategii programowania heurystycznego i dynamicznego.

Стохастическое оптимальное управление графикам производства, со многими рабочими местами и многими изделиями, со случайным временем поставок

Рассматривается проблема графика производства со многими рабочими местами и многими изделиями, со случайным временем поставок сырья и закупаемых частей при конечном горизонте планирования. Постановки имеют место в случайные моменты времени, однако для каждого сырья и закупаемой части поставка, дополняющая их начальную недостачу, реализуется не позже, чем в данный момент времени. Горизонт состоит из дискретных периодов производства, в течение каждого из которых не более, чем одно изделие может быть отведено каждому рабочену месту. Общее потребление на изделия для полного горизонта планирования заведомо известно. Все заявки должны быть реализованы и недопускаются отказы на заявки. Задача состоит в определении распределения во времени изделий по рабочим местам, а в частности минимизации ожидаемого времени окончания. Задача формулируется в виде проблемы стохастического управления. Дан эффективный алгоритм и его решения, получаемый путем объединения стратегии эвристического и динамического программирования.