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On a further extension of the method of minimally interconnected subnetworks

by

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The paper is a further extension of the authors' work [2]. In the mentioned article, the method of minimally interconnected subnetworks, as given in [6], is extended and generalized to arbitrary weighted graphs. In particular, this generalization refers to unigraphs with nonnegative real weights. Basic notions and properties of minimally interconnected subnetworks, as given in [2], are listed for convenience. Many new ones are formulated and proved. Main extension consists in a new algorithm for the determination of minimal groups. It is more efficient, but its main feature lies at the reducing of computer storage requirement which makes it possible to solve substantially larger problems.

1. Introduction.

The idea of minimally interconnected subnetworks was introduced by R. Luccio and M. Sami [6]. They dealt with the problem of some decomposition for electrical networks. In the setting of graph theory, their method referred to the partitioning of a multigraph with edge weight equal one.

The idea mentioned above is generalized and extended in the authors' last paper [2] to arbitrary graphs. In particular, it refers to unigraphs with nonnegative real edge weights. It proved to be a relatively efficient technique for solving the problems of graph partitioning type consisting in the decomposition of a set of vertices into subsets. The decomposition mentioned is performed such that the strength of mutual connections between vertices in a subset is greater than the analogous parameter computed for these vertices and the ones not belonging to this subset. The applications are shown in three recent papers of authors et al.: in [4] for the decomposition of the telephone interexchange network structure, in [1] for the partitioning of a group of enterprises into subgroups and in [3] for the partitioning of a computer network into subnetworks.

This paper presents a further extension and generalization of the method described in [2]. For convenience, all notions and properties formulated and proved there are repeated. The proofs are omitted, though. New properties, on which the extension and generalization is based, are formulated in the form of appropriate lemmas, propositions, theorems etc. and proved.

The algorithm given in [2] for the determination of minimal groups is modified. Main feature of this modification consists in the reduction of computer storage requirement which makes it possible to handle much larger problems.

2. Basic notions and properties of minimal groups

Let us consider a graph G, complete, undirected and without loops. Denote the set of its vertices by V, where $V=\{1, 2, ..., n)$. E will be the set of edges of G. Let us define a function:

$$f: E \rightarrow R^{+} \cup \{0\} \tag{1}$$

mapping the set of edges E into the set of positive real numbers R^{+} completed by zero. A particular value of f, f_{ij}, is the weight of the edge connecting the i-th vertex with the j-th one. All f_{ij}, $i, j \in V$, can be conveniently represented in the form of a matrix F, $\dim F = |V| \times |V|$, where $|V|$ is the cardinality of V. The matrix F is evidently a symmetric one, i.e. $f_{ij}=f_{ji}$ and — moreover — $f_{ii} \stackrel{\text{df}}{=} 0$, for all $i, j \in V$.

Let us denote in the sequel the ordered pair consisting of the graph G and the function f as in (1) by $\langle G, f \rangle$ or by $\langle G, F \rangle$.

Definition 1. For a given $\langle G, F \rangle$, any subset $W \subset V$ taken with all the edges connecting each pair of its elements is called a group W.

In the sequal, the groups as well as the corresponding sets of vertices are denoted by capital Latin letters. Thus, all set — theoretic operations performed on groups, i.e. the inclusion ($\subset$), the union ($\cup$), the intersection ($\cap$), the difference ($\setminus$) refer, if not otherwise indicated, to the corresponding sets of vertices. The corresponding sets of edges are only added to the product of an operation mentioned above. The same refers to the cardinality ($|\cdot|$).

Remark. In order to simplify later notations, let us write:

$$f(R, S) = \sum_{\substack{i \in R \\ j \in S}} f_{ij} \tag{2}$$

where: $S, R \subset V$, $S \cap R = \emptyset$.

In the case, where S is the complement of R to V, $f(R, S) = f(R, V \setminus R)$ corresponds to the group R and is denoted by r.

DEFINITION 2. For a given $\langle G, f \rangle$, a nonempty group S, such that for every nonempty $R \subset S$, $R \neq S$, the inequality:

$$r > s \tag{3}$$

holds, is called the minimal group. Moreover, each single vertex of G is the minimal group by definition.

Now it is expedient to present formally the class of problems considered in [1, 3, 4, 6].

The problem, in terms of the graph theory, concerns the partitioning of the set of vertices V into subsets $V_1, V_2, \ldots, V_k$, such that:

$$V_i \cap V_j = \emptyset \text{ for all } i, j \in [1, k], i \neq j \tag{4}$$

$$\bigcup_{i=1}^{k} V_i = V \tag{5}$$

$$f(R, V_i \setminus W) > f(R, V \setminus V_i) \text{ for all } R \subset V_i, \; \emptyset \neq R \neq V_i \text{ and for all } i \in [1, k]. \tag{6}$$

As we will see, the determination of minimal groups will lead to the solution of the problem defined by (4), (5) and (6).

COROLLARY 1. If S is a minimal groups in $\langle G, f \rangle$, then for every nonempty $R \subset S$, $R \neq S$:

$$r > 0 \tag{7}$$

This corollary results directly from the Definition 2.

LEMMA 1. For a given $\langle G, f \rangle$, a group S is minimal iff for every nonempty $R \subset S$, $R \neq S$, the following inequality holds:

$$f(R, S \setminus R) > f(S \setminus R, V \setminus S). \tag{8}$$

Proof. The proof of the necessity is given in [2]. Let us prove the sufficiency. Adding $f(R, V \setminus S)$ to both sides of (8), we obtain:

$$f(R, S \setminus R) + f(R, V \setminus S) > f(S \setminus R, V \setminus S) + f(R, V \setminus S). \tag{9}$$

From (2) there follows:

$$r = f(R, S \setminus R) + f(R, V \setminus S), \tag{10}$$

$$s = f(S \setminus R, V \setminus S) + f(R, V \setminus S), \tag{11}$$

which, due to (3), terminates the proof. Q.E.D.

The formula (8) can be interpreted in such a way that the entire dependence of a nonempty proper subset $Q = S \setminus R$ of the group S on its complemet R in this group is greater than the analogous parameter for Q and $V \setminus S$, respectively. It is one of basic properties of minimal groups, because it indicates their usefulness for applications.

LEMMA 2. Two minimal groups in $\langle G, f \rangle$ are either disjoint or one of them is contained in the other.

This lemma is of great importance for the construction of an efficient algorithm for the determination of minimal groups, because it makes the inclusion relation order partially the set of all possible minimal groups for a given $\langle G, f \rangle$.

3. Further properties of minimal groups

In this section, some new properties of minimal groups are presented in addition to those given in [2] and repeated here for convenience.

PROPOSITION 1. Let V_i, $i \in I = \{1, 2, ..., m\}$, be given pairwise disjoint minimal groups in $\langle G, f \rangle$ and R_i, $i \in I$ — their proper parts. We denote by S a group, such that $S \cap V_i \neq \emptyset$ for every $i \in I$. Then, if there exists an index $j \in I$, for which $R_j \neq \emptyset$, the group:

$$S \cup \bigcup_{i \in I} R_i \tag{12}$$

is not minimal.

Proposition 1 is an extension and consequence of Lemma 2. It states that a group containing a nonempty proper part of at least one another minimal group cannot be minimal.

PROPOSITION 2. Let V_i, I and R_i have the same meaning as in the previous proposition. We assume that there exists an index $j \in I$ such that $R_j \neq \emptyset$. Let us denote:

$$S = \bigcup_{i \in I} R_i . \tag{13}$$

Then, the following inequality holds:

$$s > \max \{v_i : i \in I\} . \tag{14}$$

Proposition 2 is of a similar importance for the construction of an efficient algorithm for the determination of minimal groups as Proposition 1.

THEOREM 1. Let us denote: $I = \{1, 2, ..., m\}$ and $J \subset I$, $J \neq I$, $|J| \geqslant 2$. If V_i, $i \in I$, are pairwise disjoint minimal groups in $\langle G, f \rangle$ and if for every J the group:

$$S_J = \bigcup_{i \in J} V_i \tag{15}$$

is not minimal, then the following inequality holds:

$$s_I \geqslant \min \{v_i : i \in I\} . \tag{16}$$

An immediate consequence of Theorem 1 is the following proposition.

Proposition 3. Let the notation be the same as in Theorem 1. Then, if for every $J \neq I$, S_J is not minimal, the necessary and sufficient condition for S_I to be a minimal groups is that the following inequality holds:

$$s_I < \min \{v_i : i \in I\}. \tag{17}$$

Proof. The necessity is proved in [2]. Let us now proceed to the proof of sufficiency which is very simple in fact. Namely, for every $J \neq I$, $|J| \geqslant 2$, let S_J be not a minimal group and S_I — a minimal one. Then, due to Definition 2, it has to be: $s_I < v_i$ for every $i \in I$. Q.E.D.

In fact, if S_I would not be a minimal group then, by Theorem 1, the inequality (16) would have to hold.

Proposition 1 and Proposition 2 refer mainly to the proper parts of minimal groups. Theorem 1 and Proposition 3 make it possible to extend the properties of vertices in $\langle G, f \rangle$ (minimal groups by definition) onto all minimal groups. More precisely, it follows from them that every minimal group actually found can be considered as a vertex of a new, modified weighted graph.

The properties of minimal groups mentioned above are sufficient for devising an algorithm for the generation of all minimal groups in $\langle G, f \rangle$. The new algorithm will be described in section 6. Now we proceed to a brief presentation of further properties of minimal groups.

Proposition 4. Let R_i, S_i, $i \in I = \{1, 2, ..., m\}$, be pairwise disjoint groups. For all $i \in I$, let $R_i \neq \emptyset$ and $V_i = R_i \cup S_i$ be a minimal group in $\langle G, f \rangle$. Let us denote:

$$\begin{aligned} P &= \bigcup_{i \in I} V_i, \\ Q &= \bigcup_{i \in I} R_i. \end{aligned} \tag{18}$$

If there exists such $j \in I$ that $S_j \neq \emptyset$, then the following inequality holds:

$$p < q \tag{19}$$

4. On some specific groups

The properties of groups given in previous sections are quite sufficient for the construction of an efficient algorithm for the determination of minimal groups. Relations given below can substantially improve, however, the efficency of the algorithm, especially for large scale problems.

Proposition 5. If in $\langle G, f \rangle$ there exists such $Q \subset V$, $|Q| \geqslant 3$, that for every $i, j \in Q$, $f_{ij} = f_0 = \text{const.}$, then every group $S \subset Q$, $S \neq Q$, $|S| > 1$, is not minimal.

This proposition makes it possible to eliminate from considerations the subset of sets of vertices connected by edges with the same weight.

PROPOSITION 6. Let, for a given $\langle G, f\rangle$, such a $Q \subset V$, $|Q| \geqslant 2$, exists that for every $i, j \in Q$ the following relation holds:

$$f_{ij} = f_0 = \max \{f_{ij}: i, j \in V\}. \tag{20}$$

We assume that $R \subset Q$, $R \neq Q$, $P \subset V \setminus Q$, $P \neq V \setminus Q$. If the inequality:

$$|Q| \geqslant |R| + |P| \tag{21}$$

holds, then the group $H = R \cup P$ is not minimal.

PROPOSITION 7. Let, for a given $\langle G, f\rangle$, such a $Q \subset V$, $|Q| \geqslant 2$, exists that for every $i, j \in Q$ the equality (20) holds. We assume $f_0 \neq f_{ij}$ for other pairs $i, j \in V$. Then, Q is minimal group iff for every $x \in Q$:

$$(|Q| - 1) f_0 > f(Q \setminus \{x\}, V \setminus Q). \tag{22}$$

Propositions 5, 6 and 7 concern some specific groups. In these groups every pair of vertices is connected by the edge with the same weight. This case is of importance from the practical point of view. Indeed, in many applications there often occur cases with a great number of approximately equal weights. The weights mentioned are taken from estimations and, for practical purposes, one may assume that they are equal one to another.

5. Remarks on the generation of minimal groups

In applications of the method discussed (see e.g. [1, 3, 4]), the problem represented in terms of partitioning a graph consists in finding a family of minimal groups, which satisfies the conditions (4), (5). Let us denote this family by B_j, where:

$$B_j = \{V_i^{(j)}: i \in [1, k(j)]\}, j \in A. \tag{23}$$

A is the symbol of the set of indices referred to an arbitrary pair $\langle G_j, F_j\rangle$ consisting of a graph G_j and a matrix F_j. The equality holds:

$$A = A^* \cup \{0\} \tag{24}$$

where A^* is a set of consecutive natural numbers beginning from one. The method of the construction of $B_j's$ will be discussed later.

Let us now assume that there are found $i, i \geqslant 0$, minimal groups. Let us denote:

$$S_i^{(j)} = \begin{cases} \emptyset & \text{if } i = 0 \\ \bigcup_{p=1}^{i} V_p^{(j)} & \text{otherwise.} \end{cases} \tag{25}$$

Then, the next minimal groups is sought among the elements of the set:

$$R_i^{(j)} = V^{(j)} \setminus S_i^{(j)}, \tag{26}$$

where $V^{(j)}$ denotes the set of vertices of G_j. The procedure is continued until the condition is satisfied that any subset of $R_m^{(j)}$, $m \geq 1$, with the cardinality greater than 2 does not create a minimal group. Then, the solution is in the form:

$$B_j = \{V_i^{(j)} : i \in [1, m]\} \cup \{\{v_i\} : v_i \in R_m^{(j)}\} \tag{27}$$

where $R_m^{(j)} \neq \emptyset$ and, otherwise:

$$B_j = \{V_i^{(j)} : i \in [1, m]\}. \tag{28}$$

In the first case, the following condition holds:

$$k(j) = m + |R_m^{(j)}|. \tag{29}$$

In the second one, we have:

$$k(j) = m. \tag{30}$$

The procedure described above refers to a fixed pair $\langle G_j, F_j \rangle$. Due to Theorem 1 and Proposition 3, the properties of vertices are extended to arbitrary minimal groups. Then, the graph to be partitioned is modified at every stage, i.e. after finding any $B_j, j \in A$.

The process starts from $\langle G_1, F_1 \rangle = \langle G, F \rangle$ (e.g. Fig. 1). In the preliminary step one has:

$$B_0 = \{\{v_i\} : v_i \in V\}. \tag{31}$$

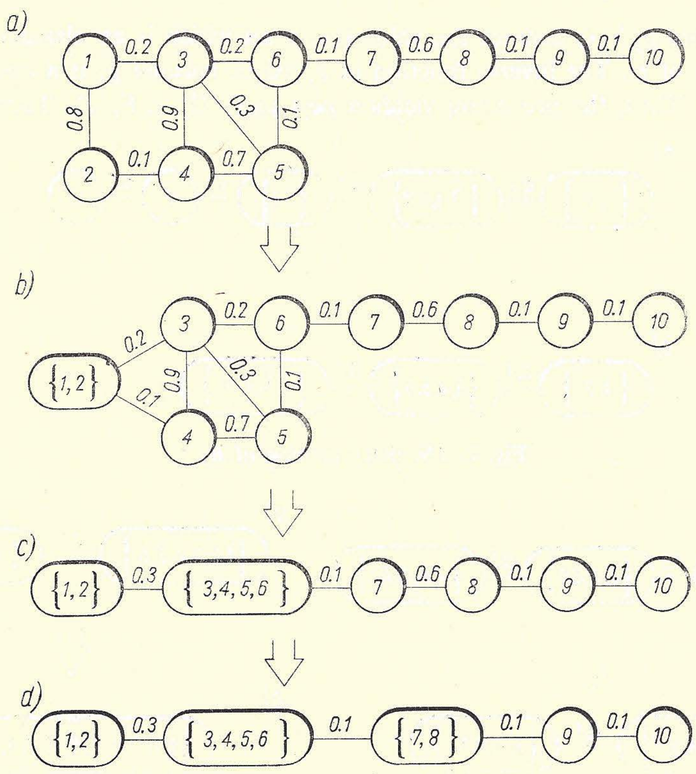

Fig. 1. The determination of B_1

B_1 is constructed in the way described above. Then, to be more general, let us assume to have a pair $\langle G_j, F_j \rangle$. We obtained B_j in the form described by the formula (27) or (28). The function:

$$g_j: B_j \to V^{(j+1)} \tag{32}$$

is now defined. It assigns to each minimal group from B_j a single point of a set $V^{(j+1)}$, where $|V^{(j+1)}| = |B_j|$. For convenience, one can assume that:

$$g_j(V_i^{(j)}) = i. \tag{33}$$

It is easy to note that the mapping g_j is one-onto-one. The graph G_{j+1} is constructed by connecting every two different vertices $v_1, v_2 \in V^{(j+1)}$ by an nonoriented edge. Further, a new matrix $F_{j+1} = [f_{rp}^{(j+1)}]$ is defined, $\dim F_{j+1} = |B_j| \times |B_j|$. Its elements are expressed by:

$$f_{rp}^{(j+1)} = \begin{cases} f_{r_1 p_1}^{(j)} & \text{if } \{r_1\}, \{p_1\} \in B_j \text{ and } g_j(\{r_1\}) = r, g_j(\{p_1\}) = p; \\ 0 & \text{if } i = p; \\ \sum\limits_{\substack{\langle s, t \rangle \\ s \in g_j^{-1}(r) \\ t \in g_j^{-1}(p)}} f_{st} & \text{in all other cases.} \end{cases} \tag{34}$$

The symbol $g_j^{-1}(\cdot)$ denotes the value (i.e. a set which is an element of B_j) of the inversion of g_j. The inverse function of g_j exists, because g_j is a one-onto-one mapping [5]. Thus, the procedure yields a new pair $\langle G_{j+1}, F_{j+1} \rangle$. Then, one can

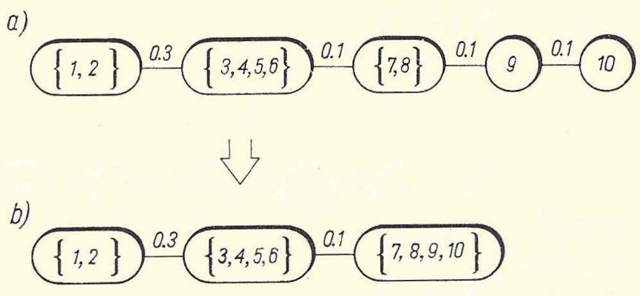

Fig. 2. The determination of B_2

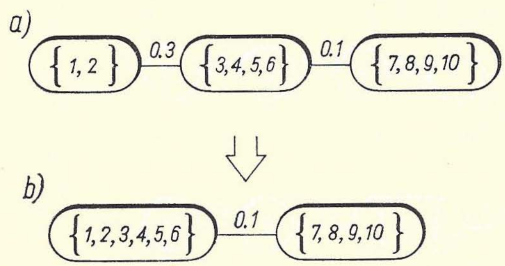

Fig. 3. The determination of B_3

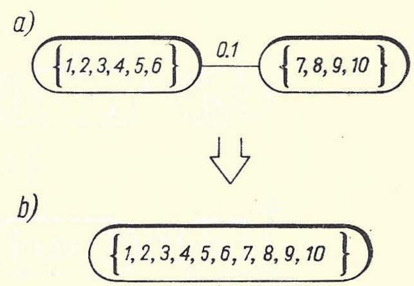

Fig. 4. The determination of B_4

obtain B_{j+1}, etc. It is clear that the algorithm terminates, when $|B_l|=1$. The other case is, when the matrix F_{l+1} contains all zeros (Corollary 1). An example is shown in Figures 2, 3 and 4. They present a continuation of the process shown in Fig. 1. Then, one can write: $A^*=\{1, 2, ..., l\}$.

6. On some properties of the new algorithm for the determination of minimal groups

The purpose of this section is to discuss some properties of the method for generating subsequent B_j' s. It serves as an algorithm for the determination of minimal groups. Our aim will mainly be to show that the method mentioned really gives all the minimal goups sought and, in addition, how often some minimal groups are generated. Other properties proved help to formulate and prove those principal ones.

THEOREM 2. Let the method of searching of minimal groups be fixed in a certain way. Then, for a given $\langle G, F\rangle$, there exists one and only one B.

Proof. Let us assume that we have two B' s: B' and B''. Let t denote the smallest index, for which:

$$V_t' \neq V_t''. \tag{35}$$

On the other side, we have two T_t' s: R_t' and R_t''. Because t is the smallest one, then:

$$\{V_i' : i<t\}=\{V_i'' : i<t\}.$$

That means: $R_t'=R_t''$. In consequence:

$$V_t'=V_t'', \tag{36}$$

which is impossible. Hence, we have $B'=B''$. Q.E.D.

The following corollary is a natural consequence of the above theorem:

COROLLARY 2. Let the method of searching of minimal groups be fixed. For a given $\langle G, F\rangle$ and the family of mappings $\{g_j : j\in A^*\}$ there exists one and only one family $\{B_j : j\in A\}$.

For the proof let us denote that in every case $B_0=V$. The further part of reasoning proceeds by the mathematical induction due to the Theorem 2.

Then, one can write that the result of the algorithm depends, maybe, only on the enumeration of vertices and on the method of searching of minimal groups.

REMARK. Till the end of this section it will be assumed that the method of searching of minimal groups is fixed and the same in the whole algorithm.

Let a family of one-onto-one mappings be set up:

$$y_p : V^{(p)}\to B_p, \quad p\in A. \tag{37}$$

Then, there exist inverse functions [5]:

$$y_p^{-1}: B_p \to V^{(p)}, \quad p \in A. \tag{38}$$

It is evident that:

$$y_0 = y_0^{-1} = I \tag{39}$$

where by I we denote the indentity transformation.

One can see that $\{y_p: p \in A\}$ depends on the methods of searching of minimal groups. The value of the transformation y_p^{-1} for particular $p \in A$ is given by the formula:

$$y_p^{-1}(B_p) = \bigcup_{i=1}^{k(p)} V_i^{(p)}. \tag{40}$$

For convenience, we also assume that:

$$g_0 = I. \tag{41}$$

Let us denote by $g \circ f$ the composition of two mappings g and f:

$$g[f(x)] = (g \circ f)(x). \tag{42}$$

Now, we define four families of mappings: $\{Y_p: p \in A\}$, $\{Y_p^{-1}: p \in A\}$, $\{h_p: p \in A\}$ and $\{q_p: p \in A\}$. The last two of those mentioned are given by recurrence relations:

$$Y_p = g_{p-1} \circ y_{p-1}, \quad p \in A^*; \tag{43}$$

$$Y_p^{-1} = y_{p-1}^{-1} \circ g_{p-1}^{-1}, \quad p \in A^*; \tag{44}$$

$$\left.\begin{array}{ll} 1. & h_0 = I, \\ 2. & h_1 = Y_p^{-1}, \\ 3. & h_p = h_{p-1} \circ Y_p^{-1}, \quad p \in A^*; \end{array}\right\} \tag{45}$$

$$\left.\begin{array}{ll} 1. & q_0 = I, \\ 2. & q_p = Y_{p-1} \circ q_{p-1}, \quad p \in A^*. \end{array}\right\} \tag{46}$$

It is clear that for a given $p \in A^*$, $Y_p: V^{(p-1)} \to V^{(p)}$. Y_p^{-1} is the inverse of Y_p. Analogously, $h_p: V^{(p)} \to V$ and $q_p: V \to V^{(p)}, p \in A$.

Let us now discuss some properties of $\{h_p: p \in A\}$.

LEMMA 3. If $C, D \subset V^{(p)}, p \in A$, then:

$$h_p(C) \cap h_p(D) = h_p(C \cap D), \tag{47}$$

and

$$h_p(C) \cup h_p(D) = h_p(C \cup D). \tag{48}$$

Proof. The mapping h_p is one-onto-one because it is the composition of one-onto--one mappings. Then, the formulae (47) and (48) hold. Q.E.D.

THEOREM 3. Let $j \in A$. Then, for each $i \in [1, k(j)]$ and each $p \in [1, k(j)] \setminus \{i\}$:

$$h_j(V_i^{(j)}) \cap h_j(V_p^{(j)}) = \emptyset. \tag{49}$$

Proof. Let us assume that $i<p$. Then, directly from formulae (25) and (26), we have [5]:

$$h_j(V_i^{(j)}) \cap h_j(V_p^{(j)}) \subset h_j(V_i^{(j)}) \cap h_j(V^{(j)} \setminus S_i^{(j)}) =$$
$$= h_j(V_i^{(j)} \cap [V^{(j)} \setminus \bigcup_{r=1}^{i} V_r^{(j)}]) = h_j(\emptyset) = \emptyset. \tag{50}$$

The case $i>p$ is accomplished in the same way. Q.E.D.

COROLLARY 3. Each system $\{h_j(V_i^{(j)}) \colon V_i^{(j)} \in B_j\}$, $j \in A^*$ satisfies the criteria (4), (5) and (6).

Proof. The criterion (4) is satisfied for each $B_j, j \in A^*$, due to the Theorem 3. Now we consider the criterion (5). From Lemma 3 one deduces that:

$$\bigcup_{i=1}^{k(j)} h_j(V_i^{(j)}) = h_j(\bigcup_{i=1}^{k(j)} V_i^{(j)}) = h_j(V^{(j)}). \tag{51}$$

From the definition of $\{h_j \colon j \in A\}$ the following equality holds

$$h_j(V^{(j)}) = V.$$

Due to the formulae (41) and (42), the criterion (5) holds.

The criterion (6) is satisfied due to the method of the generation of B_j' s and due to Lemma 1. Q.E.D.

The above mentioned corollary gives a base for the application of the algorithm for determining consecutive B_j' s for solving the problem formulated by the criteria (4), (5) and (6). Furthermore, the Corollary 2 states that the results of the algorithm are uniquely determined.

Now, it is sufficient to prove that there exist some families of functions $\{Y_p \colon p \in \in A\}$ and $\{q_p \colon p \in A\}$ for which all the minimal groups in the weighted graph $\langle G, F \rangle$ are determined. One can notice that the families mentioned serve as the method for searching of minimal groups.

Assumption. Now, it will be assumed that for every $R_i^{(j)}$, $j \in A$, $i \in [0, k(j)-1]$, the method for searching of minimal groups is as follows. All the groups of a given cardinal number are tested. At the beginning, one starts from the cardinality equals 2, passes to that of 3, etc. until $|R_i^{(j)}|$. The testing terminates when a minimal group is found or, in an extremal case, when all the nonempty subsets of $R_i^{(j)}$ are checked out and there is no minimal group with the cardinal number greater than one. In the second case, the process of the determination of the B_j should be finished. Then, the following equality holds: $m=i$ and in this B_j there are exactly $|R_i^{(j)}|$ minimal groups with the cardinal number being one in the considered weighted graph $\langle G_j, F_j \rangle$. In the first case, one is able to proceed as it was described in the previous section of the paper.

Now, we denote by $\mathscr{L}_{\langle G, F \rangle}$ the family of all minimal groups in $\langle G, F \rangle$.

THEOREM 4. Let us consider the following algorithm. The consecutive B_j' s are generated in the way described in the previous section and the method of searching of minimal groups are the same as in the assumption. The algorithm under consideration gives the result in the form: $\{B_j : j \in A\}$. Then, the following equality holds:

$$\mathscr{L}_{\langle G, F \rangle} = \{h_j (V_i^{(j)}) : V_i^{(j)} \in B_j,\ j \in A\}. \tag{52}$$

Proof. The proof will proceed by the reductio ad absurdum. We assume that there exists a $S \in \mathscr{L}_{\langle G, F \rangle}$ which is not determined during the algorithm under consideration. In other words, there is no such p, $p \in A$, and i, $i \in [1, k(p)]$, that $q_p(S) = V_i^{(p)}$. We will consider two cases.

Case 1. There is no such minimal group $W \subset V$, that $W \subset S$ and $W \neq S$. So, if $|S| = 1$, then $S \in B_0$. Further, if $|S| > 1$, then the sequence $(R_i^{(1)} : i \in [0, k(i)])$ exists, where for every i, $i \in [0, k(i)]$, one has: $R_i^{(1)} \supset q_1(S)$. Due to the Lemma 2, any proper part of $q_1(S)$ cannot be a part of another minimal groups. Hence, either the minimal group $q_1(Q)$ is generated, where $q_1(Q) \supset q_1(S)$, $q_1(Q) \neq q_1(S)$ or the elements of $q_1(S)$ are assumed to be minimal groups with the cardinality equals one. This is in opposition to the assumption.

Case 2. The minimal group $W \subset S$, $W \neq S$, exists. Let S be the smallest minimal group satisfying the above condition. We denote by $\mathscr{W} = \{V_i\}$ the family of disjoint minimal groups such that:

$$\bigcup_{V_i \in \mathscr{W}} V_i = S. \tag{53}$$

It is evident that for every V_i mentioned there exists such a pair $p, r, p \in A$, $r \in [1, k(p)]$, that the following formula holds:

$$h_p (V_r^{(p)}) = V_i. \tag{54}$$

Let P denote the set of all such smallest p's given for every i, $V_i \in \mathscr{W}$. Let us define:

$$p^* = \max \{p : p \in P\}. \tag{55}$$

Hence, due to the above construction, we have for every i, $V_i \in \mathscr{W}$:

$$q_{p^*}(V_i) = v_i \in V^{(p)}. \tag{56}$$

Furthermore, in the similar way as in the preceeding part of the proof the contradiction is received. Q.E.D.

7. The computational algorithm

The algorithm described is a modification of that given in [2]. The basic improvement consists in the reduced computer storage requirement. The waste of time caused by the initial considering of multielement vertex combinations is cancelled, since the rewritting of large tables is avoided.

The new method consists in the determination of subsequent B_j' s as described in previous sections. A simplified flow-diagram of the algorithm is shown in Fig. 5.

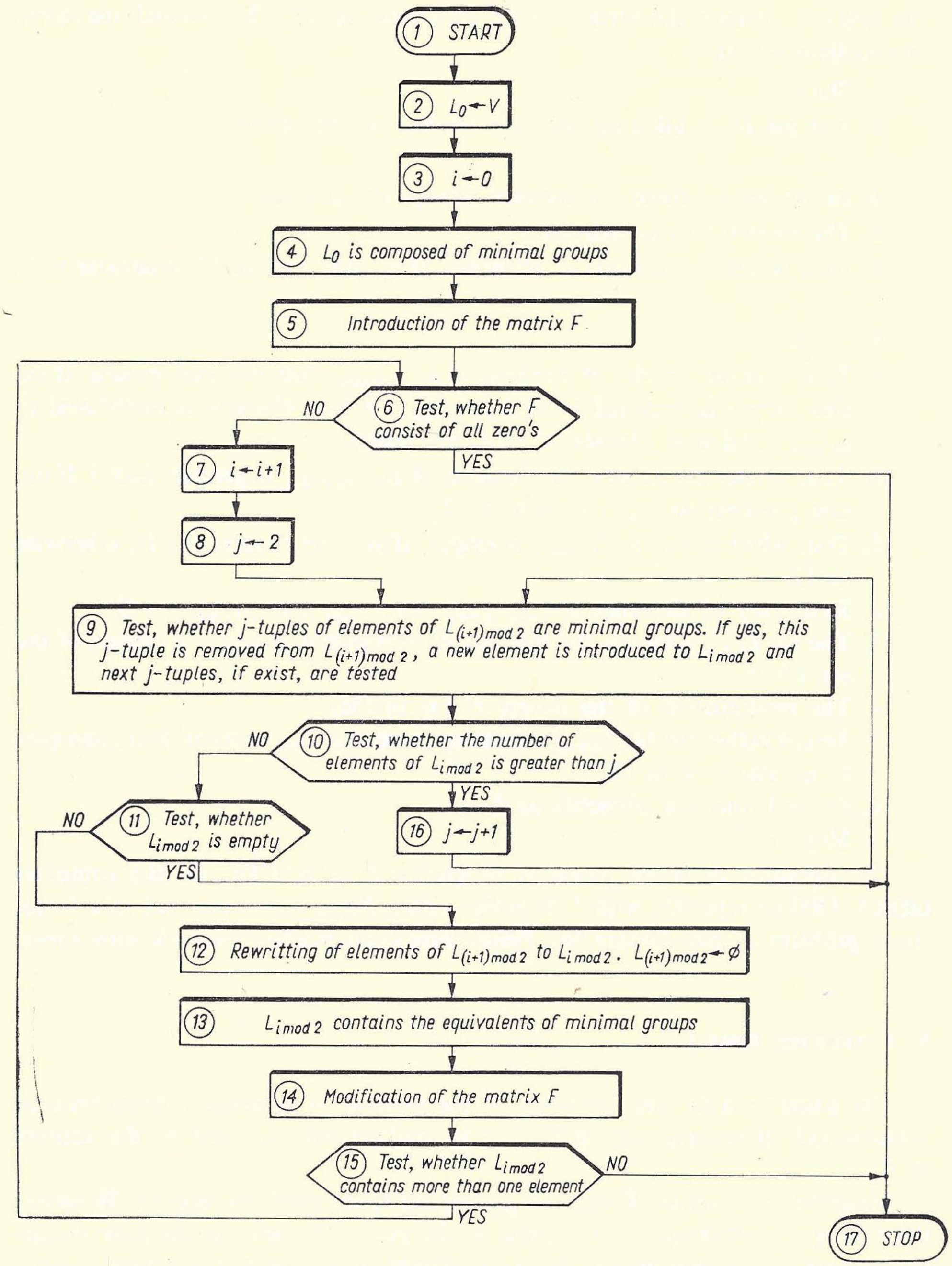

Fig. 5. A simplified flow-diagram of the new algorithm for the determination of minimal groups

Let us now describe consecutive steps of the method. Their numbers correspond to those in the flow-diagram. The algorithm uses two working lists named L_0 and L_1. The first one contains the vertices of the considered set $R_j^{(i)}$. The second one stores intermediate results.

1. Start.
2. The list L_0 is filled up with the vertices of the graph.
3. $i:=0$.
4. In the list L_0 there are minimal groups (by definition).
5. The matrix F is introduced.
6. Test, whether F consists of all zero's. If so, one goes to 17, otherwise to 7.
7. $i:=i+1$.
8. $j:=2$.
9. Test, whether j-tuples of elements of $L_{(i+1) \bmod 2}$ are minimal groups. If so, this j-tuple is removed from $L_{(i+1) \bmod 2}$, a new element is introduced to $L_{i \bmod 2}$ and next j-tuples, if exist, are tested.
10. Test, wheter the number of elements of $L_{(i+1) \bmod 2}$ is greater than j. If no, one proceeds to 16, otherwise to 11.
11. Test, whether the list $L_{i \bmod 2}$ is empty. If yes one proceeds to 17, otherwise to 12.
12. Rewritting of elements of $L_{(i+1) \bmod 2}$ to $L_{i \bmod 2}$, $L_{(i+1) \bmod 2}:=\emptyset$.
13. The list $L_{i \bmod}$ contains the equivalents of minimal groups — elements of the set $V^{(i+1)}$.
14. The modification of the matrix F due to (34).
15. Test, whether the list $L_{i \bmod 2}$ contains more than one element. If so, one goes to 6, otherwise to 17.
16. $j:=j+1$ and one proceeds to 9.
17. Stop.

The algorithm described above is programmed in ALGOL on the computer ODRA 1300 (compatible with ICL Series 1900). Now, it is examined in a large--scale problem of partitioning the Polish telecommunication network into zones.

8. Concluding remarks

The paper is a further extension of the method of minimally interconnected subnetworks, given originally in [5] and generalized and extended by the authors in [2].

Many new properties of minimal groups are formulated and proved. However, the principal contribution of the paper is the new improved algorithm for the determination of minimal groups. Its main feature is the reduced computer storage requirement, which is very relevant, making it possible to process much larger problems. It is of great importance from the viewpoint of possible applications (see e.g. [1, 3, 4, 6]).

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O dalszym rozszerzaniu metody zespołów minimalnych

Artykuł jest dalszym rozszerzeniem pracy autorów [2], w której metodę zespołów minimalnych podaną w [6] rozszerzono i uogólniono na dowolne grafy ważone. Zwłaszcza to uogólnienie odnosi się do unigrafów z nieujemnymi wagami rzeczywistymi. Dla wygody czytelnika podano podstawowe pojęcia i właściwości zespołów minimalnych zamieszczone w [2]. Sformułowano i dowiedziono wielu nowych właściwości. Główną częścią pracy jest nowy algorytm wyznaczania zespołów minimalnych Jest on bardziej efektywny, ale jego główną cechą jest zmniejszenie zajętości pamięci komputera, co umożliwia rozwiązywanie znacznie większych zadań.

О дальнейшем расширении метода минимальных групп

Статья является дальнейшим развитием работы авторов [2], в которой метод минимальных групп, представленный в [6], расширен и обобщен для произвольных взвешенных графов. В частности это обобщение относится к униграфам с неотрицательными действительными весами. Для удобства читателя даются основные понятия и свойства минимальных групп, представленные в [2]. Формулируется и доказывается много новых свойств. Основной частью работы является новый алгоритм определения минимальных групп. Он является более эффективным, а его основная черта состоит в снижении занятости памяти цифровой машины, что позволяет решать значительно более сложные задачи.