Control
and Cybernetics
VOL. 7 (1978) No. 2

Remarks on existence of selutions for parametric optimization problems for partial differential equations of parabolic type

by

JAN SOKOŁOWSKI

Polish Academy of Sciences
Systems Research Institute, Warszawa

The problem of the existence of solutions for some optimal control problems for linear partial differential equations of parabolic type is considered. Control appears in coefficients of elliptic operator.

On the basis of some results concerning weak convergence of a sequence of weak solutions of parabolic equation some sufficient conditions for existence of an optimal control are obtained.

1. Introduction

In the paper we give sufficient conditions for existence of solutions to some parametric optimization problems for linear partial differential equations of parabolic type.

In these optimization problems coefficients of operator depend on control variables.

Such problems appear in stochastic optimization [15], in control of diffusion in semiconductor [7] and in identification of coefficients of parabolic equation [2].

The problem of existence of solutions was studied by Zollezzi [14] in the case, when control depends only on space variable. We start with notation.

Let Ω be an open, bounded domain in n-dimensional Euclidean space R^n. We assume, that boundary Γ of Ω is an $(n-1)$-dimensional smooth manifold locally situated only on one side of Ω.

For given $T, 0<T<\infty$ we define

$$Q=\Omega\times(0, T) \tag{1.1}$$

$$\Sigma=\Gamma\times(0, T) \tag{1.2}$$

Denote by $\bar{\Omega}$ the closure of Ω. Take $C_0^\infty(\Omega)$ to be the standard space of infinitely differentiable functions with compact support in Ω.

We use standard notation for Sobolev spaces [6] namely $H^1(\Omega)$ is define as follows:

$$H^1(\Omega)=\left\{f\in L^2(\Omega)\,\middle|\,\frac{\partial f}{\partial x_i}\in L^2(\Omega),\ i=1,\dots,n\right\}. \tag{1.3}$$

It is well known, that $H^1(\Omega)$ is a Hilbert space, with the norm:

$$\|f\|_1^2=\int_\Omega (f(x))^2\,dx+\sum_{i=1}^n\int_\Omega\left(\frac{\partial f}{\partial x_i}\right)^2 dx. \tag{1.4}$$

Let $H_0^1(\Omega)$ denote the closure of $C_0^\infty(\Omega)$ in $H^1(\Omega)$, and let V be a given linear closed subspace of $H^1(\Omega)$ with topology induced by $H^1(\Omega)$ such that

$$H_0^1(\Omega)\subset V\subset H^1(\Omega). \tag{1.5}$$

Whence

$$V\subsetneq L^2(\Omega)\subsetneq V' \tag{1.6}$$

where by V' we denote the dual space of V.

We will use the same notation for extention of the scalar product in $L^2(\Omega)$: $(y,z)=\int_\Omega y(x)\,z(x)\,dx$ to the pair (V, V').

For given Banach space B, by $L^2(0,T,B)$ we denote the space of functions strongly measurable on $(0,T)$ with values in B, square integrable on $(0,T)$ with the norm:

$$\|f\|_{L^2(B)}=\int_0^T \|f(t)\|_B^2\,dt. \tag{1.7}$$

By $Y(0,T)$ we denote the following Hilbert space [4]

$$Y(0,T)=\left\{y\in L^2(0,T,V)\,\middle|\,\frac{dy}{dt}\in L^2(0,T,V')\right\} \tag{1.8}$$

with the norm:

$$\|y\|_Y^2=\|y\|_{L^2(V)}^2+\left\|\frac{dy}{dt}\right\|_{L^2(V')}^2. \tag{1.9}$$

It is well known [6], that

$$Y(0,T)\subset C(0,T;L^2(\Omega)) \tag{1.10}$$

with contimons imbedding (1.10).

2. One dimensional parabolic equation

In the section we will consider functions $y=y(x,t)$ of two variables $x\in(0,1)$, $t\in(0,T)$, that is $\Omega=(0,1)$. For simplicity we use the following notation

$$y_t=\frac{\partial y}{\partial t}(x,t),\qquad y_x=\frac{\partial y}{\partial x}(x,t).$$

For given function $f = f(x, t)$

$$|f| \leqslant M \text{ means } |f(x, t)| \leqslant M, \ \forall (x, t) \in (0, 1) \times (0, T).$$

Let there be given a functional:

$$I_1(., ., .): L^2(Q) \times L^2(\Omega) \times L^2(Q) \mapsto R \tag{2.1}$$

which is strongly continuous with respect to all variables. Consider the following optimization problem:

(P1) minimize $J(u) = I_1(A_1 y, A_2 y, u)$
subject to:

(i)

$$|u|, |u_x|, |u_t| \leqslant M; \tag{2.2}$$

(ii) constraints in the form of parabolic equation:

$$-p(x, t, u) y_t - (a(x, t, u)) y_x)_x + b(x, t, u) y_x + c(x, t, u) y = f(x, t, u) \tag{2.3}$$

with boundary conditions

$$y(0, t) = y(1, t) = 0 \tag{2.4}$$

and initial condition:

$$y(x, 0) = y_0(x) \tag{2.5}$$

(iii) with $A_1 \in \mathscr{L}(Y(0, T); L^2(Q))$,

$$A_2 \in \mathscr{L}(Y(0, T); L^2(\Omega))$$

in the following form:

$$(A_1 y)(x, t) = y(x, t), (x, t) \in (0, 1) \times (0, T) \tag{2.6}$$

$$(A_2 y)(x) = y(x, \Theta), x \in (0, 1) \tag{2.7}$$

where $\Theta \in (0, T]$ is given.

REMARK. For given control $u = u(x, t)$, $y = y(x, t)$ is the weak solution of the problem (2.3)–(2.5) that is:

$$\int_0^T \int_0^1 [(pz)_t y - a y_x z_x + b y_x z + cyz - fz] \, dx \, dt + \\ + \int_0^1 p(x, 0, u) z(x, 0) y_0(x) \, dx = 0 \tag{2.8}$$

$\forall z \in C^\infty([0, 1] \times [0, T])$ such that

$$z(0, t) = z(1, t) = 0, \quad t \in (0, T)$$

$$z(x, T) = 0, \quad x \in (0, 1).$$

To assure existence and uniqueness of a solution to the system (2.3)–(2.5) we assume that the following conditions are satisfied.

(A1) (i) $p(x, t, r) \geqslant \alpha_0 > 0$, $\forall (x, t) \in (0, 1) \times (0, T)$
$a(x, t, r) \geqslant \alpha_1 > 0$, $\forall r \in (-M, M)$;

(ii) $p(x, t, .)$, $a(x, t, .)$, $b(x, t, .)$, $c(x, t, .)$, $f(x, t, .)$ are uniformly Lipschitz continuous on interval $[-M, M]$ for almost every $(x, t) \in Q$;

(iii) $|p|$, $|p_t|$, $|a|$, $|b|$, $|c|$, $|f| \leqslant M$ a.e. in $(0, 1) \times (0, T) \times (-M, M)$, for sake of simplicity we write $|p|$ instead of $p(x, t, r)$;

(A2) (*i*) $|a_t|$, $|b_x| \leqslant M$ a.e. in $(0, 1) \times (0, T) \times (-M, M)$.

It can be shown [8], that under assumptions (2.2), (A1) there exists a solution to the problem (2.8) which is unique when we assume also condition (A2) to be satisfied. We recall the following result due to Markov and Olejnik [8] concerning convergence of sequence of solutions to the problem (2.8) with respect to the sequence of its coefficients. Let there be given a sequence of coefficients $p_n(x, t)$, $a_n(x, t)$, $b_n(x, t)$, $c_n(x, t)$, $f_n(x, t)$ and some elements p_0, A, B, C_0, f_0 such that the problem (2.11)–(2.13) has a unique weak solution, and we have

$$\begin{aligned} &p_n \rightharpoonup p_0 \\ &\frac{1}{a_n} \rightharpoonup \frac{1}{A} \\ &\frac{b_n}{a_n} \rightharpoonup B \quad \text{weakly in } L^2(Q) \qquad (2.9) \\ &c_n \rightharpoonup c_0 \\ &f_n \rightharpoonup f_0 \end{aligned}$$

then the corresponding sequence of solutions $\{y_n\}$ to the problem (2.8) is convergent to some function $\bar{y}$ in the following sense:

$$\text{for every } \delta > 0$$

$$y_n(x, t) \rightarrow \bar{y}(x, t) \text{ uniformly on } (0, 1) \times (\delta, T) \qquad (2.10)$$

where y is a weak solution of the following parabolic equation:

$$p_0 \bar{y}_t - (A\bar{y}_x)_x + BA\bar{y}_x + C_0 \bar{y} = f \qquad (2.11)$$

$$\bar{y}(0, t) = \bar{y}(1, t) = 0 \qquad (2.12)$$

$$\bar{y}(x, 0) = y_0 x. \qquad (2.13)$$

In the sequel we will need the following lemmas:

LEMMA 1. Assume, that there is given function $d = d(x, t, r)$ such that

(i) $|d(x, t, r)| \leqslant M_1$ a.e. in $(0, 1) \times (0, T) \times (-M, M)$

(ii) $d(x, t, .)$ is uniformly Lipschitz continuous that is

$$|d(x, t, r_1) - d(x, t, r_2)| \leqslant L\,|r_1 - r_2| \text{ a.e. in } Q. \qquad (2.14)$$

Let there be given a sequence

$$u_n = u_n(x, t), \text{ such that } |u_n(x, t)| \leqslant M$$

$$u_n \to u_0 \text{ strongly in } L^2(Q) \text{ for some } u_0 \in L^2(Q)$$

then

$$d_n \to d_0 \text{ strongly in } L^2(Q)$$

where:

$$d_n = d\big(x, t, u_n(x, t)\big), \quad d_0 = d\big(x, t, u_0(x, t)\big).$$

Proof. We have

$$||d_n - d_0||_{L^2(Q)} = \int_Q (d_n - d_0)^2 \, dQ \leqslant L^2 \int_Q (u_n - u_0) \, dQ \to 0.$$

LEMMA 2. Assume, that there is given a sequence $a_n \in L^2(Q)$ and an element $a_0 \in L^2(Q)$ such that

(i) $0 < \alpha_1 \leqslant a_n(x, t) \leqslant M$ a.e. in Q

(ii) $a_n \to a_0$ strongly in $L^2(Q)$

then

$$\frac{1}{a_n} \to \frac{1}{a_0} \text{ strongly in } L^2(Q).$$

Proof. We have

$$\left\| \frac{1}{a_n} - \frac{1}{a_0} \right\|^2_{L^2(Q)} = \int_Q \left(\frac{1}{a_n} - \frac{1}{a_0} \right)^2 dQ =$$

$$= \frac{1}{a_n^2 \, a_0^2} \int_Q (a_n - a_0)^2 \quad dQ \leqslant \frac{1}{\alpha_1^4} \int_Q (a_n - a_0)^2 \, dQ \to 0.$$

LEMMA 3. Let there ave given sequences $\{a_n\}, \{b_n\}$ such that:

$$|b_n|, \left| \frac{1}{a_n} \right| \leqslant M_2$$

$$b_n \rightharpoonup b_0 \text{ weakly in } L^2(Q)$$

$$\frac{1}{a_n} \to \frac{1}{a_0} \text{ strongly in } L^2(Q)$$

then

$$\frac{b_n}{a_n} \rightharpoonup \frac{b_0}{a_0} \text{ weakly in } L^2(Q).$$

Proof.

$$\int_Q \left(\frac{b_n}{a_n} \right)^2 dQ \leqslant M_2^4 \int_Q dQ$$

so there exists a subsequence $n \to \infty$ and an element $\psi \in L^2(Q)$ such that

$$\frac{b_n}{a_n} \rightharpoonup \psi \text{ weakly in } L^2(Q)$$

we will show, that $\psi = \frac{b_0}{a_0}$. Let $\varphi \in \mathscr{D}(Q)$ be given element, we have

$$\int_Q \left(\frac{b_n}{a_n} - \frac{b_0}{a_0}\right) \varphi \, dQ = \int_Q b_n \left(\frac{1}{a_n} - \frac{1}{a_0}\right) \varphi \, dQ +$$

$$+ \int_Q \frac{\varphi}{a_0} (b_n - b_0) \, dQ \to 0 \qquad \forall \varphi \in \mathscr{D}(Q)$$

$$\text{so } \frac{b_n}{a_n} \to \frac{b_0}{a_0} = \psi \text{ in } \mathscr{D}'(Q).$$

Now we can state a lemma concerning continuity of the cost functional $J(u)$ on the set of admissible controls which satisfy constraints in the form (2.2), namely.

LEMMA 4. Let there be given a sequence $\{u_n\}$ such that

$$|u_n|, |u_{n,x}|, |u_{n,t}| \leqslant M \tag{2.15}$$

then there exists an element $\bar{u}$, with $|\bar{u}|, |\bar{u}_x|, |\bar{u}_t| \leqslant M$ such that

$$J(u_n) \to J(\bar{u}). \tag{2.16}$$

Proof. By assumption (2.15) there exists a subsequence still denoted by $n \to \infty$ and an element $\bar{u} \in H^1(Q)$ such that

$$u_n \rightharpoonup \bar{u} \text{ weakly in } H^1(Q) \tag{2.17}$$

and

$$|\bar{u}|, |\bar{u}_x|, |\bar{u}_t| \leqslant M.$$

By compact imbedding theorem [6] (2.17) implies

$$u_n \to \bar{u} \text{ strongly in } L^2(Q) \tag{2.18}$$

let us denote

$$\begin{aligned}
p_n(x, t) &= p(x, t, u_n(x, t)) \\
p_0(x, t) &= p(x, t, u_0(x, t)) \\
a_n(x, t) &= a(x, t, u_n(x, t)) \\
a_0(x, t) &= a(x, t, u_0(x, t)) \\
b_n(x, t) &= b(x, t, u_n(x, t)) \\
b_0(x, t) &= b(x, t, u_0(x, t)) \\
c_n(x, t) &= c(x, t, u_n(x, t)) \\
c_0(x, t) &= c(x, t, u_0(x, t)) \\
f_n(x, t) &= f(x, t, u_n(x, t)) \\
f_0(x, t) &= f(x, t, u_0(x, t))
\end{aligned}$$

by (2.18), by assumption (A1) (ii) and by Lemma 1 we have:

$$\begin{aligned} &p_n \to p_0 \\ &a_n \to a_0 \\ &b_n \to b_0 \text{ strongly in } L^2(Q) \\ &c_n \to c_0 \\ &f_n \to f_0 \end{aligned} \tag{2.19}$$

so also

$$\begin{aligned} &p_n \rightharpoonup p_0 \\ &c_n \rightharpoonup c_0 \text{ weakly in } L^2(Q) \\ &f_n \rightharpoonup f_0 \end{aligned} \tag{2.20}$$

and by Lemmas 2, 3

$$\left.\begin{aligned} &\frac{1}{a_0} \rightharpoonup \frac{1}{a_0} \\ &\frac{b_n}{a_n} \rightharpoonup \frac{b_0}{a_0} \end{aligned}\right\} \text{ weakly in } L^2(Q) \tag{2.21}$$

so we can make use of the result of Markov and Olejnik, hence for every $\delta > 0$

$$y_n(x, t) \to y_0(x, t) \tag{2.22}$$

uniformly on $(0, 1) \times (\delta, T)$. But by maximum principle [8] $|y_n(x, t)| \leqslant C$, where C does not depend on n, hence

$$\Lambda_1 y_n \to \Lambda_1 y_0 \text{ strongly in } L^2(Q) \tag{2.23}$$

and by (2.22)

$$\Lambda_2 y_n \to \Lambda_2 y_0 \text{ strongly in } L^2(\Omega)$$

whence

$$J(u_n) \to J(u_0)$$

THEOREM 1. There exists an optimal solution to the problem (P1).

Proof. Take in Lemma 4 $\{u_t\}$ to be a minimizing sequence for (P1). There exists an element $\bar{u} \in H^1(Q)$ such that

$$u_n \to \bar{u} \text{ strongly in } L^2(Q)$$

$$J(u_n) \to J(\bar{u})$$

and

$$|\bar{u}|, |\bar{u}_x|, |\bar{u}_t| \leqslant M$$

whence $\bar{u}$ is a solution to (P1).

3. Parametric optimization problem

Consider the following linear parabolic equation

$$\left(\frac{\partial y}{\partial t}, z\right) + a_v (t, y, z) = (f, z), \ \forall z \in V \tag{3.1}$$

$$y(x, 0) = y_0, \quad x \in \Omega \tag{3.2}$$

where $y_0 \in L^2(\Omega)$, $f \in L^2(0, T, V')$ are given elements.

Bilinear form $a_v(t, y, z)$ is defined as follows:

$$a_v(t, y, z) = \sum_{i,j=1}^{n} \int_{\Omega} a_{ij} \frac{\partial y}{\partial x_i} \frac{\partial z}{\partial x_j} dx + \sum_{i=1}^{n} \int_{\Omega} a_i y \frac{\partial z}{\partial x_i} dx + \sum_{i=1}^{n} \int_{\Omega} b_i \frac{\partial y}{\partial x_i} z \, dx + \int_{\Omega} c y z \, dx + \int_{\Gamma} d y z \, dx, \ \forall y, z \in V \tag{3.3}$$

by

$$v = (a_{ij}, a_i, b_i, c, d; \ 1 \leqslant i, j \leqslant n) \tag{3.4}$$

we denote vector parameter, elements a_{ij}, a_i, b_i, c, d are coefficients in elliptic operator and in boundary conditions of equation (3.1), (3.2).

In particular parameter v can depend on control u in control problems, where coefficients depend on control. Let there be given functional:

$$I_2(v, y): [L^2(0, T); H^1(\Omega)]^{n^2+n} \times [L^2(Q)]^{n+1} \times L^2(\Sigma) \times Y(0, T) \to R \tag{3.5}$$

which is weakly lower semicontinuous.

Like in previous section we will consider an optimization problem, which will be denoted (P2):

(P2) minimize $J(v) = I_2(v, y_v)$

subject to constraints in the form of parabolic equation (3.1), (3.2) and some constraints on coefficients, namely

(A3) $|a_{ij}|, |a_i|, |b_i|, |c|, |d| \leqslant M$

$a_{ij}(x, t) = a_{ji}(x, t) \ \forall_{i,j} \ \forall (x, t) \in Q$

$$\nu |\xi|^2 \leqslant \sum_{i,j=1} a_{ij} \xi_i \xi_j \leqslant \mu |\xi|^2$$

$\nu > 0, \ \forall \xi \in R^n, \ \forall (x, t) \in Q$

(A4) $\left\| \frac{\partial a_{ij}}{\partial x_l} \right\|_{L^2(Q)}, \ \left\| \frac{\partial a_i}{\partial x_l} \right\|_{L^2(Q)} \leqslant C$

$l, i, j = 1, \ldots, n.$

REMARK. Let us note, that for parameters $\boldsymbol{v}$ elements of which verify to assumption (A3) there exists [6] a unique solution $y=y_{\boldsymbol{v}} \in Y(0, T)$ to the problem (3.1), (3.2). Furthermore the system (3.1), (3.2) generates the linear mapping

$$P(\boldsymbol{v}): L^2(\Omega)\times L^2(0, T, V') \ni (y_0, f) \mapsto y_{\boldsymbol{v}} \in Y(0, T) \tag{3.6}$$

which is an isomorphism [6].

We prove a lemma concerning convergence of a sequence of solutions of (3.1), (3.2) with respect to the sequence of its coefficients.

LEMMA 5. Assume, that there is given a sequence of vector parameters

$$\boldsymbol{v}_m=(a_{ij,m}, a_{i,m}, b_{i,m}, c_m, d_m; 1\leq i, j\leq n) \quad m=1, 2, \ldots$$

which satisfies to assumptions (A3), (A4) and converges to a vector parameter

$$\boldsymbol{v}^*=(a^*_{ij}, a^*_i, b^*_i, c^*, d^*)$$

in the following sense:

$$\begin{aligned} a_{ij,m} &\rightharpoonup a^*_{ij} && \text{weakly in } L^2(0, T; H^1(\Omega)) \\ a_{i,m} &\rightharpoonup a^*_i && \text{weakly in } L^2(0, T; H^1(\Omega)) \\ b_{i,m} &\rightharpoonup b^*_i && \text{weakly in } L^2(Q) \\ c_m &\rightharpoonup c^* && \text{weakly in } L^2(Q) \\ d_m &\rightharpoonup d^* && \text{weakly in } L^2(\Sigma) \end{aligned} \tag{3.7}$$

then the corresponding sequence of solutions $y_m=y_{\boldsymbol{v}m}$ converges to the solution $y^*=y_{\boldsymbol{v}^*}$ in the following sense

$$y_m \rightharpoonup y^* \text{ weakly in } Y(0, T) \tag{3.8}$$

$$y_m(t)\to y^*(t) \text{ strongly in } L^2(\Omega) \tag{3.9}$$

$$\forall t\in(0, T].$$

Proof. Let $\{\boldsymbol{v}_m\}$ be given sequence of vector parameters. Denote by $P(\boldsymbol{v}_m)$ linear operator in the form (3.6) for $m=1, 2, \ldots$, and by $y_m=y_{v_m}$ the corresponding solution of the problem (3.1), (3.2).

Let us note, that the sequence $[P(v_m)\}$ is convergent in the following sense:

$$P(\boldsymbol{v}_m)(y_0, f) \rightharpoonup P(\boldsymbol{v}^*)(y_0, f) \tag{3.10}$$

weakly in $Y(0, T)$, $\forall (y_0, f)\in L^2(\Omega)\times L^2(0, T, V')$ if and only if

$$P(\boldsymbol{v}_m)[P(\boldsymbol{v}^*)]^{-1} y \rightharpoonup y \text{ weakly in } Y(0, T) \tag{3.11}$$

for every element y in a dense subset of the space $Y(0, T)$, for example for every function y, which is smooth on Q. Hence without loss of generality we can assume, that solution of the system (3.1), (3.2) $y^*=y_{\boldsymbol{v}^*}=P(\boldsymbol{v}^*)(y_0, f)$ which corresponds to date (y_0, f) is a smooth function. Define

$$w_m=y_m-y^*. \tag{3.12}$$

We show, that the sequence $\{w_m\}$, which is bounded in $Y(0, T)$ converges to zero strongly in $L^\infty(0, T, L^2(\Omega) \cap L^2(0, T; V))$. Obviously, w_n is a solution of the following parabolic equation:

$$\left(\frac{dw_m}{dt}, z\right) + a_{vm}(t; w_m, z) = a_{vm}(t; y^*, z) - a_v(t; y^*, z), \ \forall z \in V \tag{3.13}$$

$$w_m(0) = 0. \tag{3.14}$$

By setting $z = w_m$ in (3.13) and integrating (3.13) over time interval $(0, t)$ where $t \in (0, T)$ we obtain the following estimation:

$$\|w_m\|^2_{L^\infty(0, T; L^2(\Omega))} + \|w_m\|^2_{L^2(0, T; V)} \leqslant$$

$$\leqslant C \left| \int_0^T [a_{vm}(t; y^*, w_m) - a_v(t; y^*, w_m)] \, dt \right| \tag{3.15}$$

but we can write

$$\int_0^T [a_{vm}(t; y^*, w_m) - a_v(t; y^*, w_m)] \, dt = \sum_{i=1}^7 f_i^m \tag{3.16}$$

where

$$f_1^m = \int_Q w_m \left\{ \sum_{i,j=1}^n \frac{\partial}{\partial x_j} \left[(a_{ij,m} - a_{ij}) \frac{\partial y^*}{\partial x_i} \right] \right\} dQ \tag{3.17}$$

$$f_2^m = \int_\Sigma w_m \left\{ \sum_{i,j=1}^n (a_{ij,m} - a_{ij}) \frac{\partial y^*}{\partial x_i} \cos(\vec{n}, x_j) \right\} d\Sigma \tag{3.18}$$

$$f_3^m = \int_Q w_m \left\{ \sum_{i=1}^n \frac{\partial}{\partial x_i} (a_{i,m} - a_i) y^* \right\} dQ \tag{3.19}$$

$$f_4^m = \int_\Sigma w_m \left\{ \sum_{i=1}^n (a_{i,m} - a_i) y^* \cos(\vec{n}, x_i) \right\} d\Sigma \tag{3.20}$$

$$f_5^m = \int_Q w_m \left\{ \sum_{i=1}^n (b_{i,m} - b_i) \frac{\partial y^*}{\partial x_i} \right\} dQ \tag{3.21}$$

$$f_6^m = \int_Q w_m (c_m - c) y^* \, dQ \tag{3.22}$$

$$f_7^m = \int_\Sigma w_m (d_m - d) y^* \, d\Sigma. \tag{3.23}$$

Since y^* is assumed to be a smooth function, we have

$$\frac{\partial^2 y^*}{\partial x_i \partial x_j}, \frac{\partial y^*}{\partial x_i} \in L^\infty(Q), \ i, j = 1, \ldots, n \tag{3.24}$$

$$y^* \Big|_\Sigma, \frac{\partial y^*}{\partial x_i} \Big|_\Sigma \in L^\infty(\Sigma), \ i = 1, \ldots, n \tag{3.25}$$

where by $y|_\Sigma$ we denote the trace [6] of function y on Σ. Let us recall [6], that for given element $y \in L^2(0, T, H^1(\Omega))$ its trace $y|_\Sigma \in L^2(\Sigma)$ is well defined and linear mapping

$$L^2(0, T; H^1(\Omega)) \in y \mapsto y|_\Sigma \in L^2(\Sigma) \tag{3.26}$$

is continuous [6], hence the above mapping is also continuous in weak topologies. In particular (3.7) implies, that

$$a_{ij,m}|_\Sigma \rightharpoonup a^*_{ij}|_\Sigma \text{ weakly in } L^2(\Sigma) \tag{3.27}$$

$$a_{i,m}|_\Sigma \rightharpoonup a^*_i|_\Sigma \text{ weakly in } L^2(\Sigma). \tag{3.28}$$

On the other hand the sequence $\{w_m\}$ is uniformly bounded in the space $Y(0, T)$, hence by compact imbedding theorem ([5] p. 70th. 5.1) the sequence $\{w_m\}$ belongs to a compact set of the space $L^2(0, T; H^{1-\varepsilon}(\Omega))$, $\forall \varepsilon > 0$. But for $0 < \varepsilon < \frac{1}{2}$ the linear mapping

$$L^2(0, T; H^{1-\varepsilon}(\Omega)) \ni y \mapsto y|_\Sigma \in L^2(\Sigma) \tag{3.29}$$

is continuous [6] whence

(i) the sequence $\{w_m\}$ belongs to a compact set of $L^2(Q)$,

(ii) the sequence of traces $\{w_m|_\Sigma\}$ belongs to a compact set of $L^2(\Sigma)$.

Using (3.7), (3.27), (3.28), (3.24), (3.25) and suitable convergence of the sequence $\{w_m\}$ or $\{w_m|_\Sigma\}$ one can verify that

$$f_i^m \to 0 \text{ as } m \to \infty, i = 1, \ldots, 7 \tag{3.30}$$

so $y_m \to y^*$ strongly in $L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; V)$.

But the sequence $\{y_m\}$ is uniformly bounded in the space $Y(0, T)$, whence

$$y_m \rightharpoonup y^* \text{ weakly in } Y(0, T). \tag{3.31}$$

Let there are given constants M, ν, μ, C. Denote by $U_{ad} \subset U = [L^2(0, T; H^1 \times \times (\Omega))]^{n^2+n} \times [L^2(Q)]^{n+1} \times L^2(\Sigma)$ the set of vector parameters $\boldsymbol{v}$ which satisfy to (A3) (A4). One can check, that the set U_{ad} is convex, bounded and closed subset of the space U so it is weakly compact in U. Using Lemma 5 it is easy to prove the following theorem.

THEOREM 2. There exists an optimal solution to the problem (P2).

Proof. By Lemma 5 and assumption concerning functional I_2, $J(\boldsymbol{v})$ is weakly lower semicontinuous on the set of admissible parameters U_{ad}, which is weakly compact, so there exists an optimal solution to the problem (P2).

Let us consider the case, where parameter $\boldsymbol{v}$ depends on control $u = u(x, t)$.

We consider the class of functions $h = h(x, t, r)$ defined on $Q \times R^1$ (resp. on $\Sigma \times R^1$) which satisfy the following conditions:

(A5)

(i) $|h(x, t, r)| \leq C_1$ (3.32)

a.e. in Q (resp. a.e. in Σ)

$$\forall r \in [M_1, M_2]$$

where M_1, M_2 are given constants;

(ii) $h(x, t, r)$ is Lipschitz continuous with respect to r uniformly on Q (resp. on Σ) that is

$$|h(x, t, r_1) - h(x, t, r_2)| \leqslant L\, |r_1 - r_2| \tag{3.33}$$

a.e. in Q (resp. a.e. in Σ), $\forall r_1, r_2 \in [M_1, M_2]$;

(iii)

$$\left|\frac{\partial h}{\partial x_i}(x, t, r)\right|, \left|\frac{\partial h}{\partial t}(x, t, r)\right| \leqslant C_2 \tag{3.34}$$

a.e. in Q (resp. a.e. in Σ) $\forall r \in [M_1, M_2]$.

LEMMA 6. Let there be given a sequence $u_m = u_m(x, t)$ such that:

$$M_1 \leqslant u_m(x, t) \leqslant M_2 \text{ a.e. in } Q \tag{3.35}$$

$$\int_Q \left\{\sum_{i=1}^n \left(\frac{\partial u_m}{\partial x_i}\right)^2 + \left(\frac{\partial u_m}{\partial t}\right)^2\right\} dQ \leqslant C_3. \tag{3.36}$$

M_1, M_2, C_3 — are given constants.

Denote by $h_m(x, t) = h\big(x, t, u_m(x, t)\big)$, $(x, t) \in Q$, $g_m(x, t) = g\big(x, t, u_m(x, t)\big)$, $(x, t) \in \Sigma$ where function $h(.\,,.\,,.)$ (resp. $g(.\,,.\,,.)$) is assumed to satisfy condition (A5) on Q (resp. on Σ).

Then there exists a subsequence $m' \to \infty$ and an element $u_0 \in H^1(Q)$ such that

$$u_{m'} \rightharpoonup u_0 \text{ weakly in } H^1(Q) \tag{3.37}$$

$$M_1 \leqslant u_0(x, t) \leqslant M_2 \text{ a.e. in } Q \tag{3.38}$$

$$\int_Q \left\{\sum_{i=1}^n \left(\frac{\partial u_0}{\partial x_i}\right)^2 + \left(\frac{\partial u_0}{\partial t}\right)^2\right\} dQ \leqslant C_3 \tag{3.39}$$

$$h_{m'}(x, t) \rightharpoonup h_0(x, t) \text{ weakly in } H^1(Q) \tag{3.40}$$

where $h_0(x, t) = h\big(x, t, u_0(x, t)\big)$

$$g_{m'}(x, t) \rightharpoonup g_0(x, t) \text{ weakly in } L^2(\Sigma) \tag{3.41}$$

where $g_0(x, t) = g\big(x, t, u_0, (x, t)\big)$.

Proof. By assumptions (3.35), (3.36) the sequence $\{u_m\}$ is bounded in the Sobolew space $H^1(Q)$ that is

$$\|u_m\|_{H^1(Q)} \leqslant C, \; m = 1, 2, \ldots \tag{3.42}$$

hence there exists an element $u_0 \in H^1(Q)$ which verifies (3.38), (3.39) such that

$$u_{m'} \rightharpoonup u_0 \text{ weakly in } H^1(Q) \tag{3.43}$$

for some subsequence $m' \to \infty$, furthermore by compact imbedding theorems [6], [5] (3.43) implies

$$u_{m'} \to u_0 \text{ strongly in } L^2(Q) \tag{3.44}$$

$$u_{m'}|_\Sigma \to u_0 \text{ strongly in } L^2(\Sigma) \tag{3.45}$$

so using Lemma 1 we obtain:

$$h_{m'}(x, t) \to h_0(x, t) \text{ strongly in } L^2(Q) \tag{3.46}$$

$$g_{m'}(x, t) \to g_0(x, t) \text{ strongly in } L^2(\Sigma). \tag{3.47}$$

On the other hand the sequence $\{h_m\}$ is bounded in Sobolew space $H^1(Q)$

$$\|h_m\|_{H^1(Q)} \leqslant C \tag{3.48}$$

hence for some subsequence $m'' \to \infty$ of the subsequence $m' \to \infty$ we have

$$h_{m''} \rightharpoonup \psi \text{ weakly in } H^1(Q) \tag{3.49}$$

but (3.46) implies

$$\psi = h_0$$

hence whole subsequence $\{h_{m'}\}$ is convergent weakly in $H^1(Q)$

$$h_{m'} \rightharpoonup h_0 \text{ weakly in } H^1(Q).$$

For given $u \in L^2(Q)$, by $\boldsymbol{v} = \boldsymbol{v}(u)$ let us denote the vector parameter

$$\boldsymbol{v}(u) = (a_{ij}(u), a_i(u), b_i(u), c(u), d(u)) \tag{3.50}$$

and by $y_u = y_{v(u)}$ the corresponding solution of the problem (3.1), (3.2)

We assume:

(A6)

(i) functions $a_{ij}(x, t, u)$, $a_i(x, t, u)$, $b_i(x, t, u)$, $c(x, t, u)$ satisfy condition (A5) on $Q \times [M_1, M_2]$ function $d(x, t, u)$ satisfies condition (A5) on $\Sigma \times [M_1, M_2]$

(ii)

$$a_{ij}(x, t, r) = a_{ji}(x, t, r), \; i, j = 1, \ldots, n$$

$$\text{a.e. in } Q \; \forall r \in [M_1, M_2]$$

$$\nu |\xi|^2 \leqslant \sum_{i,j=1}^{n} a_{ij}(x, t, r)\, \xi_i \xi_j \leqslant \mu |\xi|^2$$

$$\nu > 0, \; \mu < \infty, \; \forall \xi \in R^n, \text{ a.e. in } Q, \; \forall r \in [M_1, M_2]$$

where M_1, M_2, ν and μ are given constants.

For given constants M_1, M_2, M_3 denote by $W_{ad} \subset H^1(Q)$ the following set of admissible controls:

$$W_{ad} = \{u \in H^1(Q) \mid M_1 \leqslant u(x, t) \leqslant M_2 \text{ a.e. in } Q, \tag{3.51}$$

$$\int_Q \left\{ \sum_{i=1}^{n} \left(\frac{\partial u}{\partial x_i}\right)^2 + \left(\frac{\partial u}{\partial t}\right)^2 dQ \leqslant M_3 \right\}.$$

Given functional $I(.,.)$: $Y(0,T)\times H\mapsto R$ (where $H=L^2(Q)$ or $H=H^1(Q)$) which is assumed to be weakly sequentally lower semicontinuous.

Consider the following optimization problem, similar to (P1), namely (P2'): minimize $J(u)=I(y_u, u)$ over the set W_{ad} given by (3.51) and subject to constraints in the form of parabolic equation (3.1), (3.2) where $v=v(u)$ are given by (3.50).

THEOREM 3. There exists an optimal solution to (P2').

Proof. Let us note, that lemma 6 implies, that the mapping $H^1(Q)\supset W_{ad}\ni\mapsto$ $\mapsto v(u)\in U_{ad}\subset U$ is continuous in weak topologies from the space $H^1(Q)$ into the space

$$U=[L^2(0,T,H^1(\Omega))]^{n^2+n}\times[L^2(Q)]^{n+1}\times L^2(\Sigma).$$

By Lemmas 5, 6 the functional $J(u)$ is weakly sequentally lower semicontinuous on weakly sequentally compact set W_{ad}, hence there exists an element $u^*\in W_{ad}$ such that

$$J(u^*)\leqslant J(u)\ \forall u\in W_{ad}.$$

References

1. Bensoussan A.: L'indentification et le filtrage, *Cahiers IRIA*, 1 (1969).
2. Chavent G.: Analyse fonctionelle et identification de coefficients repartis dans les équations aux derivées partielles. Thése, Paris 1971.
3. Colombini F., Spagnolo S.: Sur la convergence d'equations paraboliques. To appear in Journal de Math. Pur. Appl.
4. Lions J. L.: Contrôle optimale de systemès gouvernés par des equations aux derivées partielles, Paris 1968.
5. Lions J. L.: Quelques méthodes de résolution des problèmes aux limites non linéaires, Paris 1969.
6. Lions J. L., Magenes E.: Problèmes aux limitiés non homogènes et applications, Vol. 1. Paris 1968.
7. Mały W.: Algorytm optymalizacji procesu dyfuzji stosowanego w produkcji elementów półprzewodnikowych, Rozprawa doktorska. Warszawa 1975.
8. Markow W. G., Olejnik O. A.: O rasprostranienii tiepła w odnomiernych diskrietnych sriedach. W: Prikładnaja matiematika i miechanika. T. 39. 1975.
9. Sokołowski J.: Problemy parametrycznego sterowania optymalnego układów opisywanych równaniami parabolicznymi. Rozprawa doktorska. Warszawa 1975.
10. Sokołowski J.: On parametric optimal control for partial differential equations of parabolic type. Proceedings of IFIP Conference, Nice 1975. Berlin, Springer Verlag 1976.
11. Sokołowski J.: Parametric optimization problem for abstract parabolic equation with variable operator domain, IRIA Laboria, Rapport de Recherche No. 183, Aout 1976.
12. Spagnolo S.: Convergence of parabolic equations. Lectures notes, Trieste, 1 Dec. 1976.
13. Sokołowski J.: On existence of solutions of parametric optimization problems for parabolic equations, Proceedings of the International Conference on Methods of Mathematical Programming, Zakopane 1977 (to appear).

14. Zolezzi T.: Teoremi d'esistenza per problemi di controllo ottimo retti da equazioni ellittiche o paraboliche. *Rend. Sem. Mat. Univ. Padova* 44 (1970) 155–173.
15. Zolezzi T.: Necessary conditions for optimal controls of elliptic or parabolic problems. *SIAM J. Control* **10**, 4, 594–607.

Received, January 1978

Uwagi na temat istnienia rozwiązań dla zagadnień sterowania optymalnego dla liniowego równania różniczkowego cząstkowego typu parabolicznego

Rozważono problem istnienia rozwiązań dla zagadnień sterowania optymalnego dla liniowego równania różniczkowego cząstkowego typu parabolicznego. Sterowanie występuje we współczynnikach operatora eliptycznego.

Wykorzystując wyniki dotyczące słabej zbieżności ciągu słabych rozwiązań równania parabolicznego uzyskano wyniki wystarczające dla istnienia rozwiązań rozważonych zadań sterowania parametrycznego.

Содержание

В работе рассматривается вопрос существования решений некоторых задач оптимального управления для линейного уравнения параболического типа в частных производных в случае, когда управляемыми являются коэффициенты элиптического оператора.

Используя результаты касающиеся слабой сходимости последовательности обобщённых решений параболического управления получено доставочные условия существования решений рассматриваемых задач параметрического управления.

Wskazówki dla Autorów

W wydawnictwie „Control and Cybernetics" drukuje się prace oryginalne nie publikowane w innych czasopismach. Zalecane jest nadysłanie artykułów w języku angielskim. W przypadku nadesłania artykułu w języku polskim, Redakcja może zalecić przetłumaczenie na język angielski. Objętość artykułu nie powinna przekraczać 1 arkusza wydawniczego, czyli ok. 20 stron maszynopisu formatu A4 z zachowaniem interlinii i marginesu szerokości 5 cm z lewej strony. Prace należy składać w 2 egzemplarzach. Układ pracy i forma powinny być dostosowane do niżej podanych wskazówek.

1. W nagłówku należy podać tytuł pracy, następnie imię (imiona) i nazwisko (nazwiska) autora (autorów) w porządku alfabetycznym oraz nazwę reprezentowanej instytucji i nazwę miasta. Po tytule należy umieścić krótkie streszczenie pracy (do 15 wierszy maszynopisu).

2. Materiał ilustracyjny powinien być dołączony na oddzielnych stronach. Podpisy pod rysunki należy podać oddzielnie.

3. Wzory i symbole powinny być wpisane na maszynie bardzo starannie.

Szczególną uwagę należy zwrócić na wyraźne zróżnicowanie małych i dużych liter. Litery greckie powinny być objaśnione na marginesie. Szczególnie dokładnie powinny być pisane indeksy (wskaźniki) i oznaczenia potęgowe. Należy stosować nawiasy okrągłe.

4. Spis literatury powinien być podany na końcu artykułu. Numery pozycji literatury w tekście zaopatruje się w nawiasy kwadratowe. Pozycje literatury powinny zawierać nazwisko autora (autorów) i pierwsze litery imion oraz dokładny tytuł pracy (w języku oryginału), a ponadto:

a) przy wydawnictwach zwartych (książki) — miejsce i rok wydania oraz wydawcę;

b) przy artykułach z czasopism: nazwę czasopisma, numer tomu, rok wydania i numer bieżący.

Pozycje literatury radzieckiej należy pisać alfabetem oryginalnym, czyli tzw. grażdanką.

Recommendations for the Authors

Control and Cybernetics publishes original papers which have not previously appeared in other journals. The publications of the papers in English is recommended. No paper should exceed in length 20 type written pages (210 × 297 mm) with lines spaced and a 50 mm margin on the lefthand side. Papers should be submitted in duplicate. The plan and form of the paper should be as follows:

1. The heading should include the title, the full names and surnames of the authors in alphabetic order, the name of the institution he represents and the name of the city or town. This heading should be followed by a brief summary (about 15 typewritten lines).

2. Figures, photographs tables, diagrams should be enclosed to the manuscript. The texts related to the figures should be typed on a separate page.

3. Of possible all mathematical expressions should be typewritten. Particular attention should be paid to differentiation between capital and small letters. Greek letters should as a rule be defined. Indices and exponents should be written with particular care. Round brackets should not be replaced by an inclined fraction line.

4. References should be put on the separate page. Numbers in the text identified by referenceo of the author (or authors), the complete title of the work (in the original language) and, in addition:

a) for books — the place and the year of publication and the publisher's name;

b) for journals (the name of the journal, the number of the volume, the year of the publicatins, and the original number.