Control
and Cybernetics
VOL. 6 (1977) No. 3-4

On regularity of solutions to convex optimal control problems with control constraints for parabolic systems

by

IRENA LASIECKA, KAZIMIERZ MALANOWSKI
Polish Academy of Sciences
Systems Research Institute
Warsaw, Poland

An optimal control problem for a system described by linear parabolic equation is investigated. Cost functional is convex and control functions are subject to constraints of local type. The Lagrange formalism is used to such a problem. It is shown that the Lagrange multipliers corresponding to constraints of control are some regular functions. Some regularity results for optimal controls and trajectories are derived.

1. Introduction

To estimate the rate of convergence of finite dimensional approximations to continuous optimal control problems the Lagrange formalism is used [5, 9, 10, 14]. It turns out [5] that to obtain an effective estimation of such a rate of convergence the optimal variables of an appropriate Lagrangian, both primal and dual, must be regular enough.

Very few has been done in the field of investigating the regularity of optimal solutions and in particular the regularity of optimal Lagrange multipliers. We have to mention here the pioneering papers by Hager [5, 6, 7].

In particular Hager used stability results for finite dimensional quadratic programming problems to investigate the regularity of optimal solutions for systems described by ordinary differential equations subject to control and state constraints.

As far as systems with distributed parameters are concerned, the regularity of optimal control for elliptic equations was discussed in [3].

The authors do not know any results concerning the regularity of Lagrange multipliers (other than adjoint equations) for such systems.

In this paper an optimal control problem for a system described by linear parabolic equation with control in the domain is considered. It is assumed that the cost functional is strictly convex and that control functions are subject to constraints of the local type.

Regularity of solution to such a problem is investigated.

In Section 3 a Lagrangian is introduced. It is shown that under some additional assumptions, Lagrange multipliers corresponding to constraints of controls are some regular functions and that the Lagrangian assumes its saddle point at the solution of optimal control problem.

In Section 4 the results of regularity for solutions of the state and adjoint equations are discussed. It is shown that this regularity depends on regularity of optimal controls. For particular case of constraints of amplitude type the needed regularity of the optimal control is proved.

The obtained results of the regularity will be used in the forthcomming paper of the authors [10] to investigate a finite dimensional approximation to optimal control problems.

Note that the problem of the regularity of optimal controls for parabolic systems in the case of state space constraints is an open one and seems to be hard.

Some used notations

$R^s = s$-dimensional Euclidean space

$\langle v, w \rangle$, $\|v\|$ = the inner product and the norm in R^s

$L^2(\Omega) = H^0(\Omega)$ = the space of vector-valued functions, square integrable on Ω

$(u, v) = \int_\Omega \langle u(x), v(x) \rangle \, dx$, $\|u\|_0 = (u, u)^{\frac{1}{2}}$ = the inner product and the norm in $H^0(\Omega)$

$H^r(\Omega)$ — Sobolev space of vector-valued functions, square integrable together with their (weak) derivatives up to the order r with the inner product

$$(u, v)_r = \sum_{|i|=0}^{r} (D^i, u, D^i v)$$

and the norm

$$\|u\|_r = (u. u)_r^{\frac{1}{2}}$$

where

$$D^i(u) = \frac{\partial^{|i|} u}{\partial x_1^{i_1} \partial x_2^{i_2}, \ldots, \partial x_n^{i_n}}, \quad |i| = i_1 + i_2 + \ldots + i_n .$$

By $H^r(\Omega)$ we denote the appropriate Sobolev space of scalar-valued functions

$L^2(0, T; Z)$ — the space of functions with range in a Banach space Z, square integrable on $[0, T]$,

$((y, z)) = \int_0^T (y(t), z(t))_Z \, dt$; $\|y\|_{L^2(Z)} = ((y, y))^{\frac{1}{2}}$ — the inner product and the norm in $L^2(0, T; Z)$,

$C(0, T; Z)$ — the space of functions continuous from $[0, T]$ into Z with the norm

$$\|y\|_{C(Z)} = \max_{t \in [0, T]} \|y(t)\|_Z ,$$

$\mathscr{L}(Y, Z)$ — the space of linear bounded operators from a Banach space Y into a Banach space Z.

$\delta_u J(u, y)$ — strong (Frechet) derivative of functional $J(u, y)$ with respect to u,
$\delta_{uy}^2 J(u, y)$ — second strong derivative of functional $J(u, y)$ with respect to u and y,
$\delta^2 J(u, y)$ — Hessian of functional $J(u, y)$.

2. Statement of optimal control problem

Let Ω be a bounded domain (open set) in R^n with properly regular boundary $\partial\Omega$. For the sake of simplicity we shall assume that $\partial\Omega$ is of class C^∞. Moreover we assume that locally Ω is situated on one side of $\partial\Omega$.

Let T be a fixed time.

We consider the system described in the cylinder $\Omega \times (0, T)$ by the following parabolic equation (state equation)

$$\frac{\partial y(x, t)}{\partial t} - Ay(x, t) = f(x, t) \tag{2.1}$$

where f is a function of an appropriate regularity defined on $\Omega \times (0, T)$.

Elliptic self-adjoint operator A is given by

$$Ay(x) = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_j}\left(a_{ij}(x)\frac{\partial y(x)}{\partial x_i}\right) - a_0\, y(x) \tag{2.2}$$

where the functions $a_{ij}(\cdot) = a_{ji}(\cdot)$ and $a_0(\cdot)$ are properly regular (for the sake of simplicity of class C^∞) and there exists such a constant $\rho_0 > 0$ that

$$\sum_{i,j=1}^{n} a_{ij}(x)\,\xi_i\,\xi_j \geqslant \rho_0 \sum_{i=1}^{n} \xi_i^2 \qquad \forall x \in \Omega, \quad \forall \xi_i \in R^1 \tag{2.3a}$$

and

$$a_0(x) \geqslant \rho_0 \qquad \forall x \in \Omega. \tag{2.3b}$$

For (2.1) the following homogeneous boundary conditions of Neumann type are satisfied

$$\frac{\partial y(\sigma, t)}{\partial \eta_A} = \sum_{i,j=1}^{n} a_{ij}(\sigma)\frac{\partial y(\sigma, t)}{\partial x_j}\cos(\eta, \sigma_i) = 0 \qquad \forall \sigma \in \partial\Omega,\ \forall t \in [0, T] \tag{2.4}$$

where η is the unit outward normal to $\partial\Omega$.

Moreover the initial condition

$$y(x, 0) = y^p(x) \text{ for almost all } x \in \Omega \tag{2.5}$$

is satisfied, where y^p is properly regular function defined on Ω.

The problem of regularity of solutions to (2.1) will be discuss in details in Appendix A. Here we quote only a result known from literature (cf. [2, 11]) which we shall use in the sequel:

LEMMA 2.1. *If*

$$y^p \in H^1(\Omega),\ f \in L^2\big(0, T; H^0(\Omega)\big) \tag{2.6}$$

then the solution y of (2.1) *satisfies the following conditions of regularity*

$$y \in C\big(0, T; H^1(\Omega)\big), \tag{2.7a}$$

$$\frac{dy}{dt} \in L^2\big(0, T; H^0(\Omega)\big), \tag{2.7b}$$

and the mappings

$$H^1(\Omega) \times L^2\big(0, T; H^0(\Omega)\big) \ni (y^p, f) \to y \in C\big(0, T; H^1(\Omega)\big), \tag{2.8a}$$

$$H^1(\Omega) \times L^2\big(0, T; H^0(\Omega)\big) \ni (y^p, f) \to \frac{dy}{dt} \in L^2\big(0, T; H^0(\Omega)\big) \tag{2.8b}$$

are continuous.

Let us define

$$(\mathscr{A}y)(x, t) \stackrel{\text{def}}{=} A(x)\, y(x, t) \tag{2.9}$$

and introduce the space

$$W^1(0, T) =$$

$$= \left\{ y \in L^2\big(0, T; H^1(\Omega)\big) : \mathscr{A}y \in L^2\big(0, T; H^0(\Omega)\big), \frac{dy}{dt} \in L^2\big(0, T; H^0(\Omega)\big) \right\}. \tag{2.10}$$

By Lemma 2.1 for any $f \in L^2\big(0, T; H^0(\Omega)\big)$ the solution of (2.1) belongs to $W^1(0, T)$.

In order to state the problem of optimal control we introduce the space U of control functions $\boldsymbol{u}$:

$$U = L^2(0, T; V) \tag{2.11}$$

where

$$V = \boldsymbol{H}^0(\Omega)\ ^{1)} \tag{2.11a}$$

is the space of functions with the range in R^q.

On the space R^q there is defined a r-dimensional vector function $\psi(w)$. It is assumed that ψ is convex, differentiable and its derivative satisfies the condition

$$\|\delta_u \psi(u_1) - \delta_u \psi(u_2)\| \leqslant c \|u_1 - u_2\| \qquad \forall u_1, u_2 \in R^q.\ ^{2)} \tag{2.12}$$

Moreover the set

$$W_{ad} = \{w \in R^q : \psi(w) \leqslant 0\} \tag{2.13}$$

is not empty:

$$W_{ad} = \emptyset. \tag{2.13a}$$

---

[1] Assumption (2.1a) was introduced only for the sake of simplicity of the notations. The obtained results are still valid if we put $V = L^2(S)$, where $S \subset R^m$ is some open and bounded set different from Ω. In particular if $m=0$, then control functions depend only on time.

[2] $\psi(w) \leqslant 0$ denotes $\psi_i(w) \leqslant 0$ $\forall i = 1, 2, \ldots, r$. Here and in the sequel c will denote a generic constant not necessarily the same in any two places.

In the space U we define the set U_{ad} of admissible control putting

$$U_{ad}=\{u \in L^2(0, T; V): u(x, t) \in W_{ad} \text{ for almost all } x \in \Omega, t \in [0, T]\}. \tag{2.14}$$

In (2.1) we put $f(t)=Bu(t)$, where

$$B \in \mathscr{L}(H^0(\Omega); H^0(\Omega)) \cap \mathscr{L}(H^1(\Omega); H^1(\Omega)). \tag{2.15}$$

Moreover we introduce the cost functional

$$J(u, y)=\int_0^T \int_\Omega \varphi(u(x, t), y(x, t))\, dx\, dt \tag{2.16}$$

where $\varphi(u, y)$ is a convex, twice differentiable function and the following conditions are satisfied

$$\|\delta^2_{u,u} \varphi(u, y)\|, \|\delta^2_{u,y} \varphi(u, y)\|, |\delta^2_{y,y} \varphi(u, y)| \leqslant C, \forall u \in R^q, \forall y \in R^1, \tag{2.16a}$$

$$v^r \delta^2_{u,u} \varphi(u, y)\, v \geqslant \alpha \|v\|^2, \ \alpha>0, \ \forall u, v \in R^q, \ \forall y \in R^1 \tag{2.16b}$$

From (2.16b) it follows in particular that the Hessian $\delta^2 J$ of J satisfies the following condition of coercitivity:

$$(\delta^2 J(u, y); v, z; v, z) \geqslant \alpha \|v\|^2_{L^2(H^0)}, \ \forall u, v \in L^2(0, T; H^0(\Omega)) \quad \forall y, z \in L^2(0, T; H^0(\Omega)) \tag{2.17}$$

Now we are in position to formulate our optimization problem (P–1).

(P–1) *find* $u^0 \in U_{ad}$ *such that*

$$J(u^o, y^o(u^o)) \leqslant J(u, y(u)) \quad \forall u \in U_{ad} \tag{2.18}$$

where $y(u)$ *is the solution of the state equation*

$$\frac{\partial y(x, t)}{\partial t} - A(x)\, y(x, t)=Bu(x, t) \tag{2.19}$$

along with the boundary and initial conditions

$$\frac{\partial y(\sigma, t)}{\partial \eta_A}=0, \tag{2.19a}$$

$$y(0)=y^P. \tag{2.19b}$$

Due to the assumption (2.16b) the functional $I(u)=J(u, y(u))$ is strictly convex and continuous, hence it is weakly lower semicontinuous and radially unbounded [15]. On the other hand the set U_{ad} is weakly closed in $L^2(0, T; V)$. Hence (P–1) has [1, 15] the unique solution u^o.

This paper is devoted to investigation of regularity of the optimal control u^o, the optimal trajectory $y^o=y(u^o)$ and optimal Lagrange multipliers corresponding to equality (state equation) and inequality constraints (constraints on control).

3. Lagrange formalism

Problem (P–1) of optimal control can be treated as a problem of minimization of the cost functional $J(u, y)$ in an appropriate Hilbert space subject to constraints of equality and inequality types. Introducing a Lagrangian this problem of optimization can be reduced to the problem of seeking the saddle point of the Lagrangian, hence to a problem of unconstrained minimization.

In this Section an appropriate Lagrangian will be introduced. There will be given sufficient conditions under which Lagrange multipliers corresponding to constraints on control functions are regular.

Let us introduce a Lagrange functional

$$L_1 : L^2(0, T; H_a^0(\Omega)) \times W^1(0, T) \times L^2(0, T; H^0(\Omega)) \to R^1$$

given by

$$L_1(u, y, p) \stackrel{\text{def}}{=} J(u, y) + \left(\left(p, \frac{dy}{dt} - \mathcal{A} y - \mathcal{B} u\right)\right) \tag{3.1}$$

where

$$(\mathcal{B} u)(x, t) \stackrel{\text{def}}{=} Bu(x, t). \tag{3.2}$$

By Lemma 2.1. the operator

$$\mathfrak{C}(u, y) \stackrel{\text{def}}{=} \frac{dy}{dt} - \mathcal{A} y - \mathcal{B} u$$

maps the set $L^2(0, T; H^0(\Omega)) \times \{y \in W^1(0, T) : y(0) = y^p\}$ onto $L^2(0, T; H^0(\Omega))$. Hence [8] there exists a Lagrange multiplier $p^o \in L^2(0, T; H^0(\Omega))$ such that the Lagrangian (3.1) assumes the degenerate saddle point at (u^o, y^o, p^o) i.e.

$$L_1(u^o, y^o, p) = L_1(u^o, y^o, p^o) \leqslant L_1(u, y, p^o) \tag{3.3}$$

for every

$$u \in U_{ad};\ y \in W^1(0, T),\ y(0) = y^p;\ p \in L^2(0, T; H^0(\Omega))$$

(u^o, y^o) — denotes here the solution of (P–1).

Using the fact that $A(x)$ is self-adjoint it is easy to show [12] that the element p^o satisfying (3.3) can be characterized as the solution of the following adjoint equation

$$\frac{\partial p^o(x, t)}{\partial t} + A(x) p^o(x, t) = \delta_y J(u^o, y^o)(x, t) \quad \text{in } \Omega \times (0, T) \tag{3.4}$$

$$\frac{\partial p^o(x, t)}{\partial \eta_A} = 0 \quad \text{in } \Gamma \times (0, T) \tag{3.4a}$$

$$p^o(x, T) = 0 \quad \text{in } \Omega \tag{3.4b}$$

Taking into consideration (2.16) we conclude from Lemma 2.1. that the solution p^o of the adjoint equation (3.4) satisfies the following regularity conditions

$$p^o \in C(0, T; H^1(\Omega)), \tag{3.5a}$$

$$\frac{dp^o}{dt} \in L^2(0, T; H^0(\Omega)). \tag{3.5b}$$

In order to characterize the optimal control u^o we shall use the right-hand side inequality in (3.3).

Taking into considerations definitions (2.13) and (2.16) as well as (3.1) we get

$$L_1(u^o, y^o, p^o) = \min_{u \in U_{ad}} L_1(u, y^o, p^o) =$$
$$\min_{u \in U_{ad}} \left\{ J(u, y^o) - ((\mathscr{B}^* p^o, u)) + \left(\left(p, \frac{dy^o}{dt} - \mathscr{A} y^o\right)\right)\right\} =$$
$$= \min_{u \in U_{ad}} \left\{ J(u, y^o) - ((\mathscr{B}^* p^o, u))\right\} + \left(\left(p, \frac{dy^o}{dt} - \mathscr{A} y^o\right)\right) =$$
$$= \min_{\{u:\, \psi(u(x,t)) \leqslant 0\}} \left\{ \int_0^T \int_\Omega [\varphi(u(x,t), y^o(x,t)) + \langle \mathscr{B}^* p^o(x,t), u(x,t)\rangle]\, dx\, dt + \right.$$
$$\left. + \left(\left(p, \frac{dy^o}{dt} - \mathscr{A} y^o\right)\right)\right\} = \int_0^T \int_\Omega \{ \min_{\{u(x,t):\, \psi(u(x,t)) \leqslant 0\}} [\varphi(u(x,t), y^o(x,t)) +$$
$$+ \langle \mathscr{B}^* p^o(x,t), u(x,t)\rangle]\}\, dx\, dt + \left(\left(p, \frac{dy^o}{dt} = \mathscr{A} y^o\right)\right). \tag{3.6}$$

We use here the fact that the constraints on control are of local type.

It follows from (3.6) that the optimal control u^o for almost all $x \in \Omega$ and $t \in [0, T]$ is the solution of the following local problem of optimization

(P–1) *find* $u^o(x, t)$ *satisfying condition* $\psi(u^o(x, t)) \leqslant 0$ *and such that*

$$l^1(u^o(x, t), y^o(x, t), B^* p^o(x, t)) \leqslant l^1(u, y^o(x, t), B^* p^o(x, t)) \tag{3.7}$$

for every $u \in R^q$ such that $\psi(u) \leqslant 0$, where

$$l^1(u, y, B^* p) \overset{\text{def}}{=} \varphi(u, y) + \langle B^* p, u\rangle. \tag{3.7a}$$

(P–1) is a typical problem of finite dimensional convex programming. Since the function ψ is convex and φ is strictly convex there exists the unique element $u^o(x, t) \in R^q$ statisfying (3.7) [4]. Moreover a sufficient condition of optimality of $u^o(x, t)$ is that there exists a Lagrange multiplier $\lambda^o(x, t) \in R^r$, $\lambda^o(x, t) \geqslant 0$ such that the Lagrangian

$$l(u, \lambda, y^o(x, t), B^* p^o(x, t)) \overset{\text{def}}{=} \varphi(u, y^o(x, t)) + \langle B^* p^o(x, t), u\rangle + \langle \lambda, \psi(u)\rangle \tag{3.8}$$

assumes the saddle point at $(u^o(x, t), \lambda^o(x, t))$, i.e. that

$$l(u^o(x,t), \lambda, y^o(x,t), B^* p^o(x,t)) \leqslant l(u^o(x,t), \lambda^o(x,t), y^o(x,t), B^* p^o(x,t)) \leqslant$$
$$\leqslant l(u, \lambda^o(x, t), y^o(x, t), B^* p^o(x, t)) \ \forall u \in R^q,$$
$$\forall \lambda \in R^r, \ \lambda \geqslant 0. \tag{3.9}$$

It is well known [4] that since Problem (P–1) has the unique solution then to prove that $\lambda^o(x, t)$ saisfying (3.9) exists it is enough to show that the following Slater's condition is satisfied

$$\exists \bar{u} \in R^q: \psi_i(u^o(x, t)) + \delta_u \psi_i^T(u^o(x, t))\, \bar{u} < 0, \quad i = 1, 2, \ldots, r. \tag{3.10}$$

Let $\hat{\psi}(u^o(x, t))$ denote j-dimensional ($j \leqslant r$) subvector o ψ containing all these components, which are active for $u^o(x, t)$, i.e.

$$\hat{\psi}(u^o(x, t)) = 0. \tag{3.11}$$

It is easy to see that if there is satisfied the following condition of regularity:

$$\exists \beta > 0 \colon z^T \delta_u \hat{\psi}(u^o(x, t)) \delta_u \hat{\psi}^T(u^o(x, t)) z \geqslant \beta \|z\|^2, \ \forall z \in R^j \tag{3.12}$$

then there exists a left inverse of $\delta_u \hat{\psi}^T(u^o(x, t))$. Hence we can choose an element $u_1 \in R^q$ such that

$$\delta_u \hat{\psi}^T(u^o(x, t)) u_1 < 0.$$

Taking into consideration that for all components ψ_k of the vector ψ, which do not belong to $\hat{\psi}(u^o(x, t))$ we have

$$\psi_k(u^o(x, t)) < 0,$$

we can easily see that for $\gamma > 0$ small enough the element

$$\bar{u} = \gamma u_1$$

satisfies Slater's condition (3.10).

Hence [4] there exists a Lagrange multiplier $\lambda^o(x, t) \in R^r$, $\lambda^o(x, t) \geqslant 0$ such that the following Kuhn—Tucker conditions are satisfied

$$\delta_u l(u^o(x, t), \lambda^o(x, t), y^o(x, t), B^* p^o(x, t)) = 0, \tag{3.13a}$$

$$\langle \lambda^o(x, t), \psi(u^o(x, t)) \rangle = 0 \ \lambda^o(x, t) \geqslant 0. \tag{3.13b}$$

The convexity of J and ψ together with (3.13) imply (3.9).

Note that from (3.12) and (3.13) it follows that the element $\lambda^o(x, t)$ is unique.

Now let us define on $\Omega \times (0, T)$ the function λ^o which for almost every (x, t) is equal to $\lambda^o(x, t)$. We have the following:

LEMMA 3.1. *If there exists $\beta > 0$ such that for almost all $x \in \Omega$, $t \in [0, T]$*

$$z^T \delta_u \hat{\psi}(u^o(x, t)) \delta_u \hat{\psi}^T(u^o(x, t)) z \geqslant \beta \|z\|^2, \ \forall z \in R^j \tag{3.14}$$

then

$$\lambda^o \in L^2(0, T; H^0(\Omega)).$$

Proof. First let us prove that λ^o is measurable. Let K be any subset (including the empty set) of the set $\{1, 2, ..., r\}$ of indeces and let us define

$$Q_K = \{(x, t) \in \Omega \times (0, T) \colon \psi_j(u^o(x, t)) = 0 \text{ for } j \in K \text{ and } \psi_j(u^o(x, t)) < 0 \text{ for } j \notin K\}$$

Since u^o and ψ are measurable the set Q_K is measurable.

Let us denote by $\hat{\lambda}^o_K$ and $\hat{\psi}^o_K$ subvectors of λ^o and ψ^o respectively corresponding to the set K of indeces.

Using (3.8) we rewrite (3.13a) in the form

$$\delta_u \varphi(u^o(x, t), y^o(x, t)) + B^* p^o(x, t) + \delta_u \psi^T(u^o(x, t)) \lambda^o(x, t) = 0.$$

Note that by (3.13b) this formula reduces on Q_K to

$$\delta_u \varphi(u^o(x,t), y^o(x,t)) + B^* p^o(x,t) + \delta_u \hat{\psi}_K^T(u^o(x,t))\,\hat{\lambda}_K^o(x,t) = 0. \tag{3.15}$$

By (3.14) for every $(x,t)\in Q_K$ the matrix $\delta_u \hat{\psi}_K^T(u^o(x,t))$ has a left invers $\varkappa_K(x,t)$. It is easy to see that $\varkappa_K(x,t)$ can be constructed in such a way that $\varkappa_K$ is a measurable function on Q_K.

Hence fron (3.15) we get

$$\hat{\lambda}_K^o(x,t) = \varkappa_K(x,t)\left[-\delta_u \varphi(u^o(x,t), y^o(x,t)) - B^* p^o(x,t)\right]$$

which shows that $\hat{\lambda}_T^o$ is measurable on Q_K. Since by (3.13b) all components of $\lambda^o(x,t)$ do not belonging to $\hat{\lambda}_K^o(x,t)$ are equal to zero for $(x,t)\in Q_K$ it shows that λ^o is measurable on Q_K.

Repeating the same argument for all subsets K we get measurability of λ^o on $\Omega\times(0,T)$.

Now let us show that $\lambda^o \in L^2(0,T; H^0(\Omega))$.

Multiplying (3.15) by $\hat{\lambda}_K^{oT}(x,t)\,\delta_u \hat{\psi}_K(u^o(x,t))$ and using equality

$$\hat{\lambda}_K^{oT}(x,t)\,\delta_u \hat{\psi}_K(u^o(x,t)) = -\delta_u \varphi^T(u^o(x,t), y^o(x,t)) - (B^* p^o(x,t))^T$$

we get

$$\begin{aligned} &-\delta_u \varphi^T(u^o(x,t), y^o(x,t))\,\delta_u \varphi(u^o(x,t), y^o(x,t)) + \\ &\quad -(B^* p^o(x,t))^T \delta_u \varphi(u^o(x,t), y^o(x,t)) - \delta_u \varphi^T(u^o(x,t), y^o(x,t)) \times \\ &\quad \times B^* p^o(x,t) - (B^* p^o(x,t))^T B^* p^o(x,t) + \hat{\lambda}_K^{oT}(x,t)\,\delta_u \hat{\psi}_K(u^o(x,t)) \times \\ &\quad \times \delta_u \hat{\psi}_K^T(u^o(x,t))\,\hat{\lambda}_K^o(x,t) = 0. \end{aligned}$$

Hence taking advantage of (3.14) we obtain

$$\begin{aligned} \beta\,\|\hat{\lambda}_K^o(x,t)\|^2 &\leqslant \|\delta_u \varphi(u^o(x,t), y^o(x,t))\|^2 + \\ &\quad + 2\,\|B^* p^o(x,t)\|\,\|\delta_u \varphi(u^o(x,t), y^o(x,t))\| + \|B^* p^o(x,t)\|^2 \leqslant \\ &\leqslant 2\left(\|\delta_u \varphi(u^o(x,t), y^o(x,t))\|^2 + \|B^* p^o(x,t)\|^2\right) \text{ for } (x,t)\in Q_K. \end{aligned}$$

Since for $(x,t)\in Q_K$ all components of $\lambda^o(x,t)$ which do not belong to $\hat{\lambda}_K(x,t)$ are equal to zero we have

$$\beta\,\|\lambda^o(x,t)\|^2 \leqslant 2\left(\|\delta_u \varphi(u^o(x,t), y^o(x,t))\|^2 + \|B^* p^o(x,t)\|^2\right) \text{ for } (x,t)\in Q_K.$$

Not that the same argument can be repeated for any set Q_K. Hence the above inequality holds on $\Omega\times(0,T)$. Integrating this inequality over $\Omega\times(0,T)$ and taking into consideration (2.15) and (2.16) we get

$$\begin{aligned} \beta\,\|\lambda^o\|^2_{L^2(H^0)} \leqslant 2 \Bigg[&\int_0^T \int_\Omega \|\delta_u \varphi(u^o(x,t), y^o(x,t))\|^2\,dx\,dt + \\ &+ \int_0^T \int_\Omega \|B^* p^o(x,t)\|^2\,dx\,dt \Bigg] < \infty. \end{aligned} \qquad \text{q.e.d.}$$

Now we shall define on

$$L^2(0,T;\boldsymbol{H}^0(\Omega))\times W^1(0,T)\times L^2(0,T;\boldsymbol{H}^0))\times L^2(0,T;H^0(\boldsymbol{H}))$$

a new Lagrangian $L(\boldsymbol{u},y,\boldsymbol{\lambda},p)$.

To this end let us add to $l^1(\boldsymbol{u},\boldsymbol{\lambda},y(x,t),B^*p(x,t))$ given by (3.8) the component

$$\left\langle p(x,t),\frac{dy(x,t)}{dt}-A(x)\,y(x,t)\right\rangle.$$

Integrating the result over $\Omega\times(0,T)$ for $\boldsymbol{u}\in L^2(0,T;\boldsymbol{H}^0(\Omega))$; $y\in W^1(0,T)$ and $\boldsymbol{\lambda}\in L^2(0,T:\boldsymbol{H}^0(\Omega))$, $p\in L^2(0,T;H^0(\Omega))$ we define

$$L(\boldsymbol{u},y,\boldsymbol{\lambda},p)\overset{\text{def}}{=}J(\boldsymbol{u},y)+\left(\left(p,\frac{dy}{dt}-\mathscr{A}y-\mathscr{B}\boldsymbol{u}\right)\right)+((\boldsymbol{\lambda},\Psi(\boldsymbol{u}))) \tag{3.16}$$

where

$$\Psi(\boldsymbol{u})(x,t)\overset{\text{def}}{=}\psi(\boldsymbol{u}(x,t))$$

We shall prove the following

LEMMA 3.2. *If condition of Lemma* 3.1. *are satisfied then there exist Lagrange multipliers* p^o, *which satisfies* (3.4), *and* $\boldsymbol{\lambda}^o\in L^2(0,T;\boldsymbol{H}^0(\Omega))$, $\boldsymbol{\lambda}^o\geqslant 0$, *such that Lagrangian* $L(\boldsymbol{u},y,\boldsymbol{\lambda},p)$ *assumes the saddle point at* $(\boldsymbol{u}^o,y^o,\boldsymbol{\lambda}^o,p^o)$, *i.e.*

$$L(\boldsymbol{u}^o,y^o,\boldsymbol{\lambda},p)\leqslant L(\boldsymbol{u}^o,y^o,\boldsymbol{\lambda}^o,p^o)\leqslant L(\boldsymbol{u},y,\boldsymbol{\lambda}^o,p^o) \tag{3.17}$$

$$\forall\boldsymbol{u}\in L^2(0,T;\boldsymbol{H}^0(\Omega)),\ \forall y\in W^1(0,T),\ y(0)=y^p,$$
$$\forall p\in L^2(0,T;H^0(\Omega)),\ \forall\boldsymbol{\lambda}\in L^2(0,T;\boldsymbol{H}^0(\Omega)),\ \boldsymbol{\lambda}\geqslant 0.$$ [3)]

Proof. The function $\boldsymbol{\lambda}^o\in L^2(0,T;\boldsymbol{H}^0(\Omega))$ is constructed like in Lemma 3.1. By (3.13b) it satisfies condition $\boldsymbol{\lambda}^o\geqslant 0$. Therefore it is enough to show that conditions (3.17) are satisfied.

Adding to each component of (3.9) the term

$$\left\langle p^o(x,t),\frac{dy^o(x,t)}{dt}-A(x)\,y^o(x,t)\right\rangle,$$

integrating over $\Omega\times(0,T)$ and using definition (3.16) we get

$$L(\boldsymbol{u}^o,y^o,\boldsymbol{\lambda},p^o)\leqslant L(\boldsymbol{u}^o,y^o,\boldsymbol{\lambda}^o,p^o)\leqslant L(\boldsymbol{u},y^o,\boldsymbol{\lambda}^o,p^o) \tag{3.18}$$

$$\forall\boldsymbol{u}\in L^2(0,T;\boldsymbol{H}^0(\Omega)),\ \forall\boldsymbol{\lambda}\in L^2(0,T;\boldsymbol{H}^0(\Omega)),\ \boldsymbol{\lambda}\geqslant 0.$$

Note that it follows from (3.3) as well as from (3.1) and (3.16) that:

$$L(\boldsymbol{u}^o,y^o,\boldsymbol{\lambda}^o,p^o)\leqslant L(\boldsymbol{u}^o,y,\boldsymbol{\lambda}^o,p^o)\ \forall y\in W^1(0,T),\ y(0)=y^p. \tag{3.19}$$

Convexity of $L(\boldsymbol{u},y,\boldsymbol{\lambda}^o,p^o)$ with respect to $(\boldsymbol{u},y)$ together with (3.18) and (3.19) imply

$$L(\boldsymbol{u}^o,y^o,\boldsymbol{\lambda}^o,p^o)\leqslant L(\boldsymbol{u},y,\boldsymbol{\lambda}^o,p^o). \tag{3.20}$$

---

3) $\boldsymbol{\lambda}\geqslant 0$ denotes $\boldsymbol{\lambda}(x,t)\geqslant 0$ for almost all $(x,t)\in\Omega\times(0,T)$.

On the other hand we have

$$L(u^o, y^o, \lambda, p) = L(u^o, y^o, \lambda, p^o)\ \forall p \in L^2(0, T; H^0(\Omega)). \tag{3.21}$$

Combining (3.18), (3.20) and (3.21) we arrive at (3.17). q.e.d.

REMARK 3.1. Saddle point conditions (3.17) are equivalent to the following Kuhn—Tucker conditions

$$((\delta_y L(u^o, y^o, \lambda^o, p^o), y - y^o)) = 0\ \forall y \in W^1(0, T),\ y(0) = y^p, \tag{3.22a}$$

$$\delta_u L(u^o, y^o, \lambda^o, p^o) = 0, \tag{3.22b}$$

$$((\lambda^o, \Psi(u^o))) = 0,\ \lambda^o \geqslant 0. \tag{3.22c}$$

4. Regularity of optimal controls and states

This section is devoted to investigating the regularity of the optimal solution (u^o, y^o) as well as the optimal Lagrange multipliers λ^o and p^o.

To this end we shall use the following result concerning regularity of solutions to parabolic equations

LEMMA 4.1. *If in* (2.1)

$$f \in L^2(0, T; H^1(\Omega)), \tag{4.1a}$$

$$\frac{df}{dt} \in L^2(0, T; H^0(\Omega)), \tag{4.1b}$$

$$f(0) \in H^1(\Omega), \tag{4.1c}$$

$$y^p \in H^3(\Omega), \tag{4.1d}$$

$$\frac{\partial y^p}{\partial \eta_A} = 0, \tag{4.1e}$$

then

$$y \in L^2(0, T; H^3(\Omega)), \tag{4.2a}$$

$$\frac{dy}{dt} \in L^2(0, T; H^2(\Omega)), \tag{4.2b}$$

$$\frac{d^2 y}{dt^2} \in L^2(0, T; H^1(\Omega)). \tag{4.2c}$$

Proof of Lemma 4.1. is given in Appendix A.

THEOREM 4.1. *If optimal control u^o satisfies the following conditions of regularity*

$$u^o \in L^2(0, T; H^1(\Omega)), \tag{4.3a}$$

$$\frac{du^o}{dt} \in L^2(0, T; H^0(\Omega)), \tag{4.3b}$$

$$u^o(0),\ u^o(T) \in H^1(\Omega), \tag{4.3c}$$

then the solution p^o *of the adjoint equation* (3.4) *satisfies* (4.2). *If additionally the initial condition* y^p *satisfies* (4.1d) *and* (4.1e) *then also the solution* y^o *of the state equation* (2.19) *satisfies* (4.2).

Proof. Let us start with the regularity of y^o. We shall use Lemma 4.1. Since by assumption (4.1d) and (4.1e) are satisfied it is enough to check if (4.1a), (4.1b) and (4.1c) hold, but if follows immediately from (2.15) and (4.3).

Now let us consider the adjoint equation (3.4). Changing the direction of time by substitution $\tau=T-t$ we note that also in this case Lemma 4.1. can be used. Condition $p(T)=0$ assures that (4.1d) and (4.1e) are satisfied.

Then it is enough to check if the function $f=\delta_y J(u^o, y^o)$ satisfies (4.1a), and (4.1b) as well as (4.1c) at T.

Note that

$$\delta_y J(u^o, y^o)(x, t)=\delta_y \varphi\big(u^o(x, t), y^o(x, t)\big), \tag{4.4a}$$

$$\frac{d}{dt}\delta_y J(u^o, y^o)(x, t)=\delta^2_{yu}\varphi\big(u^o(x, t), y^o(x, t)\big)\frac{\partial u^o(x,t)}{\partial t}+$$
$$+\delta^2_{yy}\varphi\big(u^o(x, t), y^o(x, t)\big)\frac{\partial y^o(x, t)}{\partial t}. \tag{4.4b}$$

Taking into account (2.7), (2.16) and (4.3) we find that the function $f=\delta_y J(u^o, y^o)$ satisfies (4.1a), (4.1b) and (4.1c). q.e.d.

It follows from Theorem 4.1 that in order to show the required regularity of optimal solutions to state and adjoint equations it is enough to show that optimal control satisfies (4.3).

Unfortunately we are not able to do that in the general case of constraints of the form (2.13) and we restrict ourself to the case where u is a scalar function with bounded amplitude. It means that we put $q=1$, $r=2$ and

$$W_{ad}=\{w\in R^1: |w|\leqslant 1\}=\{w\in R^1: \psi(w)\leqslant 0\} \tag{4.5}$$

where

$$\psi^T(w)=(w-1, -w-1) \tag{4.5a}$$

For such a function ψ condition (3.14) is trivially satisfied.

THEOREM 4.2. *If constraints function* ψ *is given by* (4.5a), *then the optimal control* u^o *satisfies condition of regularity* (4.3).

Proof. To construct the optimal control we use (3.13). To this end let us define an auxiliary function $\bar{u}(x, t)$ given by

$$\delta_u l^1\big(\bar{u}(x, t), y^o(x, t), B^* p^o(x, t)\big)=0. \tag{4.6}$$

Since by (3.7a) we have

$$\delta^2_{uu} l^1(u, y, B^* p)=\delta^2_{uu}\varphi(u, y),$$

condition (2.16b) implies

$$|[\delta_{uu} l^1(u, y, B^* p)]^{-1}|\leqslant\frac{1}{\alpha}<\infty\ \forall u, y, B^* p\in R^1. \tag{4.7}$$

Hence it follows from the implicite function theorem [13] that there exists a function $\vartheta(y, B^* p)$ continuous in a neighbourhood of $(y^0(x,t), B^* p^o(x,t))$ [13] such that

$$\bar{u}(x,t)=\vartheta\left(y^o(x,t), B^* p^o(x,t)\right). \tag{4.8}$$

It is shown in Appendix B that the function $\bar{u}$ defined on $\Omega\times(0,T)$ by (4.8) satisfies the following conditions of regularity

$$\bar{u}\in L^2\left(0,T; H^1(\Omega)\right), \tag{4.9a}$$

$$\frac{d\bar{u}}{dt}\in L^2\left(0,T; H^0(\Omega)\right), \tag{4.9b}$$

$$\bar{u}(t)\in H^1(\Omega)\ \forall t\in[0,T]. \tag{4.9c}$$

Now let us construct the optimal element $u^o(x,t)$ which satisfies (3.13). Taking into account the form (4.5) of constraints we find that $u^o(x,t)$ is given by

$$u^o(x,t)=\begin{cases}\bar{u}(x,t) & \text{if} \quad -1<\bar{u}(x,t)<1\\ 1 & \text{if} \quad \bar{u}(x,t)\geqslant 1\\ -1 & \text{if} \quad \bar{u}(x,t)\leqslant -1\end{cases}. \tag{4.10}$$

This can be rewritten in the form

$$u^o(x,t)=\max\left\{-1, \min\{\bar{u}(x,t), 1\}\right\}. \tag{4.10a}$$

It is easy to check that since $\bar{u}$ satisfies (4.9) the optimal function u^o defined on $\Omega\times(0,T)$ by (4.10) also satisfies (4.9). q.e.d.

We conclude the paper with a result concerning the regularity of Lagrange multipliers $\boldsymbol{\lambda}$.

THEOREM 4.3. *If constraints are given by* (4.5) *then the optimal Lagrange multiplier* $\boldsymbol{\lambda}^o$ *satisfies conditions*

$$\boldsymbol{\lambda}^o\in L^2\left(0,T; \boldsymbol{H}^1(\Omega)\right), \tag{4.11a}$$

$$\frac{d\boldsymbol{\lambda}^o}{dt}\in L^2\left(0,T; \boldsymbol{H}^0(\Omega)\right), \tag{4.11b}$$

$$\boldsymbol{\lambda}^o(t)\in \boldsymbol{H}^1(\Omega)\ \forall t\in[0,T]. \tag{4.11c}$$

Proof. By (3.7a), (3.8), (3.13) and (4.10) we have

$$\lambda_1^0(x,t)=\begin{cases}-\left(\delta_u\varphi\left(1, y^o(x,t)\right)+B^* p^o(x,t)\right)= & \\ \qquad =-\delta_u l^1\left(1, y^o(x,t), B^* p^o(x,t)\right) & \text{if } \bar{u}(x,t)\geqslant 1\\ 0 & \text{if } \bar{u}(x,t)<1.\end{cases}$$

Using the definition of $\bar{u}(x,t)$ and convexity of $l^1\left(\cdot, y^o(x,t), B^* p^o(x,t)\right)$ the above formula can be rewritten in the form

$$\lambda_1^0(x,t)=\max\left\{0, -\left(\delta_u\varphi\left(1, y^o(x,t)\right)+B^* p^o(x,t)\right)\right\}. \tag{4.12a}$$

In a similar way we get

$$\lambda_2^o(x,t)=\max\{0, \delta_u\varphi(-1, y^o(x,t))+B^* p^o(x,t)\}. \tag{4.12b}$$

By (2.7) and (2.16) the function $\delta_u \varphi(1, y^o)$ satisfies conditions (4.9), while by (2.15) these conditions are satisfied by $B^* p^o$. Hence, in a similar way as in the proof of Theorem 4.2. (4.12) imply (4.11). q.e.d.

APPENDIX A

Proof of Lemma 4.1. It follows from Theorem 5.2 in [12] (p. 33) that if

$$\varphi \in L^2(0, T; H^1(\Omega)), \tag{A.1a}$$

$$\frac{d\varphi}{dt} \in L^2\, 0, T; H^0(\Omega)), \tag{A.1b}$$

$$\varphi(0)=0, \tag{A.1c}$$

then the boundary value problem

$$\begin{aligned} &\frac{dz}{dt}-Az=\varphi \quad \text{on } \Omega\times(0,T)\\ &\frac{\partial z}{\partial \eta_A}=0 \quad \text{on } \partial\Omega\times(0,T)\\ &z(0)=0 \quad \text{on } \Omega \end{aligned} \tag{A.2}$$

has the solution z such that

$$z \in L^2(0, T; H^3(\Omega)) \tag{A.3a}$$

$$\frac{d^2 z}{dt^2} \in L^2(0, T; H^0(\Omega)) \tag{A.3b}$$

and z continuously depends on φ.

In order to apply this result to the case of equation (2.1) with nonhomogeneous initial condition (2.5), note that if the function y satisfies (4.1e) and if there exists a function w satisfying this condition as well as (A.3) and such that

$$w(0)=y^p \tag{A.4}$$

then we can introduce a new variable

$$z(t)=y(t)-w(t) \tag{A.5}$$

and rewrite (2.1) along with conditions (2.4) and (2.5) in the form

$$\frac{dz}{dt}-Az=f-\left(\frac{dw}{dt}-Aw\right), \text{ on } \Omega\times(0,T) \tag{A.6a}$$

$$\frac{\partial z}{\partial \eta_A}=0 \text{ on } \partial\Omega\times(0, T) \tag{A.6b}$$

$$z(0)=0 \text{ on } \Omega \tag{A.6c}$$

If the function

$$\varphi=f-\left(\frac{dw}{dt}-Aw\right) \tag{A.7}$$

satisfies (A.1) then the solution z of (A.6) satisfies conditions (A.3) and thus (A.5) implies that these conditions are also satisfied by y.

Hence taking into consideration (A.1a) and (A.1b) we conclude that the function w should satisfy the following conditions:

$$w\in L^2(0, T; H^3(\Omega)). \tag{A.8a}$$

$$\frac{d^2 w}{dt^2}\in L^2(0, T; H^0(\Omega)), \tag{A.8b}$$

$$w(0)=y^p, \tag{A.8c}$$

$$\frac{dw}{dt}-Aw\in L^2(0, T; H^1(\Omega)), \tag{A.8d}$$

$$\frac{d^2 w}{dt^2}-A\frac{dw}{dt}\in L^2(0, T; H^0(\Omega)), \tag{A.8e}$$

$$\frac{dw(0)}{dt}-Aw(0)=f(0). \tag{A.8f}$$

We are going to find conditions on y^p under which there exists a function w satisfying (A.8).

It follows from (A.8b) and (A.8e) that

$$\frac{dw}{dt}\in L^2(0, T; H^2(\Omega)) \tag{A.9}$$

Taking advantage of Proposition 2.3 in [12] (p.14) we conclude that (A.9) holds if

$$w\in H^{4,2}(\Omega\times(0, T))^{1)}. \tag{A.10}$$

Condition (A.10) assures also that (A.8a) and (A.8d) are satisfied.

From the trace theorems (Theorems 2.1 and 2.3 in [12] pp. 10 and 21) it follows that a function satisfying (A.10) exists iff the following initial conditions of regularity hold

$$w(0)\in H^3(G), \tag{A.11a}$$

$$\frac{dw(0)}{dt}\in H^1(\Omega), \tag{A.11b}$$

1) The definition of the space $H^{r,s}(\Omega\times(0, T))$ is given in [12] p. 8.

and the condition of compatibility:

$$\frac{\partial w(0)}{\partial \eta_A} = 0 \text{ on } \partial\Omega \tag{A.12}$$

is satisfied.

From (A.8c) and (A.11a) we get

$$y^p \in H^3(\Omega). \tag{A.13}$$

If (A.13) holds, then by (A.8f) and (4.1c) condition (A.11b) is also satisfied. Hence conditions (A.12) and (A.13) assures the existence of a function w satisfying (A.8).

Summarizing we conclude that if conditions (4.1) hold then the solution y of (2.1) satisfies (4.2a) and (4.2c). In turn equation (2.1) together with (4.2a) and (4.1a) imply (4.2b) q.e.d.

APPENDIX B

Proof of (4.9). According to the implicite function theorem [13] the function ϑ given by (4.8) is differentiable at $(y^o(x,t), B^* p^o(x,t))$ and

$$\delta_y \vartheta(y^o, B^* p^o) = -[\delta^2_{uu} l^1(\bar{u}, y^o, B^* p^o)]^{-1} [\delta^2_{u,y} l^1(\bar{u}, y^o, B^* p^o)], \tag{B.1a}$$

$$\delta_{B^* p^o} \vartheta(y^o, B^* p^o) = -[\delta^2_{uu} l^1(\bar{u}, y^o, B^* p^o)]^{-1} [\delta^2_{u B^* p} l^1(\bar{u}, y^o, B^* p^o)]. \tag{B.1b}$$

Note that

$$\delta^2_{u,y} l^1(u, y, B^* p) = \delta^2_{u,y} \varphi(u, y),$$

$$\delta^2_{u, B^* p} l^1(u, y, B^* p) = 1.$$

Hence using (2.16) and (4.7) we get from (B.1)

$$|\delta_y \vartheta(y^o, B^* p^o)| \leqslant C < \infty \quad \forall y^o, B^* p^o \in R^1, \tag{B.2a}$$

$$|\delta_{B^* p^o} \vartheta(y^o, B^* p^o)| \leqslant C < \infty \quad \forall y^o, B^* p^o \in R^1. \tag{B.2b}$$

From (4.8) we obtain

$$\frac{\partial \bar{u}(x,t)}{\partial t} = \delta_y \vartheta(y^o(x,t), B^* p^o(x,t)) \frac{\partial y^o(x,t)}{\partial t} + \delta_{B^* p} \vartheta(y^o(x,t), B^* p^o(x,t)) \frac{\partial B^* p^o(x,t)}{\partial t}, \tag{B.3a}$$

$$\frac{\partial \bar{u}(x,t)}{\partial x_i} = \delta_y \vartheta(y^o(x,t), B^* p^o(x,t)) \frac{\partial y^o(x,t)}{\partial x_i} + \delta_{B^* p^o} \vartheta(y^o(x,t), B^* p^o(x,t)) \frac{\partial B^* p^o(x,t)}{\partial x_i}, \quad i = 1, 2, \ldots, n. \tag{B.3b}$$

Taking advantage of (B.2) we get

$$\left|\frac{\partial \bar{u}(x,t)}{\partial t}\right|^2 \leqslant C\left(\left|\frac{\partial y^o(x,t)}{\partial t}\right|^2 + \left|\frac{\partial B^* p^o(x,t)}{\partial t}\right|^2\right),$$
$$\left|\frac{\partial \bar{u}(x,t)}{\partial x_i}\right|^2 \leqslant C\left(\left|\frac{\partial y^o(x,t)}{\partial x_i}\right|^2 + \left|\frac{\partial B^* p^o(x,t)}{x_i}\right|^2\right). \qquad \text{(B.4a)}$$

Integrating (B.4a) over $\Omega \times [0, T]$ and taking into consideration (2.15) as well as the fact that

$$\frac{dy^o}{dt}, \frac{dp^o}{dt} \in L^2(0, T; H^0(\Omega))$$

we obtain (4.9b). This in turn implies that

$$\bar{u} \in C(0, T; H^0(\Omega)). \qquad \text{(B.5)}$$

Hence for every $t \in [0, T]$ $\bar{u}(t)$ is well defined as an element of $H^0(\Omega)$. We shall show that $\bar{u}(t) \in H^1(\Omega)$. To this end let us integrate (B.4b) over Ω for a fixed value of $t \in [0, T]$. Taking advantage of the fact that by (2.7a) we have

$$y^o(t), p^o(t) \in H^1(\Omega)$$

we obtain (4.9c).

At last we get (4.9a) integrating (B.4b) over $\Omega \times [0, T]$ and taking into account (B.5).

References

1. Balakrishnan A. V.: Introduction to optimization theory in a Hilbert space. Springer, Berlin 1971.
2. Bardos C.: A regularity theorem for parabolic equations. *Journal of Func. Analysis* 7 (1971). 311–322.
3. Falk R. S.: Approximation of a class of optimal control problems with order of convergence estimates. *Journal of Math. Analysis and Appl.* 44 (1973) 28–47.
4. Hadley G.: Nonlinear and dynamic programming. Addison-Wesley. Reading, Mass. 1964.
5. Hager W. W.: The Ritz—Trefftz method for state and control constrained optimal control problems. *SIAM Jour. on Numerical Analysis* 12 (1975) 854–867.
6. Hager W. W., Mitter S. K.: Lagrange duality theory for convex control problems (to appear).
7. Hager W. W.: Stability of Quadratic Programs and Optimal Control Regularity (to appear).
8. Kurcyusz S.: On existence and nonexistence of Lagrange multipliers in Banach spaces (to be published in IOTA).
9. Lazer A. C., Landesman E. M., Meyers D. R.: On saddle point problems in the calculus of variations, the Ritz algorithm and monotone convergence. *Journal of Math. Analysis and Appl.* 52 (1975) 594–614.
10. Lasiecka I., Malanowski K.: On discrete-time Ritz—Galerkin approximation of control constrained optimal control problems for parabolic systems (to be published in this Journal).
11. Lions J. L., Magenes E.: Problèmes aux limites non homogènes. Vol. 2. Dunod, Paris 1968.
12. Lions J. L.: Contrôle optimal de systèmes gouvernés par des équations aux derivées partielles. Dunod, Paris 1968.

13. Maurin K.: Analiza. T. 1. PWN, Warszawa 1971.
14. McKnight, Bosarge W. E. Jr.: The Ritz—Galerkin procedure for parabolic control problems. *SIAM J. on Control*, **11** (1973) 510–524.
15. Вайнберг М. М.: Варнационный метод и метод монотонных операторов. Наука, Москва 1972.

Received, September 1976

O regularności rozwiązań wypukłych zadań sterowania optymalnego dla układów parabolicznych przy ograniczeniach sterowania

Rozważa się zadanie sterowania optymalnego dla układu opisywanego liniowym równaniem parabolicznym. Funkcjonał jakości jest wypukły, a na sterowanie nałożone są ograniczenia typu lokalnego. Do takiego zadania stosuje się formalizm Lagrange'a. Pokazuje się, że mnożniki Lagrange'a odpowiadające ograniczeniom sterowania są odpowiednio regularnymi funkcjami. Dowodzi się regularności sterowania i trajektorii optymalnej.

О регулярности решений выпуклых задач оптимального управления для параболических систем при ограничениях по управлению

Рассматривается задача оптимального управления для объекта описываемого линейной параболической системой. Функционал качества является выпуклым, а на управление накладываются ограничения локального типа. К такой задаче используется формализм Лагранжа. Показано, что множители Лагранжа, соответствующие ограничениям по управлению, являются регулярными функцями. Доказана регулярность управления и оптимально траектории.