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Semicontinuity in constrained optimization
Part I. 1. Metric spaces

by

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We develop (in Part I.1 and I.2) a local closed graph theory analogous to the (global) Pták theory. Applied to the class of nearly convex multifunctions (which we introduce in the paper) our theory gives especially strong results.

They generalize several classical theorems (the Banach open mapping, the Lusternik) and are successfully applied in Parts II to the optimization theory.

1. Introduction

The idea of replacement of constrained minimization problems by some related unconstrained problems (usually more handy) has been omnipresent for centuries. It was also guiding for [10], [11] of S. Kurcyusz and the present author, where a use was made of very general Lagrange functions (see also [37], [21]). Necessary and sufficient conditions for the crucial properties of the (weak, strong and strict) duality were expressed in terms of the primal functional $\overline{f\Gamma}$:

$$\overline{f\Gamma}(y)=\inf_{x\in\Gamma y} f(x), \tag{1.1}$$

where X and Y are sets, $f\colon X\to\bar{\boldsymbol{R}}$ is a minimized function and $\Gamma\colon Y\to 2^X$ is a multifunction representing a family of constraints.

Therefore, in [39] we proposed the following program of investigations:

Given a class G of real functions on Y, examine the mutual dependence of a function $f\colon X\to\bar{\boldsymbol{R}}$ and a multifunction $\Gamma\colon X_Y\to 2^X$ so that $\overline{f\Gamma}\in G$. Given a class F of real functions on X characterize those multifunctions which reassure that $\overline{f\Gamma}\in G$ for every $f\in F$.

These questions should be studied for various classes F of minimized functions and for classes G corresponding to different properties of the primal functionals with respect to varying sets of Lagrange functions.

Starting to investigate along these lines we became soon aware of the importance of sundry types of semicontinuity [7].

S. Rolewicz and the present author [12] realizing the above program for some important classes F and G obtained the mentioned characterization in terms of generalized upper semicontinuity. The results of [39] are also in the vein.

The topological thinking of that type proves useful and clarifying. It has a unifying character for numerous problems (Banach open mapping theorem, Lusternik theorem, adjoint operators, controllability and others), gives a new look on some of them and spares a lot of hard analysis arguments (compare [17], [26], [27], [55], [49]). To our knowledge the present paper is the first attempt of systematic study of the role of semicontinuity in the general theory of optimality conditions. Related papers are discussed in Parts II. A display of the structure of the article may be instrumental in reading: A brief presentation of the generalized Lagrange theory is done in Part II (Paragraph 1). It is followed by sufficient and necessary (or almost necessary) conditions for the validity of that theory in terms of diverse notions of semicontinuity of constraints multifunctions (II.2, II.3).

At this stage it is important to know when some weak and easy-to-be-checked properties of multifunctions imply the desirable semicontinuity conditions. Most of Part I (Paragraphs 3, 4 and 5) is devoted to this end. This germinates from the Banach open mapping and the Lusternik theorems and eventually comes to the controllability. Farther investigations of the controllability concept and of its dual observability are done in Part II.

Reasuming, Part I may be qualified topological, while Part II is an outline of the optimization and the systems theories.

The nonconvex and nonlinear character of these considerations should be pointed out.

Paragraph 2 recollects some basic facts about upper and lower semicontinuity (and the Hausdorff semicontinuity) of multifunctions. (Only semicontinuity of superpositions is left to Paragraph II.2.). We prove a theorem on upper semicontinuity of multifunctions $\bar{\Gamma}$ of closures of values.

Then we introduce several local types of semicontinuity that prove very useful in the sequel.

Rates of semicontinuity were introduced by Pták for the upper Hausdorff semicontinuity [24], [25]. We define rates of semicontinuity for other sorts of semicontinuity and make an extensive use of them throughout the whole paper. The notion of moduli of semicontinuity (introduced by the present author [7]) is, in a way, inverse to the rates.

We also introduce an important concept of regions of semicontinuity which is appropriate to handle some problems of uniformity. The uniformity, on the other hand, constitutes a link between upper and lower types of semicontinuity and it seems to be essential in what we can call the open mapping theory (theorems of Banach and Lusternik types).

In Paragraph 3 we prove an approximation theorem for closed multifunctions in metric spaces which extends the Pták's refinement of the closed graph theorem.

Nearly convex multifunctions are introduced in Paragraph 4. They generalize both convex closed multifunctions and continuously differentiable maps and are related to sets of the Levitin-Miljutin-Osmolovskii type.

The consequences of the approximation theorem for nearly convex multifunctions appear in Paragraph 5 as extensions of the Banach open mapping theorem, the Lusternik theorem and their generalizations by S. Robinson [26], [27].

We obtain, in conclusion, the local upper Hausdorff semicontinuity (in Theorems 5.3 and 5.11) instead of the usual weaker δ-upper Hausdorff semicontinuity, the former property being of direct applicability (see Paragraph 3 of Part II).

It still remains a challenging question, whether the assumptions of the Lusternik theorem (without our local Lipschitz continuity of the derivative) entail the local upper Hausdorff semicontinuity.

More generally

1.1. Problem

Under what additional assumptions the intersection of (two) u.H.s.c. at y_0 multifunctions is u.H.s.c.

2. Semicontinuous multifunctions

Let X and Y be sets. A mapping Γ of Y to subsets of X is called a multifunction ($\Gamma y \subset X$ for each $y \in Y$). Together with Γ we shall constantly consider its inverse multifunction Γ^{-1} acting from X to subsets of Y: $\Gamma^{-1} x = \{y : x \in \Gamma y\}$. A multifunction Γ may be represented in a unique way by its graph $G(\Gamma) = \{(x, y) : x \in \Gamma y\}$. Conversely, any subset P of $X \times Y$ determines the multifunction $\overleftarrow{P}$, $\overleftarrow{P} y = \{x : (x, y) \in P\}$ and symmetrically it determines $\overrightarrow{P}$, $\overrightarrow{P} x = \{y : (x, y) \in P\}$; of course $(\overleftarrow{P})^{-1} = \overrightarrow{P}$.

The usual convention is that for $B \subset Y$, $\Gamma B = \bigcup_{y \in B} \Gamma y$ and for $A \subset X$, $\Gamma^{-1} A = \bigcup_{x \in A} \Gamma^{-1} x = \{y : \Gamma y \cap A \neq \emptyset\}$. $\Gamma^{-1} A$ is called the preimage of A under Γ. The exponential preimage of A is defined by

$$\Gamma_{\exp}^{-1} A = \{y : \Gamma y \subset A\}, \tag{2.1}$$

and it is immediate that $\Gamma_{\exp}^{-1} A = (\Gamma^{-1} A^c)^c$, where $A^c = X \setminus A$. Note that $\Gamma B \cap A \neq \emptyset$, if and only if $B \cap \Gamma^{-1} A \neq \emptyset$; $\Gamma B \subset A$, if and only if $B \subset \Gamma_{\exp}^{-1} A$.

Upper semicontinuity. Let X and Y be topological spaces multifunction Γ: $Y \to 2^X$ is said to be upper semicontinuous (u.s.c.) at y_0, if for each open Q containing Γy_0 there is a neighborhhod W of y_0 so that $W \subset \Gamma_{\exp}^{-1} Q$. Γ is u.s.c. if for each open Q, $\Gamma_{\exp}^{-1} Q$ is open (Kuratowski [18] I p. 173).

We say that Γ is locally u.s.c. at (x_0, y_0) if for each neighbourhood P of x_0 there is a neighbourhood $Q \subset P$ such that $\bar{Q} \cap \Gamma$ is u.s.c. at y_0. Define $\tilde{\Gamma} y = \bar{Q} \cap \Gamma y$, if

$y \neq y_0$ and $\tilde{\Gamma} y_0 = \Gamma y_0$. Γ is said to be δ-u.s.c. at (x_0, y_0), if there exists a neighbourhood $Q = X$ of x_0 such that $\tilde{\Gamma}$ is u.s.c. at y_0.

Note, that if X is regular (T_3), then the upper semicontinuity implies local upper semicontinuity which becomes equivalent to δ-upper semicontinuity (Kuratowski [18], I, p. 180).

We denote $\bar{\Gamma} y = (\overline{\Gamma y})$.

2.1. Theorem

$\bar{\Gamma}$ is u.s.c. for each u.s.c. multifunction Γ into subsets of X, if and only if X is normal (T_4).

Proof. Suppose X to be normal and let $\Gamma: Y \to 2^X$ be u.s.c. at y_0. Take an open set Q with $\bar{\Gamma} y_0 \subset Q$. By normality there are open set Q_1 and a closed set F such that $\Gamma y_0 \subset Q_1 \subset F \subset Q$. Since Γ is u.s.c. at y_0 there is a neighbourhood W of y_0 such that for $y \in W$ $\Gamma y \subset Q_1$ and thus $\bar{\Gamma} y \subset Q$.

Suppose now that X is not normal. There exist sets open Q_0 and closed F_0 such that $F_0 \subset Q_0$ and for each open set Q, $F_0 \subset Q$, $\bar{Q}$ is not a subset of Q_0.

Let Y be the family composed of F_0 and all open subsets G of X such that $F_0 \subset G$. A set A in Y is open, whenever there is an open set G_0 of X so that $A = \{y \in Y; y \subset G_0\}$. Define $\Gamma: Y \to 2^X$ by $\Gamma G = G$, $\Gamma F_0 = F_0$. Γ is u.s.c. at F_0, but there is an open set, namely Q_0, such that $\bar{\Gamma} F_0 = F_0 \subset Q_0$, but for each neighbourhood W of F_0 there is $G \in W$ so that $\overline{\Gamma G} = \bar{G} \cap Q_0^c \neq \emptyset$.

Active boundary The active boundary of Γy_0 (where Γ: $Y - 2^X$) is defined by (Dolecki [7])

$$\operatorname{Frac} \Gamma y_0 = \bigcap_{W \in \mathscr{B} y_0} (\overline{\Gamma W \setminus \Gamma y_0}) \tag{2.2}$$

where $\mathscr{B}(y_0)$ is a basis of neighbourhoods of y_0. Of course, (2.2) is independent of choice of the basis. Frac Γy_0 is closed and disjoint from Int Γy_0.

Observe that $x_0 \in \operatorname{Frac} \Gamma y_0$, if for each neighbourhood Q of x_0 and for each neighbourhood W of y_0 $Q \cap \Gamma W \cap \Gamma y_0^c \neq \emptyset$. If X is regular and if Γ is u.s.c. at y_0 then Frac $\Gamma y_0 \subset \operatorname{Fr} \Gamma y_0$ (topological boundary). Indeed, let $x_0 \in \Gamma y_0$ and $x_0 \in \operatorname{Frac} \Gamma y_0$. By regularity there are disjoint open sets Q_1 and Q_2 such that $x_0 \in Q_1$ and $\overline{\Gamma y_0} \subset Q_2$ and by upper semicontinuity there us $W \in \mathscr{B}(y_0)$ with $\Gamma W \subset Q_2$. Thus $Q_1 \cap \Gamma W \cap \Gamma^c y_0 = \emptyset$.

2.2. Example [32]

For an element y of a Banach space X, Γy denotes the set of best approximations of y by $\{x: \|x\| \geqslant 1\}$. For $\|y\| \leqslant 1$, $\Gamma y \subset \Gamma 0 = \{x: \|x\| = 1\}$ and consequently Γ is u.s.c. at 0. Note that Frac $\Gamma 0 = \emptyset$.

2.3. Example

If Γ is open and closed, Y being T_1 then the whole boundary is active: Fr $\Gamma y_0 \subset$ $\subset$ Frac Γy_0.

Proof. Take any $x_0 \in \operatorname{Fr} \Gamma y_0$ and any neighbourhood W of y_0. Since ΓW is open and Γy_0 is closed ΓW is a neighbourhood of x_0; We have that for each neighbourhood Q of x_0 the open set $Q \cap (\Gamma y_0)^c$ is not empty, hence $Q \cap (\Gamma W \setminus \Gamma y_0) \neq \emptyset$.

2.4. Example

Let f be a (continuous) mapping $f: X \to Y$. We define a multifunction $\Gamma y = f^{-1}(y)$. Γ is both open and closed and thus every boundary point of $f^{-1}(y)$ is active.

Closedness. If X is regular Γ is closed-valued ($\bar{\Gamma} y = \Gamma y$, $y \in Y$) and u.s.c., then the graph $G(\Gamma)$ is closed (Kuratowski [18], I, p. 175). Before stating a partial converse of this proposition, we notice that Γ is u.s.c. at y_0, if and only if Γ^{-1} is a closed mapping at y_0: for each closed set $F \subset X$ such that $y_0 \in \overline{\Gamma^{-1} F}$, also $y_0 \in \Gamma^{-1} F$.

Consider the following property of a multifunction Γ: $Y \to 2^X$, X, Y Hausdorff spaces (T_2):

$$\text{for each compact set } K \subset X \text{ such that}$$

$$y_0 \in \overline{\Gamma^{-1} K}, \text{ we have } y_0 \in \Gamma^{-1} K. \tag{2.3}$$

2.5. Theorem (Rockafellar [49], see also [18], II, p. 57)

Let X be T_2-space and let $P \subset X \times Y$ be closed. Then $\bar{P}$ satisfies (2.3) for each $y_0 \in Y$.

Proof. Take a compact set $K \subset X$. The set $\bar{P}K = \{y: \bar{P}y \cap K \neq \emptyset\}$ is the projection of $K \times Y \cap P$ on Y and thus closed because of compactness of K.

It follows that for X compact and Y being T_2, Γ is u.s.c. closed-valued, if and only if $G(\Gamma)$ is closed (Kuratowski [18], II, p. 57).

Lower semicontinuity. Let $\mathscr{B}(x_0)$ denote a basis of x_0. Γ is lower semicontinuous (l.s.c.) at (x_0, y_0) is for each element B of $\mathscr{B}(x_0)$ there exists a neighbourhood W of y_0 such that $\Gamma^{-1} B \supset W$ (the definition does not depend on the basis). We say that Γ is l.s.c. at y_0, if for any open set Q that meets Γy_0, y_0 is an interior point of $\Gamma^{-1} Q$ (Kuratowski [18], I, p. 173).

In other words Γ is l.s.c. (at each $(x_0, y_0) \in G(\Gamma)$), if and only if Γ^{-1} is an open mapping: $\Gamma^{-1} Q$ is open for open Q.

A local character of lower semicontinuity is expressed by the fact that Γ is l.s.c. at y_0, if and only if it is l.s.c. at (x_0, y_0) for each $x_0 \in \Gamma y_0$. For sufficiency take any open Q, $Q \cap \Gamma y_0 \neq \emptyset$ and choose an element x_0 of $Q \cap \Gamma y_0$. Let $B \in \mathscr{B}(x_0)$ be such that $B \subset Q$. We have $\Gamma^{-1} Q \supset \Gamma^{-1} B \supset W$, a neighbourhood of y_0, because Γ is l.s.c. at (x_0, y_0).

2.6. Example

Let π denote the projection of $X \times Y$ on Y. π^{-1} is l.s.c. (π is open), since a basis of the product topology is composed of products of open set $Q_1 \subset X$, $Q_2 \subset Y$, and $\pi(Q_1 \times Q_2) = Q_2$.

To see the mutual dependence of lower and upper semicontinuity, let us introduce an auxiliary notion: Γ is said to be inner semicontinuous (i.s.c.) at y_0, if for each closed set F, $F \subset \Gamma y_0$, there is a neighbourhood W of y_0 so that for each $y \in W$, $F \subset \Gamma y$. Of course Γ is i.s.c. if and only if the complementary multifunction Γ^c is u.s.c. at y_0. On the other hand, if X is a T_1-space the inner semicontinuity entails the lower semicontinuity, for then $\Gamma y_0 \cap Q \neq \emptyset$ implies that for any $x_0 \in \Gamma y_0 \cap Q$ there is a neighbourhood W such that for $y \in W$, $x_0 \in \Gamma y$.

Almost lower semicontinuity. We say that Γ is almost lower semicontinuous at (x_0, y_0), whenever for each neighbourhood Q of x_0 there is a neighbourhood W of y_0 such that $\overline{\Gamma^{-1} Q} \supset W$.

The relationship to lower semicontinuity will be studied in detail in next paragraphs.

2.7. Example

Let π be the projection of $\boldsymbol{R}^2$ on the abscissa. Define $\Gamma y = \pi^{-1} y$, if y is rational and $\Gamma y = \emptyset$ otherwise.

Γ is almost l.s.c. but not l.s.c.

Hausdorff semicontinuities. Let (X, ρ) be a metric space. Denote $B(x, \varepsilon) = \{w: \rho(w, x) < \varepsilon\}$ and for $A \subset X$ $B(A, \varepsilon) = \bigcup_{x \in A} B(x, \varepsilon)$; dist $(x, A) = \inf \{r: B(x, r) \cap A \neq \emptyset\}$. Let Y be a topological space.

A multifunction $\Gamma: Y - 2^X$ is upper Hausdorff semicontinuous (u.H.s.c.) at y_0, if for each $\varepsilon > 0$ there exists a neighbourhood W of y_0 such that $\Gamma W \subset B(\Gamma y_0, \varepsilon)$.

Γ is lower Hausdorff semicontinuous (l.H.s.c.) at y_0, if for each $\varepsilon > 0$ there is a neighbourhood W of y_0, such that $W \subset \{y: \Gamma y_0 + \subset B(\Gamma y, \varepsilon)\}$. (Pollul, see [32]).

Γ is l.H.s.c. at y_0, if and only if it is l.s.c. at (x_0, y_0) uniformly for each $x_0 \in \Gamma y_0$: for each $\varepsilon > 0$ there is a neighbourhood W of y_0 so that for each $x_0 \in \Gamma y_0$, $\Gamma^{-1} B(x_0, \varepsilon) \supset W$.

In fact, the latter means that for each $x_0 \in \Gamma y_0$ and for every $y \in W$, $y \in \Gamma^{-1} B \times (x_0, \varepsilon)$, or in other words $\Gamma y \cap B(x_0, \varepsilon) \neq \emptyset$, or else $x_0 \in B(\Gamma y, \varepsilon)$, thus $\Gamma y_0 \subset B(\Gamma y, \varepsilon)$ for every $y \in W$.

As for the upper types of semicontinuity we have

2.8. Theorem (Dolecki [7])

Let X be a metrizable space and let Y be a topological space with countable local basis $\mathscr{B}(y_0)$ at y_0.

If Γ is u.s.c. at y_0, then Γ is u.H.s.c. (for each metric of X) at y_0 and the active boundary is compact.

This theorem generalizes the Vainstein lemma [33]: Let X and Y be as in the theorem and let $f: X \to Y$ be a continuous closed mapping. Then $\mathrm{Fr}\, f^{-1}(y_0)$ is compact. Indeed, the multifunction $\Gamma y = f^{-1}(y)$ is closed and open (Example 2.4), thus the whole boundary is active (Lemma 2.4). Γ is u.s.c. since f is closed.

If the whole boundary is active, then the converse theorem is true: If the boundary is compact and Γ is u.H.s.c. at y_0, then Γ is u.H.s.c. at y_0. However, in general this is no longer valid.

2.9. Example

Consider $\Gamma\colon \mathbf{R}\to R^2$, given by $\Gamma 0 = 0\times \mathbf{R}$, $\Gamma r=\{r\}\times\left[\frac{1}{r}, \infty\right]$ for $r\neq 0$. Observe that Γ is a u.H.s.c. at y_0, its active boundary at y_0 is empty, but Γ is not u.s.c. at y_0.

Local upper Hausdorff semicontinuities. We say that Γ is locally u.H.s.c. at (x_0, y_0), if for any neighbourhood P of x_0 there is $Q=B(x_0, \varepsilon)\subset P$ so that $\bar{Q}\cap\Gamma$ is u.H.s.c. at y_0. Γ is called δ-upper Hausdorff semicontinuous (δ-u.H.s.c.) at (x_0, y_0) if there is neighbourhood $Q=B(x_0, \varepsilon)$ such that for each $r>0$ there is a neighbourhood W of y_0 so that $W\subset\{y\colon \bar{Q}\cap\Gamma y\subset B(\Gamma y_0, r)\}$.

2.10. Example

For $a\in\mathbf{R}$, $\Gamma a=\{(x, y)\in\mathbf{R}^2\colon y=ax^2+a\}$. If $|a-a_0|<\frac{\varepsilon}{1+x^2}$, then $|ax^2+a-(a_0x^2+a_0)|<\varepsilon$, thus Γ is l.s.c. and locally u.H.s.c. everywhere. It is neither u.H.s.c. nor l.H.s.c.

It is interesting that a multifunction may be u.H.s.c. at y_0 (and consequently δ-u.H.s.c. at y_0) but not locally u.H.s.c. at (x_0, y_0).

2.11. Example

Let X be a non separable Hilbert space and let $\{x_\varepsilon\}$, $0<\varepsilon<1$ be a set with the property that $\|x_\varepsilon\|=\varepsilon$ and $B(x, \varepsilon/2)$ does not contain any x_η, $\eta\neq\varepsilon$.

Let A be the set composed of 0 and of open segments $\left(x_\varepsilon, \frac{5}{4}x_\varepsilon\right)$. For $r\in\mathbf{R}_+$ define $\Gamma r=B(A, r)$ and $\Gamma 0=A$. Of course Γ is u.H.s.c. at 0. Take any $0<\varepsilon<1$. The ball $B(0, \varepsilon)$ is disjoint with $\left(x_\varepsilon, \frac{5}{4}x_\varepsilon\right)$ so that $B\left(A\cap\overline{B(0, \varepsilon)}, \frac{\varepsilon}{4}\right)$ does not contain x_ε which is an element of $\overline{B(x_0, \varepsilon)}\cap B(A, r)$ for each $r>0$. Therefore Γ is not locally u.H.s.c. at (0, 0).

Therefore $K\cap\Gamma$ need not be u.H.s.c. at y_0 even if a multifunction Γ is u.H.s.c. at y_0 and if K is a closed subset of X.

This is in the contrast with properties of the upper semicontinuity (Kuratowski [18], I, p. 180). However the following simple theorem shows how an additional requirement of the shape of the set Γy_0 reassures the inverse implication.

2.12. Theorem

Suppose that the set Γy_0 possesses the following property: there is $\varepsilon>0$ such that for all $s>0$ there exists $r>0$ such that

$$B(\Gamma y_0\cap\overline{B(x_0, \varepsilon)}, s)\supset\overline{B(x_0, \varepsilon)}\cap B(\Gamma y_0, r) \tag{2.4}$$

and that Γ is δ-u.H.s.c. at (x_0, y_0). Then Γ is locally u.H.s.c. at (x_0, y_0).

We have a specification of Theorem 2.12 generalizing the estimates (IV.2.12 of Kato [45]).

2.13. Theorem

Let X be a normed space and suppose that Γy_0 is locally convex at x_0 (i.e. there is $\xi>0$ such that $B(x_0, \xi) \cap \Gamma y_0$ is convex). If Γ is δ-u.H.s.c. at (x_0, y_0) then it is locally u.H.s.c. at (x_0, y_0).

Proof. By assumptions there is $\varepsilon>0$ such that for each $r>0$ there is a neighbourhood W of y_0 such that for $y \in W$ $\Gamma y \cap \overline{B(x_0, \varepsilon)} \subset B(\Gamma y_0, r)$ (and we may assume that $\varepsilon \leqslant \xi/2$). We shall show that $\Gamma y \cap \overline{B(x_0, \varepsilon)} \subset B(\Gamma y_0 \cap \overline{B(x_0, \varepsilon)}, 2r)$, that is local upper Hausdorff semicontinuity at (x_0, y_0).

Let $||x_0 - x|| = \varepsilon_1 \leqslant \varepsilon$ and suppose that there exists $x_1 \in \Gamma y_0$ with $||x - x_1|| < r$. The interval $[x_0, x_1]$ is a subset of Γy_0 in view of local convexity and setting $\varepsilon_2 = \varepsilon_1 ||x_1 - x_0||^{-1}$.

$$\operatorname{dist}\left(x, \Gamma y_0 \cap \overline{B(x_0, \varepsilon)}\right) \leqslant ||x_0 + \varepsilon_2 (x_1 - x_0) - x|| \leqslant$$
$$\leqslant ||x - x_1|| + ||(1-\varepsilon_2)(x_1 - x_0)|| \leqslant r + ||(1-\varepsilon_2)(x_1 - x_0)||. \tag{2.5}$$

The second term may be estimated from the formula

$$||(1-\varepsilon_2)(x_1 - x_0)|| + ||\varepsilon_2 (x_1 - x_0)|| \leqslant ||x - x_0|| + ||x - x_1||, \quad \text{or}$$
$$||(1-\varepsilon_2)(x_1 - x_0)|| + \varepsilon_1 \leqslant \varepsilon_1 + r. \tag{2.6}$$

2.14. Example

Let F be a mapping from a normed space X to a normed space Y. We say that x_0 is regular for F, whenever there are $\varepsilon>0$ and $k>0$ such that for $x \in \overline{B(x_0, \varepsilon)}$

$$\operatorname{dist}\left(x, F^{-1}(F(x_0))\right) \leqslant k\, ||F(x) - F(x_0)|| \tag{2.7}$$

(Ioffe [42], Ioffe, Tikhomirov [17]).

Set $\Gamma y = F^{-1}(y)$ and $y_0 = F(x_0)$. Notice that (2.7) is equivalent to the following condition: if $y \in B(y_0, r)$, then $\Gamma y \cap \overline{B(x_0, \varepsilon)} \subset B(\Gamma y_0, kr)$, that is, Γ is δ-u.H.s.c. at (x_0, y_0). The fact that in Banach spaces, x_0 is regular for a continuously Fréchet differentiable mapping F with the surjective derivative $F'(x_0)$ (the Lusternik theorem [49]) will follow from our considerations. The proof that under a mild additional assumption Γy_0 fulfils (2.4) and thus Γ is locally u.H.s.c. at (x_0, y_0) will be postponed to the last section.

Moduli and rates of semicontinuity. Let X and Y be metric spaces. The definition of lower semicontinuity (at (x, y)) of a multifunction $\Gamma: Y \to 2^X$ may be restated as follows: for each $r>0$ there exists a number $q(r)>0$ such that

$$\Gamma^{-1} B(x, r) \supset B\left(y, q(r)\right). \tag{2.8}$$

A function $q: (0, r_0) \to \boldsymbol{R}_+$ which fulfils (2.8) is called a rate (of lower semicontinuity of Γ at (x, y)). Similarly we define rates of almost lower semicontinuity.

analogous quantitative definitions may be formulated for the upper Hausdorff semi-continuity (δ-u.H.s.c., local u.H.s.c. and so on). For instance, q is a rate of u.H.s.c. at y_0 if for each $r \in (0, r_0)$

$$\Gamma B(y_0, q(r)) \subset B(\Gamma y_0, r). \tag{2.9}$$

This notion (for u.H.s.c.) was introduced by Pták [24]. A sort of inverse notion was used by the author in [7]. The modulus of semicontinuity (of Γ at y_0) is the function $u: \boldsymbol{R}_+ \to \boldsymbol{R}_+$ given by

$$u(s) = \inf\{r: \Gamma B(y_0, s) \subset B(\Gamma y_0, r)\}. \tag{2.10}$$

Note, that if $r > u(s)$, then there exists a rate q such that $q(r) \leqslant s$.

Uniformity. Γ is said to be l.s.c. uniformly at (x_0, y_0) if there are $\varepsilon > 0, \eta > 0$ and a function $q: (0, r_0) \to \boldsymbol{R}_+$ such that for each $x \in B(x_0, \varepsilon)$ and each $y \in \Gamma^{-1} x \cap \cap B(y_0, \eta)$ (2.8) holds.

Similarly we define the uniform almost lower semicontinuity. Γ is δ-u.H.s.c. uniformly at (x_0, y_0), if there are $\varepsilon > 0, \eta > 0$ and a function q such that for $y \in \in B(y_0, \eta)$

$$\Gamma B(y, q(r)) \cap \overline{B(x_0, \varepsilon)} \subset B(\Gamma y, r). \tag{2.11}$$

Analogous definitions we introduce for the upper Hausdorff (and local upper Hausdorff) semicontinuity.

2.15. Theorem

Γ is l.s.c. uniformly at (x_0, y_0), if and only if Γ is δ-u.H.s.c. uniformly at (x_0, y_0). Besides the rates of semicontinuity are the same (perhaps restricted to an interval $(0, r_1)$).

Proof. Suppose the former and take positive numbers η_1, η_2 with $\eta_1 + \eta_2 = \eta$. Let $y' \in B(y_0, \eta_1)$ and take $r > 0$ ($a \vee b$ denotes the smaller number). For each $y \in B(y', \eta_2 \vee q(r))$ and for $x \in \Gamma y \cap B(x_0, \varepsilon)$, $\Gamma^{-1} B(x, r) \supset B(y, q(r))$ that is $y' \in \Gamma^{-1} B(x, r)$ and thus $x \in B(\Gamma y', r)$.

Rewriting, $\Gamma B(y', \eta_2 \vee q(r)) \cap \overline{B(x_0, \varepsilon')} \subset B(\Gamma y', r)$ for $\varepsilon' < \varepsilon$. Conversely, assume that there are $\varepsilon > 0$, $\eta > 0$, and q so that for each $y \in B(y_0, \eta)$, (2.11) holds. Split $\eta = \eta_1 + \eta_2, \eta_1 > 0, \eta_2 > 0$.

Let $y' \in B(y_0, \eta_1)$, $x \in \Gamma y' \cap B(x_0, \varepsilon)$ and $y \in B(y', q(r) \vee \eta_2)$. Therefore $y \in B(y_0, \eta)$ and fulfils (2.11), hence $x \in B(\Gamma y, r)$ or $B(x, r) \cap \Gamma y \neq \emptyset$ or else $y \in \Gamma^{-1} B(x, r)$. Thus $B(y', q(r) \vee \eta_2) \subset \Gamma^{-1} B(x, r)$.

Regions of uniform semicontinuity. Let (X, ρ) and (Y, ϑ) be metric spaces and let $\Gamma: Y \to 2^X$ be a multifunction.

Consider a positive function q defined on $(0, r_0)$ for some $r_0 > 0$.

We define a multifunction $\Gamma_q: Y \to 2^X$:

$$\Gamma_q y = \{x \in \Gamma y: \Gamma^{-1} B(x, r) \supset B(y, q(r)) \text{ for } r < r_0\}. \tag{2.12}$$

The graph $G(\Gamma_q)$ is the set of all these (x, y) at which Γ is l.s.c. at rate q. $G(\Gamma_q)$ is called the region of q-lower semicontinuity of Γ. Similarly we define

$$\Gamma_{q\sim} y=\{x \in \Gamma y: \overline{\Gamma^{-1} B(x, r)}\supset B(y, q(r)), r<r_0\}. \tag{2.13}$$

Let ω be another positive function on $(0, r_0)$. Let us introduce another multifunction

$$\Gamma_{q,\omega} y=\{x \in \Gamma y: B(\Gamma^{-1} B(x, r), \omega(r))\supset B(y, q(r)) \text{ for } r<r_0\}. \tag{2.14}$$

We point out that

$$G(\Gamma_q)\subset G(\Gamma_{q\sim})=\bigcap_{\omega} G(\Gamma_{q,\omega})\subset G(\Gamma_{q,\omega})\subset G(\Gamma) \tag{2.15}$$

where the intersection is taken over all ω such that $\omega(r)\to 0$, as $r\to 0$.

Observe that Γ is uniformly l.s.c. at (x_0, y_0) with rate q if and only if there are balls $B(x_0, \varepsilon)$, $B(y_0, \eta)$ such that for $B=B(x_0, \varepsilon)\times B(y_0, \eta)$

$$B\cap G(\Gamma)\subset G(\Gamma_q). \tag{2.16}$$

The uniform almost lower semicontinuity at (x_0, y_0) (with rate q) admits the following interpretation in terms of $\Gamma_{q\sim}$

$$B\cap G(\Gamma)\subset G(\Gamma_{q\sim}). \tag{2.17}$$

3. Approximation theorem

The purpose of this section is to show that for closed multifunctions the uniform almost lower semicontinuity entails the (uniform) lower semicontinuity, and equivalently (in virtue of Theorem 2.15) the uniform δ-upper Hausdorff semicontinuity. In fact, we shall derive the above conclusions from a more general approximation property for a multifunction. We start with the Pták nondiscrete induction theorem.

Following Pták [24] we say that a mapping ω: $(0, r_0)\to(0, r_0)$ is a small function, whenever the sum

$$\sigma(r)=r+\omega(r)+\omega(\omega(r))+\dots \tag{3.1}$$

is finite for each $r \in (0, r_0)$.

Let Z be a multifunction from $(0, r_0)$ with $Z(r)\subset X$, where (X, ρ) is a complete metric space. The limit of Z as r tends to 0 is given by

$$Z(0)=\bigcap_{s>0} \overline{\bigcup_{r\leq s} Z(r)}. \tag{3.2}$$

3.1. Theorem (Pták [24], [25])

Suppose that for each $0<r<r_0$

$$Z(r)\subset B(Z(\omega(r)), r). \tag{3.3}$$

Then for each $0<r<r_0$

$$Z(r)\subset B(Z(0), \sigma(r)). \tag{3.4}$$

We shall be concerned with tiny functions: those small functions that are non-decreasing and tend to zero as r does. For such the functions the sum σ tends to zero with r.

3.2. Theorem

Let Γ be a closed multifunction from a metric space Y to subsets of a complete metric space X. Let q be a positive function tending to zero with r and let ω be a tiny function (both defined on $(0, r_0)$). Assume that $x_0 \in \Gamma y_0$. If there are $\varepsilon>0, \eta>0$ such that

$$B(x_0, \varepsilon)\times B(y_0, \eta)\cap G(\Gamma)\subset G(\Gamma_{q, q\omega}) \tag{3.5}$$

then there is $r_2>0$ such that Γ is δ-u.H.s.c. uniformly at (x_0, y_0) at rate p such that

$$p(r)\leq q(m(r)),\ r<r_2, \text{ where } m(r)<\sup_{\delta(t)\leq r} t. \tag{3.6}$$

Proof. Let $y\in B(y_0, \eta)$ and let $x\in \Gamma y\cap B(x_0, \varepsilon)$. In view of the assumptions $(x, y)\in G\times\times(\Gamma_{q,q\omega})$, hence for $y'\in B(y, q(r))$ one has that $B(y', q(\omega(r)))\cap\Gamma^{-1}\times B(x, r)\neq\emptyset$. Equivalently $\Gamma B(y', q(\omega(r)))\cap B(x, r)\neq\emptyset$ or $x\in B(\Gamma B(y', q\times(\omega(r))), r)$.

Represent $\eta=\eta_1+\eta_2$, $\eta_1>0$, $\eta_2>0$. We have just proved that for $y'\in B(y_0, \eta_1)$ and $y\in B(y', q(r)\vee\eta_2)$, $\Gamma y\cap B(x_0, \varepsilon)\subset B(\Gamma B(y', q(\omega(r))), r)$, that is

$$\Gamma B(y', q(r)\vee\eta_2)\cap B(x_0, \varepsilon)\subset B(\Gamma B(y', q(\omega(r))), r). \tag{3.7}$$

Set

$$Z(r)=\Gamma B(y', q(r))\cap B(x_0, \varepsilon-\sigma(r)). \tag{3.8}$$

In order to apply the Pták theorem we shall show that (3.7) entails (3.3) with Z given by (3.8). In fact, from (3.7) it follows that for $r<r_1$, $q(r_1)\leq\eta_2$, $\Gamma B(y', q(r))\cap\cap B(x_0, \varepsilon-\sigma(r))\subset B(\Gamma B(y', q(\omega(r)))\cap B(x_0, \varepsilon-\sigma(r)+r), r)$, which in view of the equality $\sigma(\omega(r))=\sigma(r)-r$ becomes

$$\Gamma B(y', q(r))\cap B(x_0, \varepsilon-\sigma(r))\subset B(\Gamma B(y', q(\omega(r)))\cap B(x_0, \varepsilon-\sigma(\omega(r))), r). \tag{3.9}$$

Recalling the definition of Z (3.8) we recognize the first step of the Pták theorem. A standard argument shows that, because of closedness of $G(\Gamma)$, $\bigcap_{s<0}\overline{\bigcup_{t\leq s}\Gamma B(y', q(t))}=\Gamma y'$, thus $Z(0)\subset\Gamma y'$ and applying the second step we have for $r\leq r_1$

$$\Gamma B(y', q(r))\cap B(x_0, \varepsilon-\sigma(r))\subset B(\Gamma y', \sigma(r)). \tag{3.10}$$

For any $\varepsilon_1<\varepsilon$ we may find $r_2\leq r_1$ such that for $r<r_2$ (3.10) is valid with $B(x_0, \varepsilon-\sigma(r))$ replaced by $\overline{B(x_0, \varepsilon')}$. For any $p\colon (0, r_2)\to\boldsymbol{R}_+$ such that (3.6) holds

$$\Gamma B(y', p(r))\cap B(x_0, \varepsilon')\subset B(\Gamma y', r),\ r<r_2. \tag{3.11}$$

3.3. Corollary

Under the above assumptions there is a neighbourhood B of (x_0, y_0) such that

$$B\cap G(\Gamma)\subset G(\Gamma_p). \tag{2.16'}$$

3.4. Corollary

Let X, Y, q and Γ be as in Theorem 3.2 and let $x_0 \in \Gamma y_0$. Suppose that there are $\varepsilon>0$, $\eta>0$ such that for each $x \in B(x_0, \varepsilon)$ and every $y \in \Gamma^{-1} x \cap B(y_0, \eta)$,

$$\overline{\Gamma^{-1} B(x, r)} \supset B(y, q(r)). \tag{3.12}$$

Then there are $\varepsilon_1>0, \eta_1>0$ and $r_1 \leqslant r_0$ such that for each $r<r'<r_1$ and for each $x \in B(x_0, \varepsilon), y \in \Gamma^{-1} x \cap B(y_0, \eta_1)$

$$\Gamma^{-1} B(x, r') \supset B(y, q(r)). \tag{3.13}$$

Indeed (3.12) entails the assumptions of Theorem 3.2 for arbitrarily tiny ω. Thus σ may be arbitrarily close to 1 and p to q.

The corollary is a local version of the closed graph theorem of Pták [24]. The assumptions (and the thesis) of the Pták theorem are (3.12) (and (3.13)) uniform everywhere: for each $x \in X$ and $y \in \Gamma^{-1} x$. The closed graph theorem (with the additional assumption that q tends to zero with r) follows immediately from Corollary 3.4.

Strivings to derive the corollary directly from the closed graph theorem by restricting it to the metric space $\overline{B(x_0, \varepsilon)} \times B(y_0, \eta)$ encounter strong difficulties in replacing (3.12) by $\overline{\Gamma^{-1} (B(x, r) \cap B(x_0, \varepsilon))} \supset B(y, q(r)) \cap B(y_0, \eta)$.

The course of the proof of Theorem 3.2 resembles the first part of the original proof of the Pták closed graph theorem [24], but our argument requires several refinements imposed especially by the localization.

The first part of Pták's proof concludes with, a (not stated explicitely) u.H.s.c. result. The second part of Pták's proof is related to the necessity part of out Theorem 2.15.

Półciągłość w optymalizacji z ograniczeniami. Część I

W pracy rozwinięto lokalną teorię o domkniętym wykresie analogiczną do (globalnej) teorii Ptáka. Stosując tę teorię do klasy multifunkcji prawie wypukłych (którą wprowadza się w pracy) otrzymuje się szczególnie silne wyniki. Uogólniają one kilka klasycznych twierdzeń (twierdzenie Banacha o odwzorowaniu otwartym, twierdzenie Lusternika). Stosuje się je z powodzeniem w Części II do teorii optymalizacji.

Полунепрерывность воптимизаnии с ограничениями. (Часть I)

В работе представлена расширенная локальная теория о замкнутом графике, аналогично (глобальной) теории Птака. Используя эту теорию для класса квазивыпуклых мультифункций (вводимого в данной работе) достигаются особо сильные результаты. Они обобщают несколько классических теорем (теорема Банаха об открытом отображении, теорема Люстерника).Они с успехом используются во 2-й части в теории оптимизации.