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On linear systems described by right invertible operators acting in a linear space

by

NGUYÊN DINH QUYÊT

(Warszawa)

In a series of papers ([2]–[4]) and in [5] the controllability and observability is considered for systems described by differential equations in Banach spaces. In this paper we shall consider the notion of controllability and other properties of systems described by right invertible operators in linear spaces without any topology (se [1a–c]). This approach permits to unify the language for very different problems and to study some problems never considered before.

1. Linear systems and their solutions

DEFINITION 1.1. We recall (cf. [1]) that operator $D \in L(X)$ where $L(X)$ is the set of all linear operators defined on linear subsets D_1 of X and with values in X is said to be right invertible if there exists operator $R \in L_0(X) = \{A \in L(X): \mathscr{D}_A = X\}$, such that $DR = I$, where I is identity operator.

Now consider a system

$$Dx = Ax + Bu \tag{1.1}$$

$$y = A_1 x + B_1 u \tag{1'.1}$$

where $A, A_1 \in L_0(X)$, $A_1 \neq 0$, and $B, B_1 \in L_0(U \to X)$; X, U are linear spaces.

The spaces X and U are called *space of trajectories and space of controls*, respectively.

We admit for system (1.1) an initial conditions

$$F_0 x = x_0 \tag{1.2}$$

where $x_0 \in Z_D = \ker D = \{x \in \mathscr{D}_D : Dx = 0\}$ and Z_D is called the space of constants for D.

Operator $F_0 \in L_0(X)$ satisfies by definition the following conditions

(i) $F_0^2 = F_0$ and $F_0 X = Z_D$

(ii) $F_0 R = 0$ on X.

i.e. F_0 is an initial operator of D corresponding to R, where R is a right inverse of D (cf. [1]).

Observe that every initial state x_0 is, by our assumption, a constant.

Suppose that the operator $I-RA$ is invertible. Then the system (1.1), (1.2) has a unique solution of the form

$$x=(I-RA)^{-1} x_0+(I-RA)^{-1} RBu, \tag{1.3}$$

for every $u \in U$ (compare [1]).

If we substitute x given by formula (1.3) into equation (1'.1), we obtain $y= =A_1 (I-RA)^{-1} x_0+[A_1 (I-RA)^{-1} RB+B_1] u$.

Consider the system

$$Q(D) x=Bu \tag{1.4}$$

$$y=A_1 x+B_1 u \tag{1.5}$$

with initial conditions

$$F_0 D^k x=x_k^0, \quad k=0, 1, \ldots, N-1, \tag{1.6}$$

where $x_k^0 \in Z_D$, $Q(D)=\sum_{k=0}^{N} Q_k D^k$, $Q_k \in L(X)$ and such that $\mathscr{D}_{Q_k} \subset D^k X$ for $k= =0, 1, \ldots, N-1$ and $Q_N=I$.

If operator $Q_0(R)=\sum_{k=0}^{N} Q_k R^{N-k}$ is invertible then the system (1.4)-(1.6) has a unique solution $x=R^N [Q_0(R)]^{-1} Bu+\sum_{k=0}^{N-1} R^k x_k^0$.

Indeed, equation (1.4) implies $Q_0(R) D^N x=Bu$ thus $D^N x=[Q_0(R)]^{-1} Bu$ and $x=R^N [Q_0(R)]^{-1} Bu+\sum_{k=0}^{N-1} R^k z_k$, $z_k \in Z_D$ for $k=0, 1, \ldots, N-1$.

Conditions (1.6) imply for $k=0, 1, \ldots, N-1$:

$$F_0 D^k x=x_k^0=F_0 D^k R^N [Q_0(R)]^{-1} Bu+F_0 D^k \sum_{l=0}^{N-1} R^l z_l=$$

$$=F_0 R^{N-k} [Q_0(R)]^{-1} Bu+F_0 D^k \sum_{l=0}^{N-1} R^l z_l=z_k$$

because $F_0 R=0$.

In our case the output is $y=[A_1 R^N Q_0(R)^{-1} B+B_1] u+A_1 \sum_{k=0}^{N-1} R^k x_k^0$.

Suppose that the operator B is of the form $B=P(D)$, where $P(D)=\sum_{k=0}^{M} P_k D^k$ with $P_k \in L(X)$ and $\mathscr{D}_{P_k} \subset D^k X$ for $k=0, \ldots, M-1$ and $P_M=I$. Then the system (1.4)-(1.6) can be written in the following form

$$Q(D) x=P(D) u \tag{1.7}$$

$$y=A_1 x+B_1 u \tag{1.8}$$

$$F_0 D^k x=x_k^0, \quad k=0, 1, \ldots, N-1. \tag{1.9}$$

Suppose that $Q_0(R)$ is an invertible operator, where $Q_0(R)=\sum_{k=0}^{N-1} Q_k R^k$ as before. Write $P_0(R)=\sum_{k=0}^{M-1} P_k R^{M-k}$ the system (1.7)–(1.9) has a unique solution $x=W(R)D^M u+\sum_{k=0}^{N-1} R^k x_k^0$, where $W(R)=R^N[Q_0(R)]^{-1}[P_0(R)]$.

Observe that the operator $W(R)$ is a rational function of R.

Moreover, if we consider the system

$$Q(D)D^M x=Bu, M\geqslant 0 \tag{1.10}$$

$$F_0 D^k x=x_k^0, x_k^0\in Z_D, k=0, \ldots, M+N-1. \tag{1.11}$$

In this case the solution of (1.10)–(1.11) is of the form $x=R^{M+N}[Q_0(R)]^{-1}Bu+ +\sum_{k=0}^{M+N-1} R^k x_k^0$ provided that operator $Q_0(R)$ is invertible.

Indeed, define $D^M x=v$. Then $F_0 D^k v=F_0 D^{k+M} x=x_{k+M}^0$, $k=0, \ldots, N-1$.

We therefore can rewrite (1.10)–(1.11) as follows:

$$Q(D)v=Bu \tag{1.12}$$

$$F_0 D^k v=x_{k+M}^0, \quad k=0, \ldots, N-1. \tag{1.13}$$

Since operator $Q_0(R)$ is invertible by our assumption we obtain

$$v=R^N[Q_0(R)]^{-1}Bu+\sum_{k=0}^{N-1} R^k x_{k+M}^0. \tag{*}$$

We consider the problem $D^M x=v$, $F_0 D^k x=x_k^0$, $k=0, \ldots, M-1$. This problem is well possed and has solution (cf. [1])

$$x=R^M v+\sum_{k=0}^{M-1} R^k x_k^0. \tag{**}$$

Substitute v given by formula (*) into (**), we obtain

$$x=R^M\left\{R^N[Q_0(R)]^{-1}Bu+\sum_{k=0}^{N-1} R^k x_{k+M}^0\right\}+\sum_{k=0}^{M-1} R^k x_k^0=$$

$$=R^{M+N}[Q_0(R)]^{-1}Bu+\sum_{k=0}^{N-1} R^{k+M} x_{k+M}^0+\sum_{k=0}^{M-1} R^k x_k^0$$

and hence we have

$$x=R^{M+N}[Q_0(R)]^{-1}Bu+\sum_{k=0}^{M+N-1} R^k x_k^0$$

the output $y=A_1 x+B_1 u$ is of the form

$$y=\{A_1 R^{M+N}[Q_0(R)]^{-1}B+B_1\}u+ +A_1\sum_{k=0}^{M+N-1} R^k x_k^0.$$

If the coefficients $Q_0, ..., Q_{N-1}$ are commutative with D and $Q_0(R)$ is invertible then the system

$$D^M Q(D)\, x = Bu, \quad M \geqslant 0 \tag{1.14}$$

$$F_0 D^k x = x_k^0, \; x_k^0 \in Z_D, \; k = 0, ..., M+M-1 \tag{1.15}$$

has the solution as before:

$$x = R^{M+N} [Q_0(R)]^{-1} Bu + \sum_{\kappa=0}^{M+N-1} R^k x_k^0$$

since $D^M Q(D) = Q(D) D^M$.

Equation with superposition of right invertible operators. Suppose that $D_1, D_2, ..., D_m \in R(X)$ and $R_1, ..., R_m$ are right inverses of $D_1, ..., D_m$ respectively.

Consider the superposition $D = D_1, D_2, ..., D_{m-1}, D_m$. It is easy to check that operator $R = R_m \cdot R_{m-1} \cdot ... \cdot R_1$ is a right inverse of D, i.e. $D \cdot R = I$ (cf. [1b, 1g]) and an initial operator for D corresponding to R is of the form $F = F_m + R_m F_{m-1} D_m + ... + R_m R_{m-1} ... R_2 F_1 D_2 ... D_m$, where $F_j, j = 1, ..., m$, are initial operators for D_j corresponding to R_j.

Instead of (1.1), (1'.1) and (1.2) we consider the system

$$\tilde{D} x = Ax + Bu \tag{1.16}$$

$$y = A_1 x + B_1 u \tag{1.17}$$

with initial condition

$$\tilde{F} x = x_0, \quad x_0 \in Z_D. \tag{1.18}$$

Assume that operatore $I - \tilde{R}A$ is invertible. Then the problem (1.16)–(1.17) is well posed and its solution is as for system (1.1), (1'.1), (1.2) with R replaced by $\tilde{R}$.

REMARK 1.1. If $D_1 = D_2 = ... = D_m = D$ then $\tilde{D} = D^m$ and $\tilde{R} = R^m, F_1 = ... = F_m = F$. The operator $\tilde{F} = F + RFD + ... + R^{m-1} FD^{m-1}$.

The Taylor formula (see [1]) implies $F^{(m)} = \tilde{F} = I - R^m D^m$, we obtain the system

$$D^m x = Ax + Bu \tag{1.16'}$$

$$F^{(m)} x = x_0, \; x_0 \in Z_{D^m}. \tag{1.17'}$$

But $Z_{D^m} = \{x \in \mathscr{D}_{D^m} : x = \sum_{k=0}^{m-1} R^k x_k, x_k \in Z_D, k = 0, ..., m-1\}$, hence $x_0 = \sum_{k=0}^{m-1} R^k x_k^0$, $x_k^0 \in Z_D$. Moreover

$$x_k^0 = FD^k x_0, \quad k = 0, ..., m-1. \tag{1.19}$$

Indeed, since $x_0 = x_0^0 + Rx_1^0 + ... + R^{m-1} x_{m-1}^0$, we have $FD^k x_0 = FD^k (x_0^0 + Rx_1^0 + ... + R^{m-1} x_{m-1}^0)$, i.e. $FD^k x_0 = FD^k \cdot R^k x_k^0 = x_k^0$, p $= 0, 1, ..., m-1$.

Conversely, it is easy to prove that the problem (1.16')–(1.19) is equivalent to the problem (1.16')–(1.17').

For the superposition $D_1, ..., D_m$ of the right invertible operator we have the following:

THEOREM 1.1. Suppose that $D_1, \ldots, D_m$ are right invertible operators and $R_1, \ldots, R_m$ are right inverse of $D_1, \ldots, D_m$ respectively. Then

$$Z_{D_1, \ldots, D_m} \overset{\text{df}}{=} \{z \in X: D_1, \ldots, D_m z = 0\} =$$
$$= \left\{ z \in X: z = z_m + \sum_{k=1}^{m-1} R_m, \ldots, R_{m-k+1} z_{m-k}, z_{m-k} \in Z_{D_{m-k}}, \ k=0, \ldots, m-1 \right\}. \quad (1.20)$$

Proof. Let $z = z_m + \sum_{k=1}^{m-1} R_m, \ldots, R_{m-k+1} z_{m-k}$, $z_{m-k} \in Z_{D_{m-k}}$, $k=1, \ldots, m-1$, and $z_m \in Z_{D_m}$. Then $D_1, \ldots, D_m z = D_1, \ldots, D_m \{z_m + R_m z_{m-1} + \ldots + R_m, \ldots, R_2 z_1\} = 0$.

Conversely if $z \in Z_{D_1, \ldots, D_m}$. We shall prove that z can be written in the form $z = z_m + R_m z_{m-1} + \ldots + R_m, \ldots, R_2 z_1$, where $z_k \in Z_{D_k}$, $k=1, \ldots, m$.

We prove this by method of induction.

Clearly, if $m=1$ the formula (1.20) is true.

Suppose that (1.20) is true for a number m, i.e. if $z \in Z_{D_1, \ldots D_m}$ then z can be written in the form $z = z_m + R_m z_{m-1} + \ldots + R_m, \ldots, R_2 z_1$, and suppose that $z \in Z_{D_1, \ldots, D_{m+1}}$. We have $D_1, \ldots, D_m (D_{m+1} z) = 0$.

Write $D_{m+1} z = w$, by assumption of induction we obtain $w = z_m + R_m z_{m-1} + \ldots + R_m, \ldots, R_2 z_1$, where $z_k \in Z_{D_k}$, $k=1, \ldots, m$, or $D_{m+1} z = z_m + R_m z_{m-1} + \ldots + R_m, \ldots, R_2 z_1$. Hence $z = z_{m+1} + R_{m+1} z_m + R_{m+1} R_m z_{m-1} + \ldots + R_{m+1}, \ldots, R_2 z_1$, where $z_k \in Z_{D_k}$, $k=1, \ldots, m+1$. Thus (1.20) is true for arbitrary m.

COROLLARY 1.1. Suppose that operator $I - R_m, \ldots, R_1 A$ is invertible. Then the equation

$$D_1, \ldots, D_m x = Ax + Bu, \ A \in L_0(x), \ B \in L(U \to X) \quad (1.21)$$

with the initial condition

$$Fx = x_0, \quad x_0 \in Z_{D_1, \ldots, D_m} \quad (1.22)$$

where $F = F_m \cdot + R_m F_{m-1} D_m + \ldots + R_m R_2 F_1 D_2, \ldots, D_m$ is an initial operator for $D_1, \ldots, D_m$ corresponding to $R_m, \ldots, R_1$ (see [1b], [1g]) and F_j are initial operators D_j corresponding to R_j for $j=1, \ldots, m$, has a solution $x = (I - R_m, \ldots, R_1 A)^{-1} \times (R_m, \ldots, R_1 Bu + z_m^0 + \sum_{k=1}^{m-1} R_m, \ldots, R_{m-k+1} z_{m-k}^0)$, where $z_m^0 = F_m x_0$ and $z_{m-k}^0 = F_{m-k} D_{m-k+1}, \ldots, D_m x_0$, $k=1, \ldots, m-1$.

Proof. Applying the Theorem 1.1, we can write the solution of (1.21)–(1.22) in the form $x = (I - R_m, \ldots, R_1 A)^{-1} (R_m, \ldots, R_1 Bu + x_0)$ $x_0 \in Z_{D_1, \ldots, D_m}$, i.e.

$$x_0 = z_m^0 + R_m z_{m-1}^0 + \ldots + R_m, \ldots, R_2 z_1^0.$$

By acting operators $F_m, F_{m-1}, D_m, \ldots, F_1, D_2, \ldots, D_m$ on the both sides of (1.20) we obtain

$$F_m x_0 = z_m^0, \ F_{m-1} D_m x_0 = z_{m-1}^0, \ldots, F_1 D_2, \ldots, D_m x_0 = z_1^0$$

because of

$$F_j R_j=0,\ D_j R_j=I,\ j=1, \ldots, m$$

$$F_j z_j=z_j \text{ on } Z_{D_j},\ j=1, \ldots, m.$$

REMARK 1.2. If $D_1=D_2=\ldots=D_m\ (=D)$, then the problem (1.21)–(1.22) is the problem (1.16′)–(1.17′).

2. Controllability of systems described by a right invertible operator

Suppose that we are given a system

$$Dx=Ax+Bu, \tag{2.1}$$

$$F_0 x=x_0,\ x_0 \in Z_D, \tag{2.2}$$

where as before $D \in L(X)$ is right invertible and F_0 is an initial operator for D corresponding to a right inverse R, such that the operator $I-RA$ is invertible and $BU \supset (D-A)\,\mathscr{D}_D$.

For each initial state $x_0 \in Z_D$ and each control $u \in U$ the system (2.1)–(2.2) has a unique solution

$$x=\Phi(x_0, u)=\Phi_0 x_0+\Phi_1 u, \tag{2.3}$$

where we write $\Phi_0 x_0=(I-RA)^{-1} x_0$, $\Phi_1 u=(I-RA)^{-1} RBu$, provided that the operator $I-RA$ is invertible.

1. Denote by $\mathscr{F}_D$ the set of all initial operators for D. We have:

DEFINITION 2.1. The system (2.1)–(2.2) is said to be F_1-controllable if for every $x_0 \in Z_D$ we have

$$\{F_1 \Phi(x_0, u) \colon u \in U\}=Z_D. \tag{2.4}$$

THEOREM 2.1. A necessary and sufficient condition for the systems (2.1)–(2.2) to be F_1-controllable is

$$\ker B^* R^* (I-RA)^{*-1} \tilde{F}_1^*=\{0\}, \tag{2.5}$$

where $*$ denotes algebraic adjoint, and $\tilde{F}_1$ is restriction of F_2 on Rang $v\{(I-RA)^{-1} \times \times RB\} \subset X$.

Proof. By definition, the system (2.1)–(2.2) is F_1-controllable if

$$\{F_1 (I-RA)^{-1} x_0+F_1 (I-RA)^{-1} RBu \colon u \in U\}=Z_D. \tag{2.6}$$

But the equality (2.6) is satisfied if and only if for every $z \in Z_D$ the equation

$$F_1 (I-RA)^{-1} x_0+F_1 (I-RA)^{-1} RBu=z, \tag{2.7}$$

has solution u or the equation

$$F_1\,(I-RA)^{-1}\,RBu=\bar{z}, \tag{2.7'}$$

where $\bar{z}=z-F_1\,(I-RA)^{-1}\,x_0\in Z_D$, has a solution u for every $\bar{z}\in Z_D$.

By the definition of $\tilde{F}_1$, (2.7') can be rewriten $\tilde{F}_1\,(I-RA)^{-1}\,RBu=\bar{z}$. Thus we obtain $\ker\{\tilde{F}_1\,(I-RA)^{-1}\,RB\}^*=\{0\}$, i.e. $\ker B^*\,R^*\,(I-RA)^{*-1}\,\tilde{F}_1^*=\{0\}$.

Corollary 2.1. If the system (2.1)–(2.2) is stationary then the condition (2.5) is of the following form:

$$\ker B^*\,(I-RA)^{*-1}\,\tilde{F}_1^*=\{0\}. \tag{2.5'}$$

2. Consider the stationary system $Dx=Ax+Bu$ (eq. (2.1)) and $F_0\,x=0$ (eg. (2.2')).

Theorem 2.2. Suppose that the stationary system (2.1)–(2.2) with $x_0=0$ is F_1-controllable and $BU\supset(I-RA)X$. Then the system (2.1)–(2.2) is F_2-controllable for every $F_2\in F_D$, where $\tilde{F}_D=\{F\in\mathscr{F}_D\colon\forall z\in Z_D,\ \exists y\in X\colon FRy=z\}$.

Proof. To begin with observe that for the stationary system (2.1)–(2.2) we have $(r-RA)\,R=R\,(I-RA)$, $(I-RA)^{-1}\,R=R\,(I-RA)^{-1}$, because that $(I-RA)$ is invertible.

The solution of the system (2.1)–(2.2) is of the form $x=(I-RA)^{-1}\,RBu$. For every $z\in Z_D$, there exists $u\in U$ such that $F_1\,(I-RA)^{-1}\,RBu=z$.

We have $(F_1-F_2)\,(I-RA)^{-1}\,RBu=(R_2-R_1)\,DR\,(I-RA)^{-1}\,Bu=F_1\,R_2\,(I+-RA)^{-1}\,Bu$, or $F_2\,(I-RA)^{-1}\,RBu=z-z_2$, where $z_2=F_1\,R_2\,(I-RA)^{-1}\,Bu$.

Since $F_2\in\mathscr{F}_D$ then there exists $y\in X$ such that $F_2\,Ry=z_2$, and there exists $w\in U$ such that $Bw=(I-RA)\,y\in X$, or there exists $w\in U$ such that $(I-RA)^{-1}\,Bw=y$ since $I-RA$ is invertible. Thus there exists a control $w\in U$ such that $F_2\,R\,(I-RA)^{-1}\,Bw=z_2$ or $F_2\,(I-RA)^{-1}\,RBw=z_2$.

Now if we choose $u_1=u+w$ then $F_2\,(I-RA)^{-1}\,RBu_1=F_2\,(I-RA)^{-1}\,RB(u+w)=F_2\,(I-RA)^{-1}\,RBu+F_2\,(I-RA)^{-1}\,RBw$.

But $F_2\,(I-RA)^{-1}\,RBu=z-z_2$ and $F_2\,(I-RA)^{-1}\,RBw=z_2$. Thus $F_2\,(I-RA)^{-1}\times RBu_1=z$, i.e. the system (2.1)–(2.2) is F_2-controllable for $F_2\in\mathscr{F}_D$ if the system (2.1)–(2.2) is F_1-controllable.

3. Examples

Consider the process described by the linear system differential equation

$$(\mathscr{L})\begin{cases}\dot{x}\,(t)=Ax\,(t)+Bu\,(t)\\ x\,(0)=0\end{cases}$$

where A and B are matrices and x, u are vectors. We assume that

(i) $A-n\times n$ constant matrix and $B-n\times m$ constant matrix,

(ii) the control $u\,(t)$ is m-vector function measurable and integrable on [0, 1] (cf. [7] and [8]).

If we define the operator $(Dx)(t)=\left(\frac{d}{dt}x_1, ..., \frac{d}{dt}x_n\right)$, where $x(t)=(x_1(t), ...,$ $..., x_n(t))$ then a right inverse of D is defined as follows: $(Rx)(t)=(\int_0^t x_1(s)\,ds, ...,$ $..., \int_0^t x_n(s)\,ds)$ and $(Fx)(t)=x(0)\,(F_1 x)(t)=x(T_1)=x_1$.

The system $\mathscr{L}$ can be written in terms of the symbol of operator

$$(\mathscr{S})\begin{cases} Dx=Ax+Bu \\ Fx=0. \end{cases}$$

By assuming (i) the system $(\mathscr{S})$ is stationary, i.e. $RA=AR$, $RB=BR$, $DA=AD$, $DB=BD$.

Moreover by the theorem Bielecki [4] it follows that the operator $I-RA$ is invertible. Hence the solution of $(\mathscr{S})$ i.e. $(\mathscr{L})$ is of the form $x_u=(I-RA)^{-1}RBu$. In this case we have

$$BU=\{\text{measurable and integrable functions on } [0,1]\}$$

$$X=\{\text{absolutely continuous functions on } [0,1]\}$$

and the conditions of Theorem 2.2 hold.

Therefore if the system $(\mathscr{S})$ if F_1-controllable for some $F_1 \in \mathscr{F}_D$ then $(\mathscr{S})$ is F_2-controllable for all F_2 belonging to $\tilde{\mathscr{F}}_D$, where $(F_2 x)(t)=x(T_2)$, $0<T_2\leqslant 1$. But the system $(\mathscr{S})$ is F_1-controllable if and only if

$$\{x_u(T_1): u \in U\}=\{F_1(I-RA)^{-1}RBu: u \in U\}=Z_D=R^n \qquad (*)$$

In our case relation (*) holds for all operators $F_\alpha \in \mathscr{F}_D$, $F_\alpha \neq F$.

One can consider also some partial differential operators, functional-differentional operators and difference operators, as it was indicated in the papers [1b–e].

4. Some generalization

1. Consider the system

$$Q(D)\,x=Bu \qquad (4.1)$$

$$F_0 D^k x=x_k^0, \quad x_k^0 \in Z_D, \quad k=0, ..., N-1 \qquad (4.2)$$

where $Q(D)=\sum_{k=0}^{N} Q_k D^k$ is a polynomial in D degree N with coefficients $Q_k \in L(X)$, $k=0, ..., N-1$ and $Q_N=I$.

Suppose that operator $Q_0(R)=\sum_{k=0}^{N} Q_k R^{N-k}$ is invertible. Then the system (4.1)–(4.2) has a unique solution for every $u \in U$ and for every initial values x_k^0, $k=0, ..., N-1$, $x=R^N[Q_0(R)]^{-1}Bu+\sum_{k=0}^{N-1} R^k x_k^0$ (see [1e]).

From Definition 2.1 it follows that the system (4.1)-(4.2) is said to be F_1-controllable if for every $z_k^1 \in Z_D$, $k=0, \ldots, N-1$, there exists a control $\bar{u} \in U$ such that $F_1 D^k \times [R^N Q_0(R)^{-1} B\bar{u} + \sum_{k=0}^{N-1} R^k x_k^0] = x_k^1$ for all $k=0, \ldots, N-1$.

THEOREM 4.1. The system (4.1)–(4.2) is F_1-controllable if and only if

$$\ker B^* [Q_0(R)]^{*-1} R^{*N-k} F_1^* = \{0\} \tag{4.3}$$

for all $k=0, 1, \ldots, N-1$, where F_1 is a restriction of $\tilde{F}_1$ on $\mathrm{Rang}_U\, R^{N-k} [Q_0(R)]^{-1} B$.

The proof of Theorem 4.1 is going on the some lines as the proof of the Theorem 2.1.

2. If $B=P(D)=\sum_{k=0}^{M} P_k D^k$, P is a polynomial in D degree M with the coefficients $P_k \in L(X)$, $k=0, \ldots, M-1$, and $P_M=I$ we have the system

$$Q(D)\, x = P(D)\, u \tag{4.4}$$

$$F_0 D^k x = x_k^0, \quad x_k^0 \in Z_D, \quad k=0, \ldots, N-1. \tag{4.5}$$

Write $P_0(R)=\sum_{k=0}^{M-1} P_k R^{M-k}$ and suppose that $Q_0(R)$ is invertible. The solution of (4.4), (4.5) for every initial state x_k^0, $k=0, \ldots, N-1$, and $u \in U$ is of the form $x=W(R) D^M u + \sum_{k=0}^{N-1} R^k x_k^0$, where $W(R)=R^N [Q_0(R)]^{-1} P_0(R)$ is a rational function of R.

The condition of the F_1-controllability of the system (4.4)–(4.5) can be written as follows: $\ker D^{*M} [P_0(R)]^* [Q_0(R)^{-1}]^* R^{*N-k} \tilde{F}_1 = \{0\}$, $k=0, 1, \ldots, N-1$, where F_1 is a restriction of $\tilde{F}_1$ on $\mathrm{Rang}_U \{R^{N-k} [Q_0(R)]^{-1} [P_0(R)] D^M\}$.

3. An initial operator $\hat{F}$ for the polynomial $Q(D) \sum_{k=0}^{N} q_k D^k$. Let a polynomial of the right inverse operator D be given $Q(D)=\sum_{k=0}^{N} q_k D^k$, where q_k — constant, $k=0, \ldots, N$.

If we denote $Q^*(R)=\sum_{k=0}^{N} q_k R^{N-k}$, where as before R is an right inverse of the operator D, then $\hat{R}=[Q^*(R)]^{-1} R^N = R^N [Q^*(R)]^{-1}$ is a right inverse of $\hat{D}=\sum_{k=0}^{N} q_k D^k$ (see [1a]). Thus an initial operator $\hat{F}$ for the operator $\hat{D}$ corresponding to $\hat{R}$ is of the form $\hat{F}=I-\hat{R}\hat{D}$, and then it can be written as follows:

$$\begin{aligned}
\hat{F} &= I-[Q^*(R)]^{-1} R^N Q(D) = [Q^*(R)]^{-1} \{[Q^*(R)] - R^N Q(D)\} = \\
&= [Q^*(R)]^{-1} \{q_0 R^N + q_1 R^{N-1} + \ldots + q_N - R^N (q_0 + q_1 D + \ldots + q_N D^N)\} = \\
&= [Q^*(R)]^{-1} \{q_1 (R^{N-1} - R^N D) + \ldots + q_N (I - R^N D^N)\} = \\
&= [Q^*(R)]^{-1} \{q_1 R^{N-1} (I-RD) + \ldots + q_N (I - R^N D^N)\}.
\end{aligned}$$

By the Taylor formula [1a] we have $R^k D^k = I - \sum_{i=0}^{k-1} R^i F D^i$ for $k=1, 2, \ldots, N$, and then $\hat{F}=[Q^*(R)]^{-1} \{q_1 R^{N-1} F + q_2 R^{N-2} (F+RFD) + \ldots + q_N (F + \sum_{k=1}^{N-1} R^k F D^k)\} =$

$=[Q^*(R)]^{-1}\{(q_1 R^{N-1}+q_2 R^{N-2}+...+q_N) F+(q_2 R^{N-2}+...+q_N) RFD+ ... +q_N\times$ $\times R^{N-1} FD^{N-1}\}$, or $\hat{F}=[Q^*(R)]^{-1}\{(\sum_{k=1}^{N} q_k R^{N-k}) F+(\sum_{k=1}^{N} q_k R^{N-k}) RFD+...+$ $+q_N R^{N-1} FD^{N-1}\}$.

PROPOSITION. If for $\forall z_i\in Z_D$, $\exists \bar{y}_i$ such that $(\sum_{k=i}^{N} q_k R_2^{N-k}) R_2^{i-1} F_2 R^{N-i+1} \bar{y}_{i-1}=$ $=R_2^{i-1} z_{i-1}$, $i=1, ..., N$, where R_2 is a right inverse of D (i.e. $DR_2=I$) and F_2 is an initial operator for D corresponding to R_2, then there exist y_i $(i=1, ..., N)$ such that $\langle\hat{F}_2, \hat{R}, y\rangle=z$ for every $u\in Z_{Q(D)}$, where $y=(y_1, ..., y_n)$ and

$$\langle\hat{F}_2, \hat{R}, y\rangle \overset{\mathrm{df}}{=} [Q^*(R_2)]^{-1}\left(\sum_{k=1}^{N} q_k R_2^{N-k}\right) F_2 R^N [Q^*(R)]^{-1} y_1+...+$$
$$+[Q^*(R_2)]^{-1} q_N R_2^{N-1} F_2 R [Q^*(R)]^{-1} y_N. \qquad (*)$$

Proof. For every $z\in Z_{Q(D)}$ we have (see [1a]): $z=[Q^*(R_2)]^{-1}\sum_{k=0}^{N-1} R_2^k z_k$, $z_k\in Z_D$, $k=0, ..., N-1$. Moreover we have

$$\tilde{F}_2 \hat{R}=[Q^*(R_2)]^{-1}\left(\sum_{k=1}^{N} q_k R_2^{N-1}\right) F_2 R^N+\left(\sum_{2}^{N} q_k R_2^{N-k}\right)(R_2 F_2 D) R^N+$$
$$...+q_N R^{N-1} F_2 D^{N-1} R^N [Q^*(R)]^{-1}=[Q^*(R_2)]^{-1}\left\{\left(\sum_{k=1}^{N} q_k R_2^{N-k}\right) F_2 R^N+\right.$$
$$\left.+\left(\sum_{2}^{N} q_k R_2^{N-k}\right) R_2 F_2 R^{N-1}+...+q_N R_2^{N-1} F_2 R [Q^*(R)]^{-1}\right\}.$$

The relation $(*)$ holds if there exist y_i, $i=1, ..., N$, such that

$$[Q^*(R_2)]^{-1}\left(\sum_{k=i}^{N} q_k R_2^{N-k}\right) R_2^{i-1} F_2 R^{N-i+1} [Q^*(R)]^{-1} y_{i=1}=$$
$$=[Q^*(R_2)]^{-1} R_2^{i-1} z_{i-1}$$

or

$$\left(\sum_{k=i}^{N} q_k R_2^{N-k}\right) R_2^{i-1} F_2 R^{N-i+1} [Q^*(R)]^{-1} y_{i=1}=R_2^{i-1} z_{i-1},\ i=1, ..., N.$$

By assumption there exist $\bar{y}_i$, $i=1, ..., N$, such that

$$\left(\sum_{k=i}^{N} q_k R_2^{N-k}\right) R_2^{i-1} F_2 R^{N-i+1} \bar{y}_{i-1}=R_2^{i-1} z_{i-1}.$$

Then y_{i-1}, $i=1, ..., N$, can be found by relation $y_{i-1}=[Q^*(R)]\bar{y}_{i-1}$.

5. Observability

1. In the notion of the Section 2, we now consider the system

$$Dx = Ax + Bu \tag{5.1}$$

$$Fx = x_0, \quad y = Hx \tag{5.2}$$

where $D \in R(X)$, $A \in L_0(X)$, $B \in L_0(U \to X)$ and $H \in L_0(X \to Y)$. $H \neq 0$.

The solution of the system (5.1) is of the form $x = (I - RA)^{-1} x_0 + (I - RA)^{-1} \times \times RBu$, and then the output (5.2) $y = H(I - RA)^{-1} x_0 + H(I - RA)^{-1} RBu$.

DEFINITION 5.1. The system (5.1)–(5.2) is said to be observable if for every given output y and input u, there exist a unique initial state x_0 such that $y = H(I - RA)^{-1} \times \times x_0 + H(I - RA)^{-1} RBu$. We have the following theorem:

THEOREM 5.1. The system (5.1)–(5.2) is observable if and only if $\ker H(I - RA)^{-1} = \{0\}$.

Proof. The output y corresponding to x_0 and u is of the form $y = H(I - RA)^{-1} x_0 + H(I - RA)^{-1} RBu$, and hence in order to equation $H(I - RA)^{-1} x_0 = y - H(I + - RA)^{-1} RBu$ has unique solution x_0 for every u and y, the necessary and sufficient condition is $\ker H(I - RA)^{-1} = \{0\}$.

2. Suppose that X, Y are Hilbert spaces. In the space X we consider the linear system described by a right invertible operator D:

$$\begin{cases} Dx = Ax \\ Fx = x_0, \quad x_0 \in Z_D \end{cases} \tag{5.1'}$$

and the output $y \in Y$ defined by the formula $y = Hx$, where H is a linear operator from X into Y.

Suppose that $f \in (Z_D)^*$ be given. We have the definition:

DEFINITION 5.2. (see [7]).

a. The functional $f \in (Z_D)^*$ is said to be SR-observable if there exists en element $\varphi \in Y^*$ such that $f = E_A^* H^* \varphi$, where $E_A = (I - RA)^{-1}$ is an operator from Z_D into X, generated by the system (5.1′), i.e. for each $x_0 \in Z_D$, $x = (I - RA)^{-1} x_0$ is a unique solution of the system (5.1′).

b. The system (5.1′) is said to be observable if all $f \in Z_D^*$ are observable.

THEOREM 5.1′. The functional f is SR-observable if and only if $\ker HE_A = \{0\}$.

Proof. For each $f \in Z_D^*$, the equation $f = E_A^* H^* \varphi$ has solution φ if and only if $\ker(E_A^* H^*) = \{0\}$ (see [6]), or if and only if $\ker(H^{**} E_A^{**}) = \{0\}$.

If we denote $\kappa_{Z_D}: Z_D \to Z_D^{**}$, $\kappa_X: X \to X^{**}$, $\kappa_Y: Y \to Y^{**}$ as canonical imbeddings from Z_D into Z_D^{**}, X into X^{**} and Y into Y^{**} respectively, then we have (see [6])

$E_A^{**} = \kappa_X E_A \kappa_{Z_D}^{-1}$, $H^{**} = \kappa_Y H \kappa_X^{-1}$. Hence $H^{**} E_A^{**} = \kappa_Y H \kappa_X^{-1} \kappa_X E_A \kappa_{Z_D}^{-1} = \kappa_Y H E_A \kappa_{Z_D}^{-1}$. Therefore the condition $\ker H^{**} E_A^{**} = \{0\}$ holds if and only if $\ker HE_A = \{0\}$.

By the Theorem it follows that the system (5.1′) is SR-observable if and only if this system is observable in the sense of the Definition 5.1.

3. The relationship between observability and controllability. Now we shall assume that all spaces X, Y, U are self-adjoint i.e. $X^* = X$, $Y^* = Y$, $U^* = U$. In such spaces the operator $A \in L_0(X)$, $H \in L_0(X \to Y)$, have the following properties (see [6] p. 77): $A^{**} = (A^*)^* = A$, $H^{**} = (H^*)^* = H$. Now we consider stationary system

$$Dx = Ax + Bu, \quad Fx = x_0 \tag{5.1}$$

$$y = Hx \tag{5.2}$$

under assumption concerning A, B, D, F, H as before.

The system

$$\begin{aligned} Dx &= A^* x + H^* u \\ Fx &= x_0 \end{aligned} \tag{5.3}$$

is called a dual system for the system (5.1)–(5.2).

THEOREM 5.2. Suppose that the operator D is self-adjoint, $D = D^*$ the spaces X, Y, U are self-adjoint and moreover, $U = Y$.

The stationary system (5.1)–(5.2) is observable if and only for dual system (5.3) the following relation holds: $\ker (H^*)^* (I - RA^*)^{*-1} = \{0\}$.

Proof. Since $D = D^*$ we have $R = R^*$. Indeed, since $DR = I$, we have $R^* D^* = R^* D = I$. Hence D has right inverse R and left inverse R^*. Therefore $R = R^*$. For the system (5.3) the relation $\ker (H^*)^* (I - RA^*)^{*-1} = \{0\}$ holds means that $\{0\} = \ker H^* (I - A^* R^*)^{*-1} = \ker H (I - RA)^{-1}$ since $H^{**} = H$, $A^{**} = A$, $R^* = R$, and $AR = RA$, i.e. $\ker H (I - RA)^{-1} = \{0\}$.

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Układy liniowe opisane za pomocą prawostronnie odwracalnych operatorów w przestrzeni liniowej

W wielu pracach [2, 3, 4, 5] rozpatruje się sterowalność i obserwowalność dla układów opisanych równaniami różniczkowymi w przestrzeniach Banacha. W niniejszym artykule rozpatrzono pojęcie sterowalności oraz inne właściwości układów opisywanych prawostronnie odwracalnymi operatorami w przestrzeniach liniowych bez topologii (patrz prace [1a—c]). Ujęcie to umożliwia ujednolicenie języka dla bardzo trudnych zagadnień oraz zbadanie niektórych problemów dotychczas nie rozpatrywanych.

Линейные системы описываемые с помощъю обратимых справа операторов в линейном пространстве

В ряде работ ([2], [3], [4]) и в [5] рассматривается вопрос управляемости и наблюдаемости систем описываемых с помощъю дифференциальных уравнений в банаховых пространствах. В данной статъе рассматривается понятие управляемости и ряд других свойств, систем описываемых посредством обратимых справа операторов в линейных пространствах вне топологии (см. [1a], [1b], [1c]). Этот подход позволяет унифицировать язык для весьма разных задач, а также дает возможность исследовать некоторые вопросы, которые еще не рассматривались до настоящего времени.